Exercise Sheet 5.

Due: **Monday, 27.10.2025, 12:00.**

Exercise 1 (Schur decomposition II). Using Schur's decomposition, prove the spectral theorem: *A matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *is normal if and only if there exists a unitary matrix* $\mathbf{Q} \in \mathbb{C}^{n\times n}$*, such that* $\mathbf{Q}^\star\mathbf{A}\mathbf{Q} = \mathbf{D}$ *is a diagonal matrix.*

Hint. *Consider Schur's decomposition. Is* $\mathbf{R}$ *normal? Is* $\mathbf{D}$ *also normal?*

Exercise 2 (Householder-Deflation). Take a diagonalizable matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$. Suppose that the largest eigenvalues λ_1 and its corresponding eigenvector $\mathbf{v}_1$ have been determined with the help of the power method.

(a) Determine a vector $\mathbf{u} \in \mathbb{R}^n$ such that for the associated Householder transformation $\mathbf{Q}_\mathbf{u}$ and a $\sigma \neq 0$ the relationship

$$\mathbf{Q}_\mathbf{u}\mathbf{v}_1 = \sigma\mathbf{e}_1$$

holds.

(b) Show that $\mathbf{A}$ can be transformed into a block structure by

$$\mathbf{Q}_\mathbf{u}\mathbf{A}\mathbf{Q}_\mathbf{u} = \begin{bmatrix} \lambda_1 & \mathbf{b}^\intercal \\ \mathbf{0} & \mathbf{A}_1 \end{bmatrix},$$

where $\mathbf{A}_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\mathbf{b} \in \mathbb{R}^{n-1}$.

(c) Show that the matrix $\mathbf{A}_1$ has the eigenvalues $\lambda_2, \dots, \lambda_n$.

The power method then can be applied again to the smaller matrix $\mathbf{A}_1$ yielding the next eigenvalue λ_2 and associated eigenvector $\mathbf{v}_2$. Hence this *Householder-Deflation* method can be applied successively to determine all eigenpairs of the matrix $\mathbf{A}$.

Exercise 3 (Orthonormal polynomials and tridiagonal matrices). Let $(p_n)_{n\geq 0}$ be a sequence of real orthonormal polynomials defined by the three-term recursion

$$\lambda p_n(\lambda) = \beta_n p_{n-1}(\lambda) + \alpha_{n+1}p_n(\lambda) + \beta_{n+1}p_{n+1}(\lambda), \qquad n \geq 0,$$

with $p_{-1} = 0$, $p_0 = 1$ and $\beta_n > 0$. Show that, for $n \geq 1$, the roots of p_n are the eigenvalues of the following tridiagonal matrix

$$\mathbf{T}_n = \begin{bmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{bmatrix}.$$

What are the corresponding eigenvectors?

Exercise 4 (Krylov subspaces). For a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{v} \in \mathbb{R}^n$ let the Krylov subspaces $\mathcal{K}_k$ be defined as

$$\mathcal{K}_k(\mathbf{A}, \mathbf{v}) := \operatorname{span}\left\{\mathbf{v}, \mathbf{A}\mathbf{v}, \dots, \mathbf{A}^{k-1}\mathbf{v}\right\}, \quad k = 1, 2, \dots.$$

(a) Let Π_m for $m \in \mathbb{N}$ be the space of all real polynomials of degree deg $\leq m$. Show

$$\mathcal{K}_k(\mathbf{A}, \mathbf{v}) = \left\{ \mathbf{x} \in \mathbb{R}^n \ : \mathbf{x} = p(\mathbf{A})\mathbf{v} \text{ with } p \in \Pi_{k-1} \right\} \quad \text{for all } k \geq 1.$$

(b) For $\mathbf{v} \in \mathbb{R}^n$, we define

$$m = \min\left\{ k \in \mathbb{N} \, : \, \text{there exists } p \in \Pi_k \setminus \{0\} \text{ with } p(\mathbf{A})\mathbf{v} = \mathbf{0} \right\}$$

as the *degree of* $\mathbf{v}$ *with respect to* $\mathbf{A}$. The corresponding polynomial $p \in \Pi_m$, that satifies $p(\mathbf{A})\mathbf{v} = \mathbf{0}$, is called the *minimal polynomial* of $\mathbf{v}$ with respect to $\mathbf{A}$.

Let $\ell \leq k$ be the degree of the minimal polynomial of $\mathbf{v}$ with respect to $\mathbf{A}$. Show

$$\mathcal{K}_k(\mathbf{A}, \mathbf{v}) = \mathcal{K}_\ell(\mathbf{A}, \mathbf{v}) \quad \text{for all } k \geq \ell.$$

(c) Let $\mathbf{A}$ be a non-zero real symmetric $n \times n$ matrix and define μ as the number of distinct eigenvalues of A. Moreover, let ℓ be the degree of the minimal polynomial of $\mathbf{v}$ with respect to $\mathbf{A}$. Show

$$\ell \leq \mu.$$

Hint. *Use the idea of Cayley-Hamilton theorem.*