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Is the Affine Space Determined by Its Automorphism Group?

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In this note we study the problem of characterizing the complex affine space $\mathbb{A}^n$ via its automorphism group. We prove the following. Let X be an irreducible quasi-projective n-dimensional variety such that $\mathrm{Aut}(X)$ and $\mathrm{Aut}(\mathbb{A}^n)$ are isomorphic as abstract groups. If X is either quasi-affine and toric or X is smooth with Euler characteristic $\chi(X) \neq 0$ and finite Picard group $\mathrm{Pic}(X)$, then X is isomorphic to $\mathbb{A}^n$.

The main ingredient is the following result. Let X be a smooth irreducible quasi-projective variety of dimension n with finite $\mathrm{Pic}(X)$. If X admits a faithful $(\mathbb{Z}/p\mathbb{Z})^n$-action for a prime p and $\chi(X)$ is not divisible by p, then the identity component of the centralizer $\mathrm{Cent}_{\mathrm{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a torus.

1 Introduction

In 1872, Felix Klein suggested as part of his Erlangen Programm to study geometrical objects through their symmetries. In the spirit of this program it is natural to ask to which extent a geometrical object is determined by its automorphism group. This is the case for compact and locally Euclidean manifolds as shown by Whittaker [30]. It also holds for differentiable manifolds, for symplectic manifolds, and for contact manifolds; see [30], [6], [27], and [28].

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We will study this question in the algebraic setting, that is, for complex algebraic varieties. For such a variety X we denote by $\mathrm{Aut}(X)$ the group of regular automorphisms of X. As this automorphism group is usually quite small, it almost never determines the variety. However, if $\mathrm{Aut}(X)$ is large, like for affine n-space $\mathbb{A}^n$, $n \geq 2$, this might be true. Our guiding question is the following.

Question. Let X be a variety. Assume that $\mathrm{Aut}(X)$ is isomorphic to the group $\mathrm{Aut}(\mathbb{A}^n)$. Does this imply that X is isomorphic to $\mathbb{A}^n$?

This question cannot have a positive answer for *all* varieties X. For example, $\mathrm{Aut}(\mathbb{A}^n)$ and $\mathrm{Aut}(\mathbb{A}^n \times Z)$ are isomorphic for any complete variety Z with a trivial automorphism group. Similarly, $\mathrm{Aut}(\mathbb{A}^n)$ and $\mathrm{Aut}(\mathbb{A}^n \dot\cup Y)$ are isomorphic for any variety Y with a trivial automorphism group. Thus, we have to impose certain assumptions on X.

In case X is affine, the group $\mathrm{Aut}(X)$ has the structure of a so-called ind-group. Using this extra structure one has the following result; see [17]. *If X is a connected affine variety, then every isomorphism of ind-groups between* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(\mathbb{A}^n)$ *is induced by an isomorphism $X \xrightarrow{\sim} \mathbb{A}^n$ of varieties.* For some generalizations of this result we refer to [25].

In dimension 2, it is shown in [22] that *if X is an irreducible normal surface and Y is an affine toric surface, then X is isomorphic to Y if the automorphism groups* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(Y)$ *are isomorphic.*

Our main result in this paper is the following.

Main Theorem. Let X be a complex irreducible quasi-projective variety of dimension n such that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$. Then $X \simeq \mathbb{A}^n$ if one of the following conditions holds.

1. X is smooth, the Euler characteristic $\chi(X)$ is nonzero and the Picard group $\mathrm{Pic}(X)$ is finite.
2. X is toric and quasi-affine.

As an immediate application we get the following result.

Corollary. If $S \subset \mathbb{A}^n$ is a closed subvariety such that $\chi(S) \neq 1$, then $\mathrm{Aut}(\mathbb{A}^n \setminus S) \not\simeq \mathrm{Aut}(\mathbb{A}^n)$.

In fact, $X := \mathbb{A}^n \setminus S$ is smooth and quasi-projective, $\chi(X) = \chi(\mathbb{A}^n) - \chi(S) \neq 0$ (Lemma 2.14(1)), and $\mathrm{Pic}(X)$ is trivial.

Outline of Proof

Let $\theta\colon \operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \operatorname{Aut}(X)$ be an isomorphism. First we show that if a torus of $\operatorname{Aut}(\mathbb{A}^n)$ of maximal dimension n is mapped onto an algebraic subgroup of $\operatorname{Aut}(X)$ and if X is quasi-affine, then $X \simeq \mathbb{A}^n$ (Proposition 4.1). Our main result in order to achieve these conditions is the following. (For the definition of the topology on $\operatorname{Aut}(X)$ see Section 2.2.)

Theorem 1.1. Let Y and Z be irreducible quasi-projective varieties, and let $\theta\colon \operatorname{Aut}(Y) \xrightarrow{\sim} \operatorname{Aut}(Z)$ be an isomorphism. Assume that $n := \dim Y \geq \dim Z$ and that the following conditions are satisfied:

(i) Y is quasi-affine and toric.
(ii) Z is smooth, $\chi(Z) \neq 0$, and $\operatorname{Pic}(Z)$ is finite.

Then $\dim Z = n$, and for each n-dimensional torus $T \subseteq \operatorname{Aut}(Y)$, the identity component of the image $\theta(T)^\circ$ is a closed torus of dimension n. Furthermore, Z is quasi-affine.

From this and Proposition 4.1 we can deduce our Main Theorem by setting $Y := \mathbb{A}^n$ and $Z := X$ in case (1) and $Y := X$ and $Z := \mathbb{A}^n$ in case (2); see Section 4.2.

For the proof of Theorem 1.1 we first remark that every torus $T \subseteq \operatorname{Aut}(X)$ of maximal dimension $n = \dim X$ is self-centralizing (Lemma 2.10). For any prime p the torus T contains a unique subgroup μ_p isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$. In particular, $T \subseteq \operatorname{Cent}_{\operatorname{Aut}(X)}(\mu_p)$, and thus the image of T under $\theta\colon \operatorname{Aut}(X) \xrightarrow{\sim} \operatorname{Aut}(Y)$ is mapped to a subgroup of the centralizer of $\theta(\mu_p)$.

Our strategy is then to show that the identity component of the centralizer $\operatorname{Cent}_{\operatorname{Aut}(Y)}(\theta(\mu_p))$ is an algebraic group. Our main result in this direction is the following generalization of [19, Proposition 3.4].

Theorem 1.2. Let X be a smooth, irreducible, quasi-projective variety of dimension n with finite Picard group $\operatorname{Pic}(X)$. Assume that X carries a faithful $(\mathbb{Z}/p\mathbb{Z})^n$-action for some prime p that does not divide $\chi(X)$. Then the centralizer $C := \operatorname{Cent}_{\operatorname{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a closed subgroup of $\operatorname{Aut}(X)$ and its identity component C° is a closed torus of dimension $\leq n$.

