Exercise Sheet 10.

Due: **Monday, 01.12.2025, 12:00.**

Exercise 1 (Levenberg-Marquardt algorithm II). For the Rosenbrock function

$$f(x_1, x_2) = 100 \cdot (x_2 - x_1^2)^2 + (x_1 - 1)^2,$$

determine the new search direction $\mathbf{d}_0$ of the Levenberg-Marquardt method for the trust region radii $\Delta_0 \in \{1, 0.5, 0.25\}$ with starting value $x_0 = [0, -0.001]^\intercal$. Use the value $y = 0$ for the data vector.

Exercise 2 (Convex functions). Let $F : \mathbb{R}^n \to \mathbb{R}$ be convex, that is for all pairs of points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and for all $\lambda \in [0, 1]$ one has

$$F\big(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}\big) \le \lambda F(\mathbf{x}) + (1 - \lambda)F(\mathbf{y}).$$

(a) Show that the following functions are convex

$$F(\mathbf{x}) = \|\mathbf{x}\|, \quad F(\mathbf{x}) = 4x_1^2 + x_2^2, \quad F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal\mathbf{A}\mathbf{x} - \mathbf{b}^\intercal\mathbf{x} \text{ with } \mathbf{b} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{n\times n} \text{ spd.}$$

(b) By studying the convexity of the function $f(x) = (1 + x)^n$ defined on $[-1, \infty]$ for $n \ge 2$, show that the inequality $(1 + x)^n \ge 1 + nx$ holds for all $x \ge -1$.

(c) Prove that if a convex function $F : \mathbb{R}^n \to \mathbb{R}$ has a local minimum, then it also has a global minimum.

Exercise 3 (Hölder-inequality). Let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Let a and b be two positive real numbers. Use the concavity of the logarithm function to show that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

(b) Conclude that for every $\mathbf{x} = [x_1, \dots, x_n] \in \mathbb{R}^n$ and $\mathbf{y} = [y_1, \dots, y_n] \in \mathbb{R}^n$, Hölder's inequality

$$\sum_{k=1}^{n} |x_k y_k| \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}$$

holds.

(c) Using Hölder's inequality, show the that the Minkowsky inequality

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}$$

holds.

(d) For $\mathbf{x} = [x_1, \dots, x_n] \in \mathbb{R}^n$, we define

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}.$$

Show that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.