ALGEBRAIC TRANSFORMATION GROUPS

- AN INTRODUCTION -

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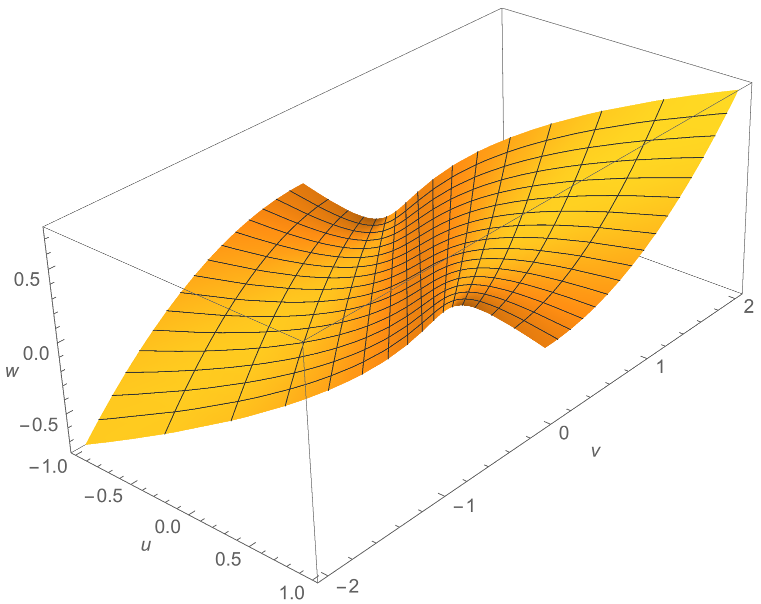
(Preliminary version from February 2009, updated in June 2011, July 2014, October 2016, January 2017)



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2010 *Mathematics Subject Classification.* Primary

Key words and phrases. algebraic groups, group actions, transformation groups, representation theory, invariants, algebraic quotient

To Renate, Marcel, Christian and Claudine

Contents

Preface

In the Fall term 1977/78 I gave a course "Geometrische Methoden in der Invariantentheorie" at the University of Bonn.

CHAPTER I

First Examples and Basic Concepts

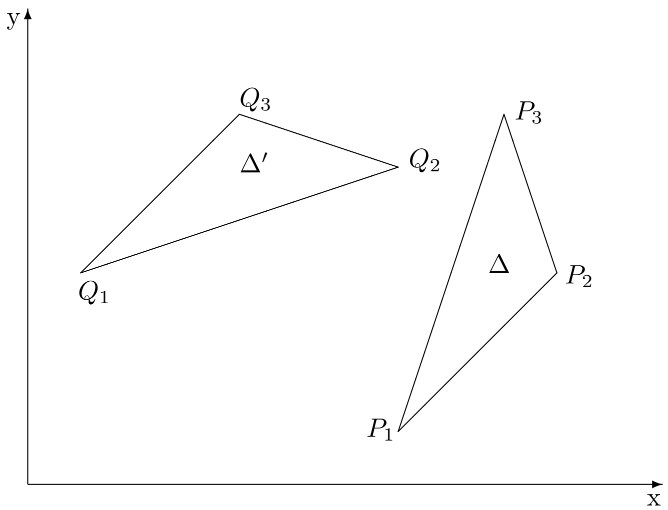
Contents

Introduction. In this first chapter, we introduce and discuss a few simple and sometimes well-known geometric examples. Since we don't develop the basics and the methods until the following chapters, we sometimes have to refer to later and must be satisfied with an intuitive justification and notions introduced ad hoc. Nevertheless, it is still worth while to make a detailed study of these examples at this point. One recognizes the necessity of making the intuitive notions and the basics precise and also of developing new methods. Moreover, in the remaining part of the book we can test our newly won knowledge on the examples which are given here.

There are also a number of exercises included in the text, some with hints. The reader is strongly advised to work out the solutions. At the end of each paragraph, we recollect them for the convenience of the reader.

1. Elementary Euclidean Geometry

1.1. Triangles. We denote by $E := \mathbb{R}^2$ the Euclidean plane endowed with the standard metric where the distance of two points $P = (x, y)$ and $P' = (x', y')$ is defined by $|P - P'| := \sqrt{(x - x')^2 + (y - y')^2}$. A *triangle* Δ is given by its vertices P_1, P_2, P_3 and will be denoted by $\Delta(P_1, P_2, P_3)$. Thus the set $\mathfrak{T} := E^3$ describes all triangles in the plane E, including the degenerated ones where all vertices are on a line or even partially coincide.



