Exercise Sheet 2.

Due: **Monday, 06.10.2025, 12:00.**

Exercise 1 (Location of eigenvalues). Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & 4 & 4 & 0 \\ 0 & -2 & -2 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}.$$

With the help of Gerschgorin's theorem (applied to $\mathbf{A}$ and $\mathbf{A}^\intercal$) and Bendixson's theorem, give sets as small as possible that contain the spectrum of $\mathbf{A}$. Sketch these sets in the complex plane.

Exercise 2 (Rayleigh quotient). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $f : \mathbb{R}^n \to \mathbb{R}$ be defined through the Rayleigh quotient as:

$$f(\mathbf{x}) = \frac{\mathbf{x}^\intercal \mathbf{A}\mathbf{x}}{\|\mathbf{x}\|_2^2}.$$

Show the following statements:

(a) For an $\mathbf{x} \in \mathbb{R}^n$, $\nabla f(\mathbf{x}) = \mathbf{0}$ holds if and only if $\mathbf{x}$ is an eigenvector of the matrix $\frac{1}{2}(\mathbf{A}+\mathbf{A}^\intercal)$ with eigenvalue $f(\mathbf{x})$.

(b) If $\mathbf{A}$ is symmetric with real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then

$$\lambda_1 = \max_{\mathbf{x}\neq\mathbf{0}} f(\mathbf{x}), \qquad \lambda_n = \min_{\mathbf{x}\neq\mathbf{0}} f(\mathbf{x}).$$

(c) For any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ one has

$$\inf_{t\in\mathbb{R}} \|\mathbf{A}\mathbf{x} - t\mathbf{x}\|_2^2 = \|\mathbf{A}\mathbf{x} - f(\mathbf{x})\mathbf{x}\|_2^2.$$

Exercise 3 (Min-Max-Principle). Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be hermitian und $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ its real eigenvalues. For $1 \leq k \leq n$ we define $V_k := \operatorname{span}\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$, where $\mathbf{v}_j$ is an eigenvector corresponding to the eigenvalue λ_j.

(a) Show that:

$$\lambda_{k+1} = \sup_{\substack{\mathbf{x}\neq\mathbf{0} \\ \mathbf{x}\perp V_k}} \frac{\mathbf{x}^\star \mathbf{A}\mathbf{x}}{\|\mathbf{x}\|_2^2}, \qquad k = 1, \ldots, n-1.$$

(b) Let $W \subset \mathbb{C}^n$ be a k-dimensional subspace. Show that $\dim(W^\perp \cap V_{k+1}) \geq 1$.

(c) Conclude that

$$\lambda_{k+1} = \inf_{\substack{W\subset\mathbb{C}^n \\ \dim W\leq k}} \sup_{\substack{\mathbf{x}\neq\mathbf{0} \\ \mathbf{x}\perp W}} \frac{\mathbf{x}^\star \mathbf{A}\mathbf{x}}{\|\mathbf{x}\|_2^2}.$$

Exercise 4 (Power method). Consider the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Compute the eigenvalues of $\mathbf{A}$.

(b) Compute the first five terms $(\mathbf{z}_k)_k$ of the Power method with starting vector $\mathbf{z}_0 = [1, 0, 0]^\mathsf{T}$ and using the 2-norm, $\|\cdot\| = \|\cdot\|_2$. Also, compute the approximation $\mu_k = \mathbf{z}_k^\mathsf{T} \mathbf{A} \mathbf{z}_k$ of the largest eigenvalue of $\mathbf{A}$ in every step.