For the proof we first show that the fixed-point set $X^{(\mathbb{Z}/p\mathbb{Z})^n}$ contains an isolated point x_0. This follows from the smoothness of X and the assumption that p does not divide $\chi(X)$. Now we study the tangent representation of $(\mathbb{Z}/p\mathbb{Z})^n$ in x_0 and show that the homomorphism $C^\circ \to \operatorname{GL}(T_{x_0}X)$ is regular and has a finite kernel.

2 Preliminary Results

Throughout this note we work over the field $\mathbb{C}$ of complex numbers. A variety will be a reduced separated scheme of finite type over $\mathbb{C}$.

2.1 Quasi-affine varieties

Let us recall some well-known results about quasi-affine varieties.

Lemma 2.1 ([10, Chapter II, Proposition 5.1.2]). A variety X is quasi-affine if and only if the canonical morphism $\eta\colon X \to \operatorname{Spec}\mathcal{O}(X)$ is a dominant open immersion of schemes.

Lemma 2.2 ([5, Chapter I, Section 2, Proposition 2.6]). Let X and Y be varieties. Then the natural homomorphism

$$\mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$$

is an isomorphism of $\mathbb{C}$-algebras.

Lemma 2.3. Let X and Y be varieties where X is quasi-affine. Then every morphism $Y \times X \to X$ extends uniquely to a morphism $Y \times \operatorname{Spec}\mathcal{O}(X) \to \operatorname{Spec}\mathcal{O}(X)$. In particular, every regular action of an algebraic group on X extends to a regular action on $\operatorname{Spec}\mathcal{O}(X)$.

Proof. We can assume that Y is affine. By Lemma 2.2 we have $\mathcal{O}(Y\times X) = \mathcal{O}(Y)\otimes_{\mathbb{C}}\mathcal{O}(X)$. Hence, $Y \times X \to X$ induces a homomorphism of $\mathbb{C}$-algebras $\mathcal{O}(X) \to \mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{O}(X)$ that in turn gives the desired extension $Y \times \operatorname{Spec}\mathcal{O}(X) \to \operatorname{Spec}\mathcal{O}(X)$. ■

2.2 Algebraic structure on the group of automorphisms

In this subsection, we recall some basic results about the automorphism group $\operatorname{Aut}(X)$ of a variety X. The survey [2] and the article [24] will serve as references. Recall that a *morphism* $\nu\colon A \to \operatorname{Aut}(X)$ is a map from a variety A to $\operatorname{Aut}(X)$ such that the associated map

$$\tilde{\nu}\colon A \times X \to X, \quad (a, x) \mapsto ax := \nu(a)(x)$$

is a morphism of varieties. We get a topology on $\operatorname{Aut}(X)$, called *Zariski topology*, by declaring a subset $F \subset \operatorname{Aut}(X)$ to be *closed*, if for every variety A the preimage $\nu^{-1}(F)$ under every morphism $\nu\colon A \to \operatorname{Aut}(X)$ is closed in A. In particular, a morphism $\nu\colon A \to \operatorname{Aut}(X)$ is continuous with respect to the Zariski topology.

Similarly, a *morphism* $\nu = (\nu_1, \nu_2)\colon A \to \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ is a map from a variety A into $\mathrm{Aut}(X) \times \mathrm{Aut}(X)$ such that ν_1 and ν_2 are morphisms. Thus, we get analogously as before a topology on $\mathrm{Aut}(X) \times \mathrm{Aut}(X)$. Note that for morphisms $\nu, \nu_1, \nu_2 \colon A \to \mathrm{Aut}(X)$ the following maps are again morphisms

$$A \to \mathrm{Aut}(X),\ a \mapsto \nu_1(a) \circ \nu_2(a),$$
$$A \to \mathrm{Aut}(X),\ a \mapsto \nu(a)^{-1},$$

and that $\nu^{-1}(\Delta)$ is closed in A where $\Delta \subset \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ denotes the diagonal. It follows that $\mathrm{Aut}(X)$ behaves like an algebraic group.

Lemma 2.4. For any variety X the maps

$$\mathrm{Aut}(X) \times \mathrm{Aut}(X) \to \mathrm{Aut}(X)\,,\ (\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2$$
$$\mathrm{Aut}(X) \to \mathrm{Aut}(X)\,,\ \varphi \mapsto \varphi^{-1}$$

are continuous, and the diagonal Δ is closed in $\mathrm{Aut}(X) \times \mathrm{Aut}(X)$.

Example 2.5. For any set $S \subseteq \mathrm{Aut}(X)$ the centralizer $\mathrm{Cent}(S)$ is a closed subgroup of $\mathrm{Aut}(X)$. This is a consequence of Lemma 2.4.

Definition 2.6. For a subset $S \subseteq \mathrm{Aut}(X)$ its *dimension* is defined by

$$\dim S := \sup \left\{ d \;\middle|\; \begin{array}{l} \text{there exists a variety } A \text{ of dimension } d \text{ and an} \\ \text{injective morphism } \nu\colon A \to \mathrm{Aut}(X) \text{ with image in } S \end{array} \right\}.$$

The following lemma generalizes the classical dimension estimate to morphisms $A \to \mathrm{Aut}(X)$.

Lemma 2.7. If $\nu\colon A \to \mathrm{Aut}(X)$ is a morphism, then $\dim \nu(A) \le \dim A$.

Proof. Let $\eta\colon B \to \mathrm{Aut}(X)$ be an injective morphism such that $\eta(B) \subseteq \nu(A)$. We have to show that $\dim B \le \dim A$. For this consider the fiber product

$$\begin{array}{ccc} A \times_{\mathrm{Aut}(X)} B & \xrightarrow[\text{surjective}]{\bar{\nu}} & B \\ {\scriptstyle \bar{\eta}}\downarrow{\scriptstyle \text{injective}} & & {\scriptstyle \eta}\downarrow{\scriptstyle \text{injective}} \\ A & \xrightarrow{\nu} & \mathrm{Aut}(X). \end{array}$$

By definition, we have $A \times_{\mathrm{Aut}(X)} B := \{(a,b) \in A \times B \mid \nu(a) = \eta(b)\}$. Since $\nu \times \eta \colon A \times B \to \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ is a morphism, hence continuous, and $\Delta \subset \mathrm{Aut}(X) \times \mathrm{Aut}(X)$ is closed, it follows that $A \times_{\mathrm{Aut}(X)} B \subset A \times B$ is closed. Thus, the fiber product is a variety, and the two maps $\bar{\nu}$ and $\bar{\eta}$ are morphisms. By assumption, $\bar{\nu}$ is surjective and $\bar{\eta}$ is injective, and the claim follows. ■

For a subgroup $G \subseteq \mathrm{Aut}(X)$, the *identity component* $G^\circ \subseteq G$ is defined by

$$G^\circ = \left\{ g \in G \;\middle|\; \begin{array}{l} \text{there exists an irreducible variety } A \text{ and a morphism} \\ \nu\colon A \to \mathrm{Aut}(X) \text{ with image in } G \text{ such that } g, e \in \nu(A) \end{array} \right\}.$$

We call a subgroup $G \subseteq \mathrm{Aut}(X)$ *connected* if $G = G^\circ$. In the next proposition, we list several properties of the identity component of a subgroup of $\mathrm{Aut}(X)$. If G is an ind-group, then these properties are known; see [9, Proposition 2.2.1].