Two triangles Δ and Δ' are called *congruent* if there is an *isometry* φ of the plane such that $\varphi(\Delta) = \Delta'$. Recall that an isometry φ of the plane E is a map $\varphi \colon E \to E$ which preserves lengths: $|\varphi(P) - \varphi(Q)| = |P - Q|$ for all $P, Q \in E$ (see [**Art91**, Chapter 5.1]). The isometries form a group, the *isometry group* which we denote by . Using the description of the triangles by their vertices, we have the following definition.

Definition 1.1.1. Two triangles $\Delta(P_1, P_2, P_3)$ and $\Delta(Q_1, Q_2, Q_3)$ are *congruent*,

$$\Delta(P_1, P_2, P_3) \sim \Delta(Q_1, Q_2, Q_3),$$

if there is an isometry $\beta \in \mathrm{Iso}(E)$ and a permutation $\sigma \in \mathcal{S}_3$ such that $Q_{\sigma(i)} = \beta(P_i)$ for $i = 1, 2, 3$.

In terms of coordinates this means the following. Let $P_i = (x_i, y_i)$ and $Q_i = (x'_i, y'_i)$, $i = 1, 2, 3$. Then there is an orthogonal matrix $A \in \mathrm{O}_2(\mathbb{R})$, a vector $t =$

$(t_x, t_y) \in \mathbb{R}^2$ and a permutation $\sigma \in \mathcal{S}_3$ such that

$$\begin{bmatrix} x'_{\sigma(i)} \\ y'_{\sigma(i)} \end{bmatrix} = A \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad \text{for } i = 1, 2, 3.$$

1.2. Invariants. A well-known classical result says that *two triangles are congruent if and only if their edges have the same lengths.* This statement has two parts. Firstly, it says that the set of lengths of the edges of a triangle is an *"invariant"* of the congruence class, and secondly that this invariant *completely determines* the triangle up to congruence. To be more precise let us make the following definition.

DEFINITION 1.2.1. An *invariant* of the triangles is a map $f \colon \mathfrak{T} \to \mathbb{R}$ with the following property: If $\Delta, \Delta' \in \mathfrak{T}$ are congruent, $\Delta \sim \Delta'$, then $f(\Delta) = f(\Delta')$.

This means that the map f is invariant under isometries and permutations, i.e. $f(\beta(P_1), \beta(P_2), \beta(P_3)) = f(P_1, P_2, P_3)$ for $\beta \in \operatorname{Iso}(E)$, and $f(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)}) = f(P_1, P_2, P_3)$ for $\sigma \in \mathcal{S}_3$. Clearly, the invariants form a subalgebra of the algebra of real functions on $\mathfrak{T}$, and this algebra is closed under substitution, i.e. if $h \colon \mathbb{R} \to \mathbb{R}$ is any function and f an invariant, then so is $h \circ f \colon \Delta \mapsto h(f(\Delta))$.

EXAMPLES 1.2.2. (a) The *circumference* of a triangle $\Delta = \Delta(P_1, P_2, P_3)$, given by

$$u(\Delta) := |P_1 - P_2| + |P_2 - P_3| + |P_3 - P_1| = \ell_{12} + \ell_{23} + \ell_{31},$$

is an invariant. Here $\ell_{ij} := |P_i - P_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$.

(b) The *area* $F(\Delta)$ of the triangle $\Delta = \Delta(P_1, P_2, P_3)$ is an invariant. It is given by the formula

$$F(\Delta) = \left| \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \right| = \left| \frac{1}{2} (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \right|.$$

Note that the length ℓ_{ij} of a single edge of Δ is not an invariant in the sense of our definition; only the *symmetric* functions of the three lengths $\ell_{12}, \ell_{23}, \ell_{31}$ are invariants! The classical result mentioned above can now be formulated in the following way.

PROPOSITION 1.2.3. *The congruence class of a triangle $\Delta \in \mathfrak{T}$ is completely determined by the three invariants*

$$f_1(\Delta) := u(\Delta), \;\; f_2(\Delta) := \ell_{12}\ell_{23} + \ell_{12}\ell_{31} + \ell_{23}\ell_{31}, \;\; f_3(\Delta) := \ell_{12}\ell_{23}\ell_{31}$$

which are the elementary symmetric functions in the lengths of the edges.

EXERCISE 1.2.4. Verify the formula for the area of a triangle given in Example 1.2.2(b).

