Exercise Sheet 1.

Due: **Monday, 29.09.2025, 12:00.**

Exercise 1 (Eigenvalues). Let $n \in \mathbb{N}$ and $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a matrix. Show:

(a) If $\lambda \in \sigma(\mathbf{A})$ is a right eigenvalue, then λ is also a left eigenvalue.

(b) If $\mathbf{A}$ is hermitian, then it follows that $\sigma(\mathbf{A}) \subset \mathbb{R}$.

(c) If $\mathbf{A}$ is a diagonal matrix, then the eigenvalues are located on the diagonal.

(d) If $\mathbf{A}$ is real and $\lambda \in \sigma(\mathbf{A})$ is an eigenvalue, then $\overline{\lambda} \in \sigma(\mathbf{A})$ also is an eigenvalue.

Hint. *We call λ a right eigenvalue of $\mathbf{A}$ if there exist a vector $\mathbf{v} \in \mathbb{C}^n$, such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. We call μ a left eigenvalue of $\mathbf{A}$ if there exist a vector $\mathbf{u} \in \mathbb{C}^n$, such that $\mathbf{u}^\star\mathbf{A} = \mu\mathbf{u}^\star$.*

Exercise 2 (Square roots of positive definite matrices). Consider the symmetric positive definite matrix

$$\mathbf{A} = \begin{bmatrix} 20 & -4 \\ -4 & 20 \end{bmatrix}.$$

(a) Compute the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Compute a positive definite matrix $\mathbf{B}$, such that $\mathbf{B}^2 = \mathbf{A}$.

Hint. *Use the spectral decomposition of* $\mathbf{A}$.

(c) Do there exist other square roots of $\mathbf{A}$? Are they positive definite?

Exercise 3 (Vandermonde matrix). Let $n \in \mathbb{N}$ and $x_i \in \mathbb{R}$, $1 \le i \le n+1$ with $x_i \neq x_j$ for $i \neq j$. Then the Vandermonde matrix of these points $x_1, \ldots, x_{n+1}$ is defined as

$$\mathbf{V} = \begin{bmatrix} 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_{n+1} & \cdots & x_{n+1}^n \end{bmatrix}.$$

(a) Show that $\det(\mathbf{V}) = \prod_{i\neq j}(x_i - x_j)$. Conclude that $\mathbf{V}$ is invertible.

Hint. *Use a recursion on the dimension of the matrix, starting with a 2×2 Vandermonde matrix.*

(b) Let $L_i(x) = \prod_{j\neq i} \frac{x-x_j}{x_i-x_j}$ be defined as the i-th Lagrange polynomial of the points $x_1, \ldots, x_{n+1}$. Show that

$$L_i(x_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

(c) Let $L_i(x) = L_{1,i} + L_{2,i}x + \cdots + L_{n+1,i}x^n$ and define the matrix

$$\mathbf{L} = \begin{bmatrix} L_{1,1} & \cdots & L_{1,n+1} \\ \vdots & & \vdots \\ L_{n+1,1} & \cdots & L_{n+1,n+1} \end{bmatrix}.$$

Show that $\mathbf{L} = \mathbf{V}^{-1}$.

Exercise 4 (Gerschgorin's theorem). A matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is called *strictly diagonally dominant*, if

$$|a_{k,k}| > \sum_{\ell \neq k} |a_{k,\ell}|, \qquad k = 1, \dots, n.$$

Use Gerschgorin's theorem to prove the following statements: If $\mathbf{A}$ is strictly diagonally dominant, then $\mathbf{A}$ is invertible. If $\mathbf{A}$ is real, symmetric and $a_{k,k} > 0$ for all $k = 1, \dots, n$, then $\mathbf{A}$ is positive definite.

Exercise 5 (Companion matrix). Let $n \in \mathbb{N}$ and $c_i \in \mathbb{R}$, $0 \leq i \leq n$. The Frobenius companion matrix for these coefficients $c_0, \dots, c_n$ is defined as

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_n \end{bmatrix}.$$

(a) Show that $(-1)^{n+1} \det(\mathbf{C} - \lambda \mathbf{I}_{n+1}) = c_0 + c_1\lambda + \cdots + c_n\lambda^n + \lambda^{n+1}$.

Hint. *Develop the determinant using the last row of* $\mathbf{C}$ *and apply an inductive argument using a well chosen submatrix.*

(b) Let λ_j, $1 \leq j \leq n+1$ be the eigenvalues of $\mathbf{C}$. Provided that the λ_j's are all different, show that $\mathbf{C}$ is diagonalizable with spectral decomposition

$$\mathbf{C} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V},$$

where $\mathbf{D}$ is the diagonal matrix containing the eigenvalues of $\mathbf{C}$ and $\mathbf{V}$ is the Vandermonde matrix based on the eigenvalues of $\mathbf{C}$.