Exercise Sheet 11.

Due: **Monday, 08.12.2025, 12:00.**

Exercise 1 (Optimization methods under affine transformation). Let $F : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Furthermore, consdier a $\mathbf{x} \in \mathbb{R}^n$, such that $\nabla^2 F(\mathbf{x})$ is positive definite. Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be an invertible matrix and $\mathbf{v} \in \mathbb{R}^n$ an arbitrary vector. The new iterate $\mathbf{x}^+ = \mathbf{x} + \mathbf{d}$ is calculated as follows:

i. Transform to a new "$\mathbf{y}$"coordinate system: $\mathbf{x} = \mathbf{x}(\mathbf{y}) := \mathbf{M}\mathbf{y} + \mathbf{v}$.
ii. Take a gradient step in the "$\mathbf{y}$" coordinate system: $\mathbf{y}^+ = \mathbf{y} - \nabla_{\mathbf{y}} F\big(\mathbf{x}(\mathbf{y})\big)$.
iii. Transform back to the "$\mathbf{x}$" coordinate system: $\mathbf{x}^+ = \mathbf{x}(\mathbf{y}^+)$.

(a) Give a formula for $\mathbf{d}$ which depends only on $\mathbf{M}$ and $\nabla F(\mathbf{x})$.

(b) How $\mathbf{M}$ should be chosen so that $\mathbf{d}$ is a gradient step at the point $\mathbf{x}$? For which $\mathbf{M}$ does $\mathbf{d}$ correspond to a Newton step at the point $\mathbf{x}$?

Exercise 2 (Optimization of a quadratic function). We consider the iteration

$$x_{k+1} = x_k + \alpha_k p_k$$

with $x_0 = 2$ to minimize $f(x) = x^2$ and want to investigate the behavior of the sequence $\{x_k\}$ for $k \to \infty$.

(a) Show that for $p_k = (-1)^{k+1}$ and $\alpha_k = 2 + 3/2^{k+1}$, $x_k = (-1)^k + (-1/2)^k$ holds.

(b) Show that for $p_k = -1$ and $\alpha_k = 1/2^{k+1}$, $x_k = 1 + 1/2^k$ holds.

Exercise 3 (Sherman-Morrison-Woodbury formula). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an invertible matrix and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ arbitrary vectors. Show that:

(a) If $\mathbf{v}^\top \mathbf{A}^{-1}\mathbf{u} = -1$, then $(\mathbf{A} + \mathbf{u}\mathbf{v}^\top)$ is singular.

(b) If $\mathbf{v}^\top \mathbf{A}^{-1}\mathbf{u} \neq -1$, then

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^\top)^{-1} = \mathbf{A}^{-1} - \frac{1}{1 + \mathbf{v}^\top \mathbf{A}^{-1}\mathbf{u}} \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\top \mathbf{A}^{-1}.$$