EXERCISE 1.2.5. Using the identification $\mathfrak{T} = \mathbb{R}^6$ show that a function $f \colon \mathbb{R}^6 \to \mathbb{R}$ is an invariant if and only if the following three conditions hold:

(1) $f(x_1, y_1, x_2, y_2, x_3, y_3) = f(x_{\sigma(1)}, y_{\sigma(1)}, x_{\sigma(2)}, y_{\sigma(2)}, x_{\sigma(3)}, y_{\sigma(3)})$ for all permutations $\sigma \in \mathcal{S}_3$;

(2) $f(x_1, y_1, x_2, y_2, x_3, y_3) = f(x_1 - x_3, y_1 - y_3, x_2 - x_3, y_2 - y_3, 0, 0)$;

(3) $f(x_1, y_1, x_2, y_2, x_3, y_3) = f(x'_1, y'_1, x'_2, y'_2, x'_3, y'_3)$ if $\begin{bmatrix} x'_i \\ y'_i \end{bmatrix} := A \begin{bmatrix} x_i \\ y_i \end{bmatrix}$ for $i = 1, 2, 3$ and $A \in \mathrm{O}_2(\mathbb{R})$.

EXERCISE 1.2.6. Consider the following set of triples of real numbers:

$$D := \{(a, b, c) \in \mathbb{R}^3 \mid a, b, c \text{ are the lengths of the edges of a triangle}\}.$$

Describe D as a subset of $\mathbb{R}^3$ by inequalities and show that there is a homeomorphism $(\mathbb{R}_{\geq 0})^3 \xrightarrow{\sim} D$ which is equivariant with respect to permutations from $\mathcal{S}_3$.

1.3. Congruence classes. It turns out that all the formulas become nicer and easier to handle if we replace the lengths ℓ_{ij} of the edges by their squares

$$\ell_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2.$$

Then, for a triangle $\Delta = \Delta(P_1, P_2, P_3) \in \mathfrak{T}$, the three *elementary symmetric functions* in $\ell_{12}^2, \ell_{23}^2, \ell_{31}^2$ are homogeneous polynomials in $x_1, y_1, x_2, y_2, x_3, y_3$ of degree 2, 4 and 6:

$$\begin{aligned} s_1(\Delta) &:= \ell_{12}^2 + \ell_{23}^2 + \ell_{31}^2, \\ s_2(\Delta) &:= \ell_{12}^2\ell_{23}^2 + \ell_{12}^2\ell_{31}^2 + \ell_{23}^2\ell_{31}^2, \\ s_3(\Delta) &:= \ell_{12}^2\ell_{23}^2\ell_{31}^2. \end{aligned}$$

As above, $s_1(\Delta), s_2(\Delta), s_3(\Delta)$ are invariants of Δ, and they determine Δ up to congruence. Here is a first example.

Example 1.3.1. For the square of the area $F(\Delta)$ of the triangle Δ we have the following expression:

$$16\, F(\Delta)^2 = 4(\ell_{12}^2\ell_{23}^2 + \ell_{12}^2\ell_{31}^2 + \ell_{23}^2\ell_{31}^2) - (\ell_{12}^2 + \ell_{23}^2 + \ell_{31}^2)^2 = 4\, s_2(\Delta) - s_1(\Delta)^2.$$

In particular, $F(\Delta)$ is a polynomial in the first two elementary symmetric functions of $\ell_{12}^2, \ell_{23}^2, \ell_{31}^2$.

Exercise 1.3.2. Verify the polynomial expression for $F(\Delta)^2$ given above. Give an expression of $F(\Delta)^2$ as a polynomial in $\ell_{12}^2 + \ell_{23}^2 + \ell_{31}^2$ and $\ell_{12}^4 + \ell_{23}^4 + \ell_{31}^4$.

Now consider the map

$$\pi\colon \mathfrak{T} \to \mathbb{R}^3,\ \ \Delta \mapsto (s_1(\Delta), s_2(\Delta), s_3(\Delta)).$$

Proposition 1.2.3 says that $\Delta \sim \Delta'$ if and only if $\pi(\Delta) = \pi(\Delta')$. In other words, the fiber $\pi^{-1}(\pi(\Delta))$ is equal to the congruence class C_Δ of Δ. Therefore, we can identify the image $\pi(\mathfrak{T})$ with the set of congruence classes:

$$\pi\colon \mathfrak{T} \to \pi(\mathfrak{T}) \simeq \mathfrak{T}/\!\sim .$$

In this way the set of congruence classes $\mathfrak{T}/\sim$ appears as a subset of $\mathbb{R}^3$. It is an interesting task to work out the shape of this subset. E.g. the image of a non-degenerate triangle is an interior point of $\pi(\mathfrak{T})$ where as the image of a degenerate triangle is a boundary point.

