Exercise Sheet 6.

Due: **Monday, 03.11.2025, 12:00.**

Exercise 1 (Tridiagonal matrices). Consider the tridiagonal matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$ given by

$$\mathbf{D} = \begin{bmatrix} \alpha & \gamma & & 0 \\ \beta & \alpha & \ddots & \\ & \ddots & \ddots & \gamma \\ 0 & & \beta & \alpha \end{bmatrix}, \qquad \beta, \gamma > 0.$$

Show that, the eigenvalues of $\mathbf{D}$ are

$$\lambda_k = \alpha + 2\sqrt{\beta\gamma}\,\operatorname{sign}(\beta)\cos\left(\frac{k\pi}{n+1}\right), \qquad \text{for } k = 1, \ldots, n.$$

Furthermore, show that the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ fulfil

$$[\mathbf{v}_k]_i = \left(\frac{\beta}{\gamma}\right)^{\frac{i-1}{2}} \sin\left(\frac{ik\pi}{n+1}\right), \qquad \text{for } i, k = 1, \ldots, n,$$

where $[\mathbf{v}_k]_i$ denotes the i-th entry of $\mathbf{v}_k$.

Exercise 2 (Lanczos algorithm). The matrix

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 4 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 1 & 3 \\ -\sqrt{2} & 3 & 1 \end{bmatrix}$$

is given. Apply the Lanczos algorithm on the matrix $\mathbf{A}$ with starting vectors $\mathbf{z}_1 = [1, 0, 0]^\top$ and $\mathbf{z}_2 = \frac{1}{2}[\sqrt{2}, -1, 1]^\top$ and explain the observed behavior.

Exercise 3 (Lucky break in the Arnoldi iteration). Consider the Arnoldi algorithm for $\mathbf{A} \in \mathbb{R}^{n\times n}$ invertible and an arbitrary $\mathbf{z} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Show that in the case $h_{m+1,m} = 0$, i.e. when the Arnoldi algorithm has ended, the m-th step gives $\mathbf{x} = \mathbf{A}^{-1}\mathbf{z} \in \mathcal{K}_m(\mathbf{A}, \mathbf{z})$.

Exercise 4 (Linear least squares problem).

Consider the matrix

$$\mathbf{A} = \frac{1}{5}\begin{bmatrix} 8 & 2 & 4 & 4 \\ 2 & 2 & -4 & -1 \end{bmatrix}^\top.$$

(a) Derive the singular value decomposition of $\mathbf{A}$.

(b) Derive the pseudoinverse $\mathbf{A}^+$ with the help of the singular value decomposition as well as with the equality $\mathbf{A}^+ = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top$.

(c) Use $\mathbf{A}^+$, to solve the linear least squares problem defined by $\mathbf{A}$ and $\mathbf{b} = [1, 2, 3, 4]^\top$.