Exercise Sheet 7.

Due: **Monday, 10.11.2025, 12:00.**

Exercise 1 (Computation of the singular value decomposition). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Consider a sequence of matrices $\mathbf{A} = \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \ldots$, where $\mathbf{A}_{k+1}$ is derived from $\mathbf{A}_k$ by applying an orthogonal transformation $\mathbf{P}_k$, $\mathbf{A}_{k+1} = \mathbf{P}_k\mathbf{A}_k$, and, for an even integer k, the matrices are of the form

$$\mathbf{A}_k = \begin{bmatrix} a_{k,1} & b_{k,1} & & 0 \\ & a_{k,2} & \ddots & \\ & & \ddots & b_{k,n-1} \\ 0 & & & a_{k,n} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_{k+1} = \begin{bmatrix} a_{k+1,1} & & & 0 \\ b_{k+1,1} & \ddots & & \\ & \ddots & a_{k+1,n-1} & \\ 0 & & b_{k+1,n-1} & a_{k+1,n} \end{bmatrix}.$$

Show that:

(a) There exists orthogonal matrices $\mathbf{Q}_1, \mathbf{Q}_2 \in \mathbb{R}^{n\times n}$ such that $\mathbf{Q}_1\mathbf{A}\mathbf{Q}_2 = \tilde{\mathbf{A}}$, where $\tilde{\mathbf{A}}$ is a bidiagonal matrix.

(b) The limit $\lim_{k\to\infty} b_{k,j} = 0$ holds for all $j = 1, \ldots, n-1$.

(c) There exists an integer $K \in \mathbb{N}$, such that $a_{k,1} \geq a_{k,2} \geq \cdots \geq a_{k,n}$ holds for all $k > K$.

Exercise 2 (Estimates of singular values). Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Prove the following inequalities, where $\sigma_{\max}$ and $\sigma_{\min}$ are the biggest and smallest singular values, respectively.

(a) For a matrix $\mathbf{E} \in \mathbb{R}^{m\times n}$,

$$\sigma_{\max}(\mathbf{A}+\mathbf{E}) \leq \sigma_{\max}(\mathbf{A}) + \|\mathbf{E}\|_2 \qquad \text{and} \qquad \sigma_{\min}(\mathbf{A}+\mathbf{E}) \geq \sigma_{\min}(\mathbf{A}) - \|\mathbf{E}\|_2.$$

(b) For $\mathbf{z} \in \mathbb{R}^m$ let $[\,\mathbf{A}\,|\,\mathbf{z}\,] \in \mathbb{R}^{m\times(n+1)}$ be the matrix defined by adding the column $\mathbf{z}$ to the matrix $\mathbf{A}$. Show that

$$\sigma_{\max}([\,\mathbf{A}\,|\,\mathbf{z}\,]) \geq \sigma_{\max}(\mathbf{A}) \qquad \text{and} \qquad \sigma_{\min}([\,\mathbf{A}\,|\,\mathbf{z}\,]) \leq \sigma_{\min}(\mathbf{A}).$$

Exercise 3 (Eckart-Young-Mirsky theorem). Let the matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$be defined by its singular value decomposition $(\{\sigma_i\}_{i=1}^r, \{\mathbf{u}_i\}_{i=1}^m, \{\mathbf{v}_i\}_{i=1}^n)$. Consider the following low rank approximations of $\mathbf{A}$

$$\mathbf{A}_k = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^T,$$

where $1 \leq k < r$. Show the following statements:

(a) It holds $\|\mathbf{A} - \mathbf{A}_k\|_2 = \sigma_{k+1}$.

(b) For every matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(\mathbf{B}) = k$, there exists a vector $\mathbf{z} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}\}$, $\mathbf{z} \neq \mathbf{0}$, such that $\mathbf{B}\mathbf{z} = \mathbf{0}$.

(c) Show that, among all matrices $\mathbf{B} \in \mathbb{R}^{m\times n}$ with rank at most equal to k, $\mathbf{A}_k$ is the best approximation of $\mathbf{A}$ with respect to the $\|\cdot\|_2$-Norm:

$$\min_{\substack{\mathbf{B}\in\mathbb{R}^{m\times n},\\ \operatorname{rank}(\mathbf{B})\leq k}} \|\mathbf{A}-\mathbf{B}\|_2 = \|\mathbf{A}-\mathbf{A}_k\|_2.$$

Hint. *Estimate* $\|\mathbf{A}-\mathbf{B}\|_2$ *using question (b) and compare it with question (a).*

Exercise 4 (Lower semi-continuity of the rank function). A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called *lower semi-continous* at $\mathbf{A}_0 \in \mathbb{R}^{m\times n}$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f(\mathbf{A}) \geq f(\mathbf{A}_0) - \varepsilon$$

holds for all matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $\mathrm{d}(\mathbf{A}, \mathbf{A}_0) \leq \delta$, where $\mathrm{d}(\cdot,\cdot)$ is a distance function. We call f *upper semi-continous* at $\mathbf{A}_0 \in \mathbb{R}^{m\times n}$ if $-f$ is lower semi-continous at $\mathbf{A}_0$. In this exercise, use the distance function induced by the matrix 2-norm, i.e. $\mathrm{d}(\mathbf{A},\mathbf{B}) = \|\mathbf{A}-\mathbf{B}\|_2$.

(a) Prove that the rank function $\mathbf{A} \mapsto \operatorname{rank}(\mathbf{A})$ is lower semi-continuous at every matrix $\mathbf{A}_0 \in \mathbb{R}^{m\times n}$.

Hint. *Use the Eckart-Young-Mirsky theorem to prove a contradiction.*

(b) Construct an example to show that in general, the rank of a matrix is not upper semi-continuous.