Proposition 2.8. Let X be a variety, and let $G \subseteq \mathrm{Aut}(X)$ be a subgroup. Then the following holds.

1. G° is a normal subgroup of G.
2. The cosets of G° in G are the equivalence classes under the relation

$$g_1 \sim g_2 \iff \left\{ \begin{array}{l} \text{there exists an irreducible variety } A \\ \text{and a morphism } \nu\colon A \to \mathrm{Aut}(X) \\ \text{with image in } G \text{ such that } g_1, g_2 \in \nu(A)\,. \end{array} \right.$$

3. For each morphism $\nu\colon A \to \mathrm{Aut}(X)$ with image in G the preimage $\nu^{-1}(G^\circ)$ is closed in A. In particular, if G is closed in $\mathrm{Aut}(X)$, then G° is also closed in $\mathrm{Aut}(X)$.
4. If X is quasi-projective and G is closed in $\mathrm{Aut}(X)$, then the index of G° in G is countable.

Proof. (1) This follows immediately from the definition of G°.

(2) We have to show that "$\sim$" is an equivalence relation on G. Reflexivity and symmetry are obvious. For the transitivity, let $g \sim h$ and $h \sim k$. By definition, there exist irreducible varieties A and B, morphisms $\nu\colon A \to \mathrm{Aut}(X)$ and $\eta\colon B \to \mathrm{Aut}(X)$ with image

in G, and $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $\nu(a_1) = g$, $\nu(a_2) = h$, $\eta(b_1) = h$, $\eta(b_2) = k$. Then the map

$$A \times B \to \mathrm{Aut}(X), \quad (a, b) \mapsto \nu(a) \circ h^{-1} \circ \eta(b)$$

is a morphism with image in G that sends (a_1, b_1) to g and (a_2, b_2) to k. Thus, $g \sim k$, proving the transitivity.

(3) Let

$$\bigcup_{i=1}^{k} B_i = \overline{\nu^{-1}(G^\circ)} \subseteq A$$

be the decomposition of the closure of $\nu^{-1}(G^\circ)$ into irreducible components $B_1, \ldots, B_k$. Thus, $B_i \cap \nu^{-1}(G^\circ)$ is nonempty. Since ν has image in G it follows from the transitivity of "$\sim$" that $\nu(B_i) \subseteq G^\circ$. Thus, $B_i \subseteq \nu^{-1}(G^\circ)$ for all i. Hence, $\nu^{-1}(G^\circ)$ is closed in A.

(4) Let $\nu \colon A \to \mathrm{Aut}(X)$ be a morphism. Since $\nu^{-1}(G) \subseteq A$ is closed, it has only finitely many irreducible components. This implies that its image $\nu(A)$ meets only finitely many cosets of G° in G. The claim follows if we show that there exist countably many morphisms of varieties into $\mathrm{Aut}(X)$ whose images cover $\mathrm{Aut}(X)$.

Since X is quasi-projective, there exists a projective variety $\overline{X}$ and an open embedding $X \subseteq \overline{X}$. For each polynomial $p \in \mathbb{Q}[x]$ we denote by Hilb^p the Hilbert scheme of $\overline{X} \times \overline{X}$ associated with the Hilbert polynomial p and denote by $\mathcal{U}^p \subseteq \mathrm{Hilb}^p \times \overline{X} \times \overline{X}$ the universal family, which is by definition flat over Hilb^p. By [14, Theorem 3.2], Hilb^p is a projective scheme over $\mathbb{C}$. For $i = 1, 2$ consider the following morphisms:

$$q_i \colon (\mathrm{Hilb}^p \times X \times X) \cap \mathcal{U}^p \to \mathrm{Hilb}^p \times X, \quad (h, x_1, x_2) \mapsto (h, x_i),$$

which are defined over Hilb^p. By [12, Proposition 9.6.1], the points $h \in \mathrm{Hilb}^p$ where the restriction

$$q_i|_{\{h\}} \colon (\{h\} \times X \times X) \cap \mathcal{U}^p \to \{h\} \times X$$

is an isomorphism form a constructible subset S^p of Hilb^p. Now choose locally closed subsets S_j^p, $j = 1, \ldots, k_p$ of Hilb^p that cover S^p. We equip each S_j^p with the underlying reduced scheme structure of Hilb^p. Note that $(\mathrm{Hilb}^p \times X \times X) \cap \mathcal{U}^p$ and $\mathrm{Hilb}^p \times X$ are both flat over Hilb^p. Therefore, we can apply [13, Proposition 5.7] and we get that q_i restricts to an isomorphism over S_j^p. Thus, for each j we get a morphism of varieties

$$S_j^p \times X \xrightarrow{(q_1|_{S_j^p})^{-1}} (S_j^p \times X \times X) \cap \mathcal{U}^p \xrightarrow{q_2|_{S_j^p}} S_j^p \times X \longrightarrow X,$$

which defines a morphism $S_j^p \to \mathrm{Aut}(X)$. For each automorphism φ in $\mathrm{Aut}(X)$, the closure in $\overline{X} \times \overline{X}$ of the graph $\Gamma_\varphi \subseteq X \times X$ defines a (closed) point in the Hilbert scheme Hilb^p for a certain rational polynomial p, which belongs to S^p. Thus, the images of the morphisms $S_j^p \to \mathrm{Aut}(X)$ cover $\mathrm{Aut}(X)$. Since there are only countably many rational polynomials, the claim follows. ■

We say that G is an *algebraic subgroup* of $\mathrm{Aut}(X)$ if there exists a morphism $\nu\colon H \to \mathrm{Aut}(X)$ of an algebraic group H with image G, which is a homomorphism of groups.

The next result gives a criterion for a subgroup of $\mathrm{Aut}(X)$ to be algebraic. The main argument is due to Ramanujam [24].