Remark 1.3.3. Since π is a polynomial map the image $\pi(\mathfrak{T}) \subseteq \mathbb{R}^3$ is a which is defined by certain inequalities (see [**PS85**]). We will see below (Remark 1.4.2) that the quotient topology on $\pi(\mathfrak{T})$ coincides with the topology induced from the embedding $\pi(\mathfrak{T}) \subseteq \mathbb{R}^3$. As a consequence, every *continuous* invariant $f\colon \mathfrak{T} \to \mathbb{R}$ is a continuous function in s_1, s_2, s_3: $f(\Delta) = \bar{f}(s_1(\Delta), s_2(\Delta), s_3(\Delta))$ where $\bar{f}\colon \mathbb{R}^3 \to \mathbb{R}$ is continuous. (This follows from Tietze's extension theorem which asserts that for a closed subset A of a normal topological space X every continuous function $f\colon A \to \mathbb{R}$ extends to a continuous function on X.)

1.4. Orbit space and quotient map. We now give a different description of the congruence classes, namely as orbits under a certain group action. The group of isometries of the plane E is the semidirect product of the *orthogonal group* $\mathrm{O}_2(\mathbb{R})$ and the subgroup of *translations* $T \simeq (\mathbb{R}_+)^2$: $\mathrm{Iso}(E) = \mathrm{O}_2(\mathbb{R}) \ltimes T$. This group acts simultaneously on any number of copies of E, in particular on $\mathfrak{T} = E^3$:

$$\varphi(P_1, P_2, P_3) = (\varphi(P_1), \varphi(P_2), \varphi(P_3)).$$

There is also an action of the symmetric group $\mathcal{S}_3$ on E^3 by permuting the factors,

$$\sigma(P_1, P_2, P_3) = (P_{\sigma^{-1}(1)}, P_{\sigma^{-1}(2)}, P_{\sigma^{-1}(3)}),$$

and these two actions obviously commute. Thus we obtain an action of the product

$$G := \mathrm{Iso}(E) \times \mathcal{S}_3$$

on $\mathfrak{T} = E^3$ whose orbits are the congruence classes (see Definition 1.1.1). Therefore, the space $\mathfrak{T}/\sim$ can be regarded as the *orbit space* $\mathfrak{T}/G$, and the map

$$\pi\colon \mathfrak{T} \to \mathbb{R}^3, \quad \Delta \mapsto (s_1(\Delta), s_2(\Delta), s_3(\Delta))$$

already considered above identifies the image $\pi(\mathfrak{T})$ with the orbit space $\mathfrak{T}/G$. Let us call such a map π a *quotient map*. A more precise definition will be given later in the algebraic context.

We have already mentioned above that the invariants s_1, s_2, s_3 are polynomials. Denote by $\mathbb{R}[\mathfrak{T}] = \mathbb{R}[x_1, y_1, x_2, y_2, x_3, y_3]$ the algebra of polynomial functions on $\mathfrak{T}$.

Proposition 1.4.1. *The algebra $\mathbb{R}[\mathfrak{T}]^G$ of invariant polynomial functions on $\mathfrak{T}$ is a polynomial ring generated by s_1, s_2, s_3. More precisely, the pull-back map of the quotient map,*

$$\pi^*\colon \mathbb{R}[x, y, z] \to \mathbb{R}[\mathfrak{T}],\ p \mapsto p \circ \pi,$$

satisfies $\pi^*(x) = s_1, \pi^*(y) = s_2, \pi^*(z) = s_3$, and it induces an isomorphism

$$\mathbb{R}[x, y, z] \xrightarrow{\sim} \mathbb{R}[\mathfrak{T}]^G.$$

Proof. Consider the three polynomial maps

$$\mathfrak{T} = E^3 \xrightarrow{\pi_1} E^2 \xrightarrow{\pi_2} \mathbb{R}^3 \xrightarrow{\pi_3} \mathbb{R}^3$$

defined in the following way:

$$\begin{aligned} \pi_1(P_1, P_2, P_3) &:= (P_2 - P_3, P_1 - P_3), \\ \pi_2(Q_1, Q_2) &:= (|Q_1|^2, |Q_2|^2, |Q_1 - Q_2|^2), \\ \pi_3(a, b, c) &:= (a + b + c, ab + ac + bc, abc). \end{aligned}$$

It is easy to see that he composition $\pi_3 \circ \pi_2 \circ \pi_1$ is equal to π.

(a) The map $\pi_1\colon E^3 \to E^2$ is linear with kernel $\{(P, P, P) \mid P \in E\}$, and so the fibers are the orbits of the normal subgroup $T \subseteq \mathrm{Iso}(E)$ of translations. Hence π_1 is a surjective quotient map with respect to T. Also, π is equivariant with respect to the linear actions of $\mathrm{O}_