Exercise Sheet 3.

Due: **Monday, 13.10.2025, 12:00.**

Exercise 1 (Cholesky decomposition). Consider a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. Show:

(a) There exists a lower triangular matrix $\mathbf{L} \in \mathbb{R}^{n\times n}$ such that $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ if and only if $\mathbf{A}$ is a symmetric positive semi-definite matrix (SPSD).

(b) If $\mathbf{A}$ is a symmetric positive definite matrix (SPD) and the diagonal entries of $\mathbf{L}$ are required to be positive, then $\mathbf{L}$ is unique.

Exercise 2 (Circulant matrix). Consider the circulant matrix

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

(a) Show that all eigenvalues from $\mathbf{S}$ satisfy $|\lambda| = 1$.

(b) Apply the power method with starting vector $\mathbf{z}_0 = \mathbf{e}_1$. What is the k-th iteration?

(c) Obviously, the iterations do not converge. Why is this not a contradiction to Theorem 1.14?

Exercise 3 (Eigenvalues approximation). Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Moreover, let $\mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.

(a) Show the estimate

$$\min_{1\le j\le n} |\mu - \lambda_j| \le \frac{\|\mathbf{A}\mathbf{x} - \mu\mathbf{x}\|_2}{\|\mathbf{x}\|_2}.$$

Hint. *Use the spectral theorem.*

(b) Conclude that

$$\min_{1\le j\le n} |a_{k,k} - \lambda_j| \le \left(\sum_{j\ne k} |a_{j,k}|^2\right)^{\frac{1}{2}}, \qquad k = 1, \dots, n.$$

(c) Apply the result from part (b) to the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 7 \end{bmatrix}.$$

Which of the eigenvalues $\lambda_1 = 13$, $\lambda_2 = 4$, $\lambda_3 = 2$ are approximated by each of the diagonal entries according to part (b)?

Exercise 4 (Frobenius-Norm). Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. The *Frobenius-norm* is defined as

$$\|\mathbf{A}\|_F := \left(\sum_{k,\ell=1}^{n} |a_{k,\ell}|^2 \right)^{\frac{1}{2}}.$$

(a) Show that $\|\mathbf{A}\|_F^2 = \mathrm{Trace}(\mathbf{A}^\star \mathbf{A})$.

(b) Show that $\mathrm{Trace}(\mathbf{AB}) = \mathrm{Trace}(\mathbf{BA})$ holds for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$.

(c) Conclude that $\|\mathbf{Q}^\star \mathbf{A}\mathbf{Q}\|_F = \|\mathbf{A}\|_F$ for all unitary matrix $\mathbf{Q}$.

(d) Show that $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F$ holds.

Hint. *Use the identity* $\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^\star \mathbf{A})}$.