Theorem 2.9. Let X be an irreducible variety, and let $G \subseteq \mathrm{Aut}(X)$ be a subgroup. Then the following statements are equivalent:

(1) G is an algebraic subgroup of $\mathrm{Aut}(X)$.
(2) There exists a morphism of a variety into $\mathrm{Aut}(X)$ with image G.
(3) $\dim G$ is finite and G° has finite index in G.
(4) There is a structure of an algebraic group on G such that for each irreducible variety A we get a bijection

$$\left\{ \begin{array}{c} \text{morphisms } A \to \mathrm{Aut}(X) \\ \text{with image in } G \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{c} \text{morphisms of} \\ \text{varieties } A \to G \end{array} \right\}.$$

Proof. The implication (1) ⇒ (2) follows from the definition.

Assume that there is a morphism $\eta\colon A \to \mathrm{Aut}(X)$ with image equal to G. By Lemma 2.7 we get $\dim G \le \dim A$; hence, $\dim G$ is finite. Since A has only finitely many irreducible components it follows from Proposition 2.8 2 that G° has finite index in G. This proves (2) ⇒ (3).

The implication (3) ⇒ (4) is proved in [24, Theorem, p. 26] in case $G = G^\circ$. This implies that G° carries the structure of an algebraic group with the required property. Since G° has finite index in G we obtain a unique structure of an algebraic group on G extending the given structure on G°. It remains to see that the required property holds for G.

By construction, the canonical inclusion $\iota\colon G \to \mathrm{Aut}(X)$ is a morphism, and thus each morphism of varieties $A \to G$ yields a morphism $A \to \mathrm{Aut}(X)$ by composing with ι. For the reverse, let $\nu\colon A \to \mathrm{Aut}(X)$ be a morphism with image in G. Since A is irreducible there is $g \in G$ such that the image of ν lies in gG° (Proposition 2.8(2)). Thus, the composition $\lambda_{g^{-1}} \circ \nu\colon A \to \mathrm{Aut}(G)$ is a morphism with image in G° where $\lambda_g \in \mathrm{Aut}(X)$ is the left multiplication with g. It follows that ν corresponds to a morphism $A \to G$ of varieties, proving (3) $\Rightarrow$ (4).

The remaining implication (4) $\Rightarrow$ (1) is obvious. ■

2.3 Ingredients from toric geometry

Recall that a toric variety is a normal irreducible variety X together with a regular faithful action of a torus of dimension $\dim X$. For details concerning toric varieties we refer to [8].

Lemma 2.10. Let X be a toric variety, and let T be a torus of dimension $\dim X$ that acts faithfully on X. Then the centralizer of T in $\mathrm{Aut}(X)$ is equal to T. In particular, the image of T in $\mathrm{Aut}(X)$ is closed.

Proof. Let $g \in \mathrm{Aut}(X)$ such that $gt = tg$ for all $t \in T$. By definition, there is an open, dense T-orbit in X, say U. Since $gU \cap U$ is nonempty, there exists $x \in U$ such that $gx \in U$. Using that $U = Tx$ we find $t_0 \in T$ with $gx = t_0x$. Thus, for each $t \in T$ we get

$$gtx = tgx = tt_0x = t_0tx\,.$$

Using that $U = Tx$ is dense in X, we get $g = t_0$. ■

Lemma 2.11. Let X be a toric variety. Then the coordinate ring $\mathcal{O}(X)$ is finitely generated and integrally closed.

Proof. This is a special case of a result of Knop; see [16, Satz, p. 33]. ■

The next proposition is based on the study of homogeneous $\mathbb{G}_a$-actions on affine toric varieties in [21]. Recall that a group action $\nu\colon G \to \mathrm{Aut}(X)$ on a toric variety is called *homogeneous* if the torus normalizes the image $\nu(G)$. Note that for any homogeneous $\mathbb{G}_a$-action ν there is a well-defined character $\chi\colon T \to \mathbb{G}_m$, defined by the formula

$$t\,\nu(s)\,t^{-1} = \nu(\chi(t)\cdot s) \quad \text{for } t \in T, s \in \mathbb{C}.$$

Proposition 2.12. Let X be an n-dimensional quasi-affine toric variety. If X is not a torus, then there exist homogeneous $\mathbb{G}_a$-actions

$$\eta_1, \dots, \eta_n \colon \mathbb{G}_a \times X \to X$$

such that the corresponding characters $\chi_1, \dots, \chi_n$ are linearly independent in the character group of T.

The proof needs some preparation. Denote by Y the spectrum of $\mathcal{O}(X)$. By Lemma 2.11, the variety Y is normal, and the faithful torus action on X extends uniquely to a faithful torus action on Y, by Lemma 2.3.

The following notation is taken from [21]. Let N be a lattice of rank n, $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, we have a natural pairing $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$, $(m, n) \mapsto \langle m, n \rangle$. Let $\sigma \subset N_{\mathbb{Q}}$ be the strongly convex polyhedral cone that describes Y and let $\sigma_M^\vee$ be the intersection of the dual cone $\sigma^\vee$ in $M_{\mathbb{Q}}$ with M. Thus, $Y = \operatorname{Spec} R$, where

$$R := \mathbb{C}[\sigma_M^\vee] = \bigoplus_{m \in \sigma_M^\vee} \mathbb{C}\chi^m \subseteq \mathbb{C}[M] .$$

For each extremal ray $\rho \subset \sigma$, denote by $\rho^\perp$ the elements $u \in M_{\mathbb{Q}}$ with $\langle u, v \rangle = 0$ for all $v \in \rho$. Moreover, let $\tau_M = \rho^\perp \cap \sigma_M^\vee$ and let

$$S_\rho = \{ e \in M \mid e \notin \sigma_M^\vee \, , \; e + m \in \sigma_M^\vee \text{ for all } m \in \sigma_M^\vee \setminus \tau_M \} .$$

By [21, Remark 2.5] we have $S_\rho \neq \varnothing$ and $e + m \in S_\rho$ for all $e \in S_\rho$ and all $m \in \tau_M$. Let us recall the description of the homogeneous locally nilpotent derivations on R.

Proposition 2.13 ([21, Lemma 2.6 and Theorem 2.7]). Let ρ be an extremal ray in σ and let $e \in S_\rho$. Then

$$\partial_{\rho,e} \colon R \to R \, , \quad \chi^m \mapsto \langle m, \rho \rangle \chi^{e+m}$$

is a homogeneous locally nilpotent derivation of degree e, and every homogeneous locally nilpotent derivation of R is a constant multiple of some $\partial_{\rho,e}$.

Proof of Proposition 2.12. Since X is not a torus, Y is also not a torus. Thus, σ contains extremal rays, say $\rho_1, \dots, \rho_k$ and $k \geq 1$. Recall that associated to these extremal rays, there exist torus-invariant divisors $V(\rho_1), \dots, V(\rho_k)$ in Y. Again, since X is not a torus, one of these divisors does intersect X. Let us assume that $\rho = \rho_1$ is an extremal ray such that $V(\rho) \cap X$ is nonempty. Then using the orbit-cone correspondence, one can see that $Y \setminus X$ is contained in the union $Z = \bigcup_{i=2}^{k} V(\rho_i)$; see [8, Section 3.1]. Let $e \in S_\rho$ be fixed. We claim that the $\mathbb{G}_a$-action on Y associated with the locally nilpotent derivation $\partial_{\rho,e+m'}$ of Proposition 2.13 fixes Z for all $m' \in \tau_M \setminus \bigcup_{i\geq 2} \rho_i^\perp$.

