Exercise Sheet 4.

Due: **Monday, 20.10.2025, 12:00.**

Exercise 1 (LR algorithm). In this exercise, we consider the LR algorithm for (2×2) matrices.

(a) Let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $a_{11} \neq 0$. Find the LR decomposition of $\mathbf{A}$, i.e. the matrices $\mathbf{L}$ and $\mathbf{R}$ such that

$$\mathbf{A} = \mathbf{LR}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \beta & \gamma \\ 0 & \delta \end{bmatrix},$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

(b) Derive $\mathbf{L}$ and $\mathbf{R}$ for the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}.$$

Why should the LR algorithm stop after the first iteration?

(c) Perform the LR algorithm for the matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

until the absolute value of the subdiagonal entry is less than 0.01.

(d) Determine the errors of the approximated eigenvalues obtained in part (c).

Exercise 2 (QR decomposition). Derive the QR decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 4 & -2 \\ 2 & 0 & 6 \end{bmatrix}.$$

Specify the matrices $\mathbf{Q}$ and $\mathbf{R}$ explicitly and verify that $\mathbf{QR} = \mathbf{A}$ holds.

Exercise 3 (Givens rotation). Given $1 \leq i < j \leq n$ and $0 \leq \theta < 2\pi$, we define

$$\mathbf{G}_{i,j}(\theta) = \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos\theta & & & & \sin\theta & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & -\sin\theta & & & & \cos\theta & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{matrix}$$

(a) Show that the matrix $\mathbf{G}_{i,j}(\theta)$ produces a rotation of the (i, j) plane with angle $-\theta$.

(b) Show that the successive multiplication of the Givens rotations $\mathbf{G}_{i,j}(\theta)$ with $\mathbf{A}$ yields a QR decomposition.

(c) Is this scheme more expensive in terms of operations needed than the Householder scheme?

Exercise 4 (Schur decomposition). Let $\mathbf{A} \in \mathbb{C}^{n\times n}$. Prove by induction over n that there exists a unitary matrix $\mathbf{Q} \in \mathbb{C}^{n\times n}$ such that

$$\mathbf{Q}^\star \mathbf{A}\mathbf{Q} = \begin{bmatrix} \lambda_1 & \star & \cdots & \cdots & \star \\ 0 & \lambda_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda_{n-1} & \star \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix} = \mathbf{R}.$$