Exercise Sheet 8.

Due: **Monday, 17.11.2025, 12:00.**

Exercise 1 (Sensitivity of linear least squares problems). For $m \geq n$, let $\mathbf{A}, \Delta\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b}, \Delta\mathbf{b} \in \mathbb{R}^m$. Assume that $\mathbf{A}$ is full rank i.e., the minimal singular value of the matrix $\mathbf{A}$ satisfies $\sigma_{\min}(\mathbf{A}) > 0$. Additionally, let the perturbation for the system matrix be small, satisfying $\|\Delta\mathbf{A}\|_2 < \sigma_{\min}(\mathbf{A})$. Now consider the linear least squares problems:

$$\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \to \min \qquad \text{and} \qquad \|(\mathbf{A} + \Delta\mathbf{A})\tilde{\mathbf{x}} - (\mathbf{b} + \Delta\mathbf{b})\|_2 \to \min.$$

(a) We define $\tilde{\mathbf{A}} := \mathbf{A} + \Delta\mathbf{A}$ and $\Delta\mathbf{x} := \tilde{\mathbf{x}} - \mathbf{x}$. Show that

$$\Delta\mathbf{x} = (\tilde{\mathbf{A}}^\top\tilde{\mathbf{A}})^{-1}\Delta\mathbf{A}^\top(\mathbf{b} - \mathbf{A}\mathbf{x}) + (\tilde{\mathbf{A}}^\top\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^\top(\Delta\mathbf{b} - \Delta\mathbf{A}\mathbf{x}).$$

(b) Show that

$$\|(\tilde{\mathbf{A}}^\top\tilde{\mathbf{A}})^{-1}\|_2 \leq (\sigma_{\min}(\mathbf{A}) - \|\Delta\mathbf{A}\|_2)^{-2}, \quad \|(\tilde{\mathbf{A}}^\top\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}\|_2 \leq (\sigma_{\min}(\mathbf{A}) - \|\Delta\mathbf{A}\|_2)^{-1}.$$

(c) Conclude that

$$\|\Delta\mathbf{x}\|_2 \leq \frac{\|\Delta\mathbf{b}\|_2 + \|\Delta\mathbf{A}\|_2\|\mathbf{x}\|_2}{\sigma_{\min}(\mathbf{A}) - \|\Delta\mathbf{A}\|_2} + \frac{\|\Delta\mathbf{A}\|_2\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2}{(\sigma_{\min}(\mathbf{A}) - \|\Delta\mathbf{A}\|_2)^2}.$$

Exercise 2 (Eckart-Young-Mirsky Theorem II). Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be a matrix with singular value decomposition given by $(\{\sigma_i\}_{i=1}^r, \{\mathbf{u}_i\}_{i=1}^m, \{\mathbf{v}_i\}_{i=1}^n)$. Consider the low-rank approximations of $\mathbf{A}$ given by

$$\mathbf{A}_k = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^\top,$$

where $1 \leq k < r$.

(a) Show that

$$\|\mathbf{A} - \mathbf{A}_k\|_F^2 = \sum_{k+1}^r \sigma_i^2.$$

(b) Show that, among all matrices $\mathbf{B} \in \mathbb{R}^{m\times n}$ with rank at most equal to k, $\mathbf{A}_k$ is the best approximation of $\mathbf{A}$ with respect to the $\|\cdot\|_F$-Norm:

$$\min_{\substack{\mathbf{B}\in\mathbb{R}^{m\times n},\\ \operatorname{rang}(\mathbf{B})\leq k}} \|\mathbf{A} - \mathbf{B}\|_F = \|\mathbf{A} - \mathbf{A}_k\|_F.$$

Hint. *Use the inequality* $\sigma_{\max}(\mathbf{V} + \mathbf{W}) \leq \sigma_{\max}(\mathbf{V}) + \sigma_{\max}(\mathbf{W})$ *that has been proven in Exercise 2 (a) of Exercise sheet 7.*

Exercise 3 (Invariant subspaces). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and $W \subset \mathbb{R}^n$ an $\mathbf{A}$-invariant subspace of dimension k. Show that if W is the smallest $\mathbf{A}$-invariant subspace such that for the initial residual of the CG method $\mathbf{r}_0$, $\mathbf{r}_0 \in W$ holds, then the CG method terminates after at most k iterations and yields the exact solution.

Exercise 4 (Gradient method). The function $g : \mathbb{R} \to \mathbb{R}$ is given by the sum of two sine functions, namely

$$g(t) = \sin(t + \phi) + \sin(2t + \psi).$$

We would like to determine the parameters ϕ, ψ. For this purpose, the values $g_1 = g(t_1)$, $g_2 = g(t_2)$ and $g_3 = g(t_3)$ are given for $t_1 < t_2 < t_3$.

(a) Write the corresponding non-linear least squares problem to determine ϕ and ψ.

(b) Perform the first two steps of the gradient descent method for the data points

i	1	2	3
t_i	0	$\pi/2$	π
g_i	1	1	-1

and starting parameters $\phi_0 = \psi_0 = 0$. Choose $\sigma = 0.1$.