Let us fix $m' \in \tau_M$ with $\langle m', v\rangle > 0$ for all $v \in \bigcup_{i\geq 2} \rho_i$. Note that the fixed-point set of the $\mathbb{G}_a$-action on Y corresponding to $\partial_{\rho,e+m'}$ is the zero set of the ideal generated by the image of $\partial_{\rho,e+m'}$. The divisor $V(\rho_i)$ is the zero set of the kernel of the canonical $\mathbb{C}$-algebra surjection

$$p_i \colon \mathbb{C}[\sigma_M^\vee] \to \mathbb{C}[\sigma_M^\vee \cap \rho_i^\perp], \quad \chi^m \mapsto \begin{cases} \chi^m, & \text{if} m \in \rho_i^\perp \\ 0, & \text{otherwise} \end{cases};$$

see [8, Section 3.1]. Thus, we have to prove that for all $i = 2, \ldots, k$ the composition

$$\mathbb{C}[\sigma_M^\vee] \xrightarrow{\partial_{\rho,e+m'}} \mathbb{C}[\sigma_M^\vee] \xrightarrow{p_i} \mathbb{C}[\sigma_M^\vee \cap \rho_i^\perp]$$

is the zero map. Since, by definition, $\partial_{\rho,e+m'}$ vanishes on $\tau_M = \rho^\perp \cap \sigma_M^\vee$, we only have to show that for all $m \in \sigma_M^\vee \setminus \tau_M$ the following holds:

$$\langle e + m' + m, v\rangle > 0 \quad \text{for all } v \in \rho_i, i = 2, \ldots, k.$$

This is satisfied because $\langle m', v\rangle > 0$ and $\langle e + m, v\rangle \geq 0$ (note that $e \in S_\rho$ implies $e + m \in \sigma_M^\vee$). This proves the claim.

Since τ_M spans a hyperplane in M and $e \notin \tau_M$, we can choose $m'_1, \ldots, m'_n \in \tau_M \setminus \bigcup_{i\geq 2} \rho_i^\perp$ such that $e + m'_1, \ldots, e + m'_n$ are linearly independent in $M_\mathbb{Q}$. Hence, the homogeneous locally nilpotent derivations

$$\partial_{\rho,e+m'_i}, \quad i = 1, \ldots, n$$

define $\mathbb{G}_a$-actions on Y that fix Z and thus restrict to $\mathbb{G}_a$-actions on X. Moreover, the character of $\partial_{\rho,e+m'_i}$ is $\chi_i = \chi^{e+m'_i}$. In particular, $\chi_1, \ldots, \chi_n$ are linearly independent, finishing the proof of Proposition 2.12. ∎

2.4 The Euler characteristic

For a variety X, the *Euler characteristic* is defined by

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}),$$

where $H^i(X, \mathbb{Q})$ denotes the i-th singular cohomology group with rational coefficients. The following results can be found in [18, Appendix].

Lemma 2.14. The Euler characteristic has the following properties.

(1) If X is a variety and $Y \subseteq X$ is a closed subvariety, then $\chi(X) = \chi(Y) + \chi(X \setminus Y)$.

(2) If $X \to Y$ is a fiber bundle, which is locally trivial in the étale topology with fiber F, then $\chi(X) = \chi(Y)\chi(F)$.

2.5 Results about the fixed-point variety

The next result gives a criterion for the existence of fixed points under the action of a finite p-group.

Proposition 2.15. Let p be a prime, and let G be finite p-group acting on a variety X. If p does not divide the Euler characteristic $\chi(X)$, then the fixed-point variety X^G is nonempty.

Proof. Assume that X^G is empty, that is, every G-orbit has cardinality p^k for some $k > 0$. We prove by induction on the dimension of X that p divides $\chi(X)$. Let $X' \subset X$ be a dense smooth open affine subset. By intersecting the G-translates gX' for $g \in G$ we can in addition assume that X' is G-invariant. Denote by $\pi \colon X' \to X'/G$ the algebraic quotient, that is, the morphism corresponding to the inclusion of the invariant ring $\mathcal{O}(X')^G$ in $\mathcal{O}(X')$. It follows from Luna's slice theorem [23, Chapter II, Section 2] that there is a smooth open dense subset $U \subset X'/G$ such that π restricts to a fiber bundle $\pi^{-1}(U) \to U$, which is locally trivial in the étale topology. Now Lemma 2.14(2) implies that p divides $\chi(\pi^{-1}(U))$. Using Lemma 2.14(1) and $\dim X \setminus \pi^{-1}(U) < \dim X$ the claim follows by induction. ■

Remark 1. The proposition above is a purely topological result and holds in a much more general setting; see, for example, [3, Chapter III, Theorem 4.4] or [4, Section III.7].

The next result is essentially due to Fogarty; see [7, Theorem 5.2].

Proposition 2.16. Let G be a reductive group acting on a variety X. Assume that X is smooth at a point $x \in X^G$. Then X^G is smooth at x and the tangent space satisfies $T_x(X^G) = (T_xX)^G$.

Remark 2. Assume that $(\mathbb{Z}/p\mathbb{Z})^n$ acts faithfully on a smooth quasi-projective variety X. If p does not divide $\chi(X)$, then $\dim X \geq n$.

In fact, by Proposition 2.15 there is a fixed-point $x \in X$, and the action of $(\mathbb{Z}/p\mathbb{Z})^n$ on the tangent space T_xX is faithful [19, Lemma 2.2]; hence, $n \leq \dim T_xX = \dim X$.

3 Proof of Theorems 1.1 and 1.2

Definition 3.1. Let X be a variety and $M \subseteq \mathrm{Aut}(X)$ a subset. A map $\eta\colon M \to Z$ into a variety Z is called *regular* if for every morphism $\nu\colon A \to \mathrm{Aut}(X)$ with image in M, the composition $\eta \circ \nu\colon A \to Z$ is a morphism of varieties.

3.1 Semi-invariant functions

Lemma 3.2. Let X be an irreducible normal variety, and let $f \in \mathcal{O}(X)$ be a non-constant function such that the zero set $Z := \mathcal{V}_X(f) \subset X$ is an irreducible hypersurface. Let $G \subseteq \mathrm{Aut}(X)$ be a connected subgroup that stabilizes Z. Then the function f is a G-semi-invariant, that is,

$$f(gx) = \chi(g)^{-1} \cdot f(x) \quad \text{for } x \in X \text{ and } g \in G,$$

where $\chi : G \to \mathbb{C}^*$ is a character and a regular map.

For the proof we need the following description of the invertible functions on a product variety, which is due to Rosenlicht [26, Theorem 2]. For a variety X we denote by $\mathcal{O}(X)^*$ the group of invertible functions on X.

Lemma 3.3. Let X_1 and X_2 be irreducible varieties. Then $\mathcal{O}(X_1 \times X_2)^* = \mathcal{O}(X_1)^* \cdot \mathcal{O}(X_2)^*$.

Proof of Lemma 3.2. Since X is normal, the local ring $R = \mathcal{O}_{X,Z}$ is a discrete valuation ring. Let $\mathfrak{m}$ be the maximal ideal of R. By assumption, $fR = \mathfrak{m}^k$ for some $k > 0$. Since $\mathfrak{m}$ is stable under G, the same is true for $\mathfrak{m}^k$. Hence, for every $g \in G$, there exists a unit $r_g \in R^*$ such that $gf = r_g \cdot f$ in R. Since f and gf have no zeroes in $X \setminus Z$, it follows that

r_g is regular and nonzero in $X \setminus Z$. Moreover, the open set where $r_g \in R$ is defined and nonzero meets Z; hence, $r_g \in \mathcal{O}(X)^*$. Consider the homomorphism

$$\chi\colon G \to \mathcal{O}(X)^*\,,\ g \mapsto r_g\,.$$

For all $x \in X \setminus Z$, $g \in G$ we get $f(gx) = \chi(g)(x)^{-1} f(x)$, and $f(gx)$ and $f(x)$ are both nonzero. Since for each morphism $\nu\colon A \to \operatorname{Aut}(X)$ with image in G, the map $\tilde{\nu}\colon A \times X \to X$, $(a, x) \mapsto \nu(a)(x)$ is a morphism, we see that

$$A \times (X \setminus Z) \to \mathbb{C}^*\,,\ (a, x) \mapsto \chi(\nu(a))(x) = f(x) \cdot f(\tilde{\nu}(a, x))^{-1}$$

is a morphism. If A is irreducible, then, by Lemma 3.3, there exist invertible functions $q \in \mathcal{O}(A)^*$ and $p \in \mathcal{O}(X \setminus Z)^*$ such that $\chi(\nu(a))(x) = q(a)p(x)$. If, moreover, $\nu(a_0) = e \in G$ for some $a_0 \in A$, then

$$1 = r_e(x) = \chi(\nu(a_0))(x) = q(a_0)p(x) \quad \text{for all}\, x \in X \setminus Z\,,$$

that is, $p \in \mathbb{C}^*$; hence, the composition $\chi \circ \nu\colon A \mapsto \mathcal{O}(X)^*$ has image in $\mathbb{C}^*$. Since G is connected, this implies that $\chi(G) \subseteq \mathbb{C}^*$ and that $\chi\colon G \to \mathbb{C}^*$ is a character.

It remains to see that χ is regular. Choose $x_0 \in X \backslash Z$. As before, for each morphism $\nu\colon A \to \operatorname{Aut}(X)$ with image in G, the map

$$A \to \mathbb{C}^*, \quad a \mapsto \chi(\nu(a)) = f(x_0) \cdot f(\nu(a)(x_0))^{-1}$$

is also a morphism. ■

Lemma 3.4. Let X be an irreducible normal variety, and let $G \subseteq \operatorname{Aut}(X)$ be a connected subgroup. Assume that $f_1, \ldots, f_n \in \mathcal{O}(X)$ have the following properties.

(1) $Z_i := \mathcal{V}_X(f_i)$, $i = 1, \ldots, n$, are irreducible G-invariant hypersurfaces.
(2) $\bigcap_i Z_i$ contains an isolated point.

If $\chi_i\colon G \to \mathbb{C}^*$ is the character of f_i (Lemma 3.2), then

$$\chi := (\chi_1, \ldots, \chi_n)\colon G \to (\mathbb{C}^*)^n$$

is a regular homomorphism with finite kernel.

Proof. Let G act on $\mathbb{A}^n$ by

$$g(a_1,\dots,a_n) := (\chi_1(g)^{-1}\cdot a_1,\dots,\chi_n(g)^{-1}\cdot a_n).$$

Then the map $f := (f_1,\dots,f_n)\colon X \to \mathbb{A}^n$ is G-equivariant. Let $Y \subseteq \mathbb{A}^n$ be the closure of $f(X)$. By assumption, $f^{-1}(0) = \bigcap_i Z_i$ contains an isolated point; hence, $f\colon X \to Y$ has a finite degree, that is, the field extension $\mathbb{C}(X) \supset \mathbb{C}(Y)$ is finite. This implies that the kernel K of $\chi\colon G \to (\mathbb{C}^*)^n$ is finite because K embeds into $\mathrm{Aut}_{\mathbb{C}(Y)}(\mathbb{C}(X))$. By Lemma 3.2, χ is regular. ■

3.2 Another centralizer result

For an irreducible normal variety X, we denote by $\mathrm{CH}^1(X)$ the *first Chow group*, that is, the free group of integral Weil divisors modulo linear equivalence [15, Chapter II, Section 6].

Proposition 3.5. Let X be an irreducible normal variety of dimension n with a faithful action of $(\mathbb{Z}/p\mathbb{Z})^n$. Assume that $\mathrm{CH}^1(X)$ is finite and that there exists a fixed-point x which is a smooth point of X. Then the centralizer $\mathrm{Cent}_{\mathrm{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a closed subgroup of $\mathrm{Aut}(X)$, and its identity component is a closed torus of dimension $\le n$.

Proof. We denote $G := \mathrm{Cent}_{\mathrm{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$. By [19, Lemma 2.2] we get a faithful representation of $(\mathbb{Z}/p\mathbb{Z})^n$ on T_xX, and thus we can find generators $\sigma_1,\dots,\sigma_n$ such that $(T_xX)^{\sigma_i} \subset T_xX$ is a hyperplane for each i and that $(T_xX)^{(\mathbb{Z}/p\mathbb{Z})^n} = 0$. By Proposition 2.16, the hypersurface $X^{\sigma_i} \subset X$ is smooth at x, with tangent space $T_x(X^{\sigma_i}) = (T_xX)^{\sigma_i}$. Hence, there is a unique irreducible hypersurface $Z_i \subseteq X$ which contains x and is contained in X^{σ_i}. It follows that Z_i is G°-stable, and that x is an isolated point of $\bigcap_i Z_i$, because $(T_xX)^{(\mathbb{Z}/p\mathbb{Z})^n} = 0$. Since a multiple of Z_i is zero in $\mathrm{CH}^1(X)$, there exist G°-semi-invariant functions $f_i \in \mathcal{O}(X)$ such that $\mathcal{V}_X(f_i) = Z_i$ (Lemma 3.2), and the corresponding characters χ_i define a regular homomorphism

$$\chi = (\chi_1,\dots,\chi_n)\colon G^\circ \to (\mathbb{C}^*)^n$$

with a finite kernel (Lemma 3.4). It follows that $\dim G^\circ \le n$. Indeed, if $\nu\colon A \to \mathrm{Aut}(X)$ is an injective morphism with image in G°, then $\chi\circ\nu\colon A \to (\mathbb{C}^*)^n$ is a morphism with finite fibers, and so $\dim A \le n$. This implies, by Theorem 2.9, that $G^\circ \subseteq \mathrm{Aut}(X)$ is an algebraic subgroup and that χ is a homomorphism of algebraic groups with a finite kernel. Hence, G° is a torus. Since G is closed in $\mathrm{Aut}(X)$ the same holds for G°, see Proposition 2.8. ■

3.3 Proof of Theorem 1.2

Now we can prove Theorem 1.2 that has the same conclusion as the proposition above, but under different assumptions. We have to show that the assumptions of Proposition 3.5 are satisfied. Since X is smooth, it follows that $\mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$ is finite, and Proposition 2.15 implies that the fixed-point variety $X^{(\mathbb{Z}/p\mathbb{Z})^n} \subseteq X$ is nonempty. Now the claims follow from Proposition 3.5.

3.4 Images of maximal tori under group isomorphisms

Proposition 3.6. Let X and Y be irreducible quasi-projective varieties such that $n := \dim X \geq \dim Y$. Assume that the following conditions are satisfied:

(1) X is quasi-affine and toric.
(2) Y is smooth, $\chi(Y) \neq 0$, and $\mathrm{Pic}(Y)$ is finite.

If $\theta\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$ is an isomorphism, then $\dim Y = n$, and for each n-dimensional torus $T \subseteq \mathrm{Aut}(X)$ the identity component of the image $\theta(T)^\circ \subset \mathrm{Aut}(Y)$ is a closed torus of dimension n.

Proof. Let $\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ be an isomorphism. Since $\chi(Y) \neq 0$ it follows that there is a prime p that does not divide $\chi(Y)$.

Let $T \subset \mathrm{Aut}(X)$ be a torus of dimension n. We have $T = \mathrm{Cent}_{\mathrm{Aut}(X)}(T)$ (Lemma 2.10), and thus $\theta(T)$ is a closed subgroup of $\mathrm{Aut}(Y)$. Let $\mu_p \subset T$ be the subgroup generated by the elements of order p, and let $G := \mathrm{Cent}_{\mathrm{Aut}(Y)}(\theta(\mu_p))$ that is closed in $\mathrm{Aut}(Y)$. By Remark 2, we have $\theta(T) \subseteq G$ and $\dim Y = n$. Now Theorem 1.2 implies that $G^\circ \subset \mathrm{Aut}(Y)$ is a closed torus of dimension $\leq n$, and by Proposition 2.8 and Theorem 2.9, we see that $\theta(T)^\circ$ is a closed connected algebraic subgroup of G°.

In order to show that $\dim \theta(T)^\circ \geq n$ we construct closed subgroups $\{1\} = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$ with the following properties:

(i) $\dim T_i = i$ for all i.
(ii) $\theta(T_i)$ is closed in $\theta(T)$ for all i.

It then follows that $\theta(T_i)^\circ$ is a connected algebraic subgroup of $\theta(T)^\circ$. Since the index of $\theta(T_i)^\circ$ in $\theta(T_i)$ is countable (Proposition 2.8), but the index of T_i in T_{i+1} is not countable, we see that $\dim \theta(T_{i+1})^\circ > \dim \theta(T_i)^\circ$, and so $\dim \theta(T)^\circ \geq n$.

(a) Assume first that X is a torus. Then $\mathrm{Aut}(X)$ contains a copy of the symmetric groups $\mathcal{S}_n$, and we can find cyclic permutations $\tau_i \in \mathrm{Aut}(X)$ such that $T_i := \mathrm{Cent}_T(\tau_i)$ is a closed subtorus of dimension i, and $T_i \subset T_{i+1}$ for all $0 < i < n$. It then follows that $\theta(T_i) = \mathrm{Cent}_{\theta(T)}(\theta(\tau_i))$ is closed in $\theta(T)$, and we are done.

(b) Now assume that X is not a torus. By Proposition 2.12 there exist one-dimensional unipotent subgroups $U_1, \ldots, U_n$ of $\mathrm{Aut}(X)$ normalized by T such that the corresponding characters $\chi_1, \ldots, \chi_n \colon T \to \mathbb{C}^*$ are linearly independent. Since

$$\ker(\chi_i) = \{t \in T \mid t \circ u_i \circ t^{-1} = u_i \text{for all} u_i \in U_i\} = \mathrm{Cent}_T(U_i)$$

it follows that

$$T_i := \bigcap_{k=1}^{n-i} \ker(\chi_k) = \mathrm{Cent}_T(U_1 \cup \cdots \cup U_{n-i}) \subseteq T$$

is a closed algebraic subgroup of T of dimension i. It follows that the image $\theta(T_i) = \mathrm{Cent}_{\theta(T)}(\theta(U_1) \cup \cdots \cup \theta(U_n))$ is closed in $\theta(T)$, and the claim follows also in this case. ■

3.5 Proof of Theorem 1.1

Using Proposition 3.6, it is enough to show that a smooth toric variety Y with finite (and hence trivial) Picard group is quasi-affine.

For proving this, let $\Sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ be the fan that describes Y where N is a lattice of rank n. Let $N' \subseteq N$ be the sublattice spanned by $\Sigma \cap N$, and let Y' be the toric variety corresponding to the fan Σ in $N'_{\mathbb{Q}} = N' \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows from [8, p. 29] that

$$Y \simeq Y' \times (\mathbb{C}^*)^k,$$

where $k = \mathrm{rank}\, N/N'$. Thus, Y' is a smooth toric variety with trivial Picard group. Hence, it is enough to prove that Y' is quasi-affine and therefore we can assume $k = 0$, that is, Σ spans $N_{\mathbb{Q}}$. By [8, Proposition in Section 3.4] we get

$$0 = \mathrm{rank}\, \mathrm{Pic}(Y) = d - n,$$

where d is the number of edges in Σ. Let $\sigma \subset N_{\mathbb{Q}}$ be the convex cone spanned by the edges of Σ and let $\sigma^\vee$ denote the dual cone of σ in $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ where $M = \mathrm{Hom}(N, \mathbb{Z})$. Since $d = n$, the edges of Σ are linearly independent in $\mathbb{N}_{\mathbb{Q}}$ and thus σ is a simplex. From the inclusion of the cones of Σ in σ we get a morphism $f \colon Y \to \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$ by [8, Section 1.4], and since each cone in Σ is a face of σ it is locally an open immersion

[8, Lemma in Section 1.3]. This implies that f is quasi-finite and birational and thus by Zariski's Main Theorem [11, Corollaire 4.4.9] it is an open immersion.

4 Proof of the Main Theorem

4.1 A first characterization

Proposition 4.1. Let X be an irreducible quasi-affine variety. If $\mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$ is an isomorphism that maps an n-dimensional torus in $\mathrm{Aut}(\mathbb{A}^n)$ to an algebraic subgroup, then $X \simeq \mathbb{A}^n$ as a variety.

Proof. Since all n-dimensional tori in $\mathrm{Aut}(\mathbb{A}^n)$ are conjugate [1], all n-dimensional tori are sent to algebraic subgroups of $\mathrm{Aut}(X)$ via θ. The standard maximal torus T in $\mathrm{Aut}(\mathbb{A}^n)$ acts via conjugation on the subgroup of standard translations $\mathrm{Tr} \subset \mathrm{Aut}(\mathbb{A}^n)$ with a dense orbit $O \subset T$ and thus we get $\mathrm{Tr} = O \circ O$.

This implies that $S := \theta(T)$ acts on $U := \theta(\mathrm{Tr})$ via conjugation and we get $U = \theta(O) \circ \theta(O)$. Hence, for fixed $u_0 \in \theta(O) \subset U$ the morphism

$$S \times S \to \mathrm{Aut}(X)\,, \quad (s_1, s_2) \mapsto s_1 \circ u_0 \circ s_1^{-1} \circ s_2 \circ u_0 \circ s_2^{-1}$$

has image equal to U. Now it follows from Theorem 2.9 that U is a closed (commutative) algebraic subgroup of $\mathrm{Aut}(X)$ with no nontrivial element of finite order, hence a unipotent subgroup.

We claim that U has no nonconstant invariants on X. Indeed, let $\rho\colon \mathbb{G}_a \times X \to X$ be the $\mathbb{G}_a$-action on X coming from a nontrivial element of U. If $f \in \mathcal{O}(X)^U$ is a U-invariant, then it is easy to see that

$$\rho_f(s, x) := \rho(f(x) \cdot s, x) \qquad (*)$$

is a $\mathbb{G}_a$-action commuting with U. Since U is self-centralizing, we see that $\rho_f(s) \in U$ for all $s \in \mathbb{G}_a$. Moreover, formula $(*)$ shows that for every finite dimensional subspace $V \subset \mathcal{O}(X)^U$ the map $V \to U$, $f \mapsto \rho_f(1)$, is a morphism, which is injective. Indeed, $\rho_f(1) = \rho_h(1)$ implies that $\rho(f(x), x) = \rho(h(x), x)$ for all $x \in X$; hence, $f(x) = h(x)$ for all $x \in X \setminus X^\rho$. It follows that $\mathcal{O}(X)^U$ is finite-dimensional. Since X is irreducible, $\mathcal{O}(X)^U$ is an integral domain and hence equal to $\mathbb{C}$, as claimed.

Since X is irreducible and quasi-affine, the unipotent group U has a dense orbit that is closed, and so X is isomorphic to an affine space $\mathbb{A}^m$. Since m is the maximal number such that there exists a faithful action of $(\mathbb{Z}/2\mathbb{Z})^m$ on $\mathbb{A}^m$ (Remark 2), we finally get $m = n$. ■

If X is an affine variety, then $\mathrm{Aut}(X)$ has the structure of a so-called affine ind-group; see, for example, [20], [29], and [9] for more details. The following result is a special case of [17, Theorem 1.1]. It is an immediate consequence of Proposition 4.1 above because a homomorphism of affine ind-groups sends algebraic groups to algebraic groups.

Corollary 4.2. Let X be an irreducible affine variety. If there is an isomorphism $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ of affine ind-groups, then $X \simeq \mathbb{A}^n$ as a variety.

Corollary 4.3. Let X be a smooth, irreducible quasi-projective variety such that $\chi(X) \neq 0$ and $\mathrm{Pic}(X)$ is finite. If there is an isomorphism $\mathrm{Aut}(\mathbb{A}^n) \simeq \mathrm{Aut}(X)$ of abstract groups and if $\dim X \leq n$, then $X \simeq \mathbb{A}^n$ as a variety.

Proof. Theorem 1.1 shows that for an isomorphism $\theta\colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$ and any n-dimensional torus $T \subseteq \mathrm{Aut}(\mathbb{A}^n)$, the identity component of the image $S := \theta(T)^\circ$ is a closed torus of dimension n in $\mathrm{Aut}(X)$ and $\dim X = n$, and X is quasi-affine. Thus, we can apply Theorem 1.1 to $\theta^{-1}\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{A}^n)$ and get that $\theta^{-1}(S)^\circ$ is a closed torus of dimension n in $\mathrm{Aut}(\mathbb{A}^n)$. Since

$$\theta^{-1}(S)^\circ \subseteq \theta^{-1}(S) \subseteq T,$$

it follows that $\theta^{-1}(S) = T$, that is, $\theta(T) = S$ is a closed n-dimensional torus in $\mathrm{Aut}(X)$. The assumptions of Proposition 4.1 are now satisfied for the isomorphism $\theta\colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$, and the claim follows. ■

4.2 Proof of the Main Theorem

If the assumptions (1) of the Main Theorem hold, that is, X is a smooth, irreducible, quasi-projective variety of dimension n such that $\chi(X) \neq 0$ and $\mathrm{Pic}(X)$ is finite, then the claim follows from Corollary 4.3.

Now assume that the assumptions (2) are satisfied, that is, X is quasi-affine and toric of dimension n. Let $T \subseteq \mathrm{Aut}(X)$ be a torus of maximal dimension. We can apply Theorem 1.1 to an isomorphism $\theta\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{A}^n)$ and find that $S := \theta(T)^\circ \subset \mathrm{Aut}(\mathbb{A}^n)$ is a closed torus of dimension n. Since the index of the standard n-dimensional torus in its normalizer in $\mathrm{Aut}(\mathbb{A}^n)$ is finite and since all n-dimensional tori in $\mathrm{Aut}(\mathbb{A}^n)$ are conjugate [1], it follows that S has finite index in $\theta(T)$. Hence, $\theta^{-1}(S)$ has finite index in T. Since T is a divisible group, $\theta^{-1}(S) = T$ is an algebraic group. Thus, we can apply Proposition 4.1 to the isomorphism $\theta^{-1}\colon \mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}(X)$ and find that $X \simeq \mathbb{A}^n$ as a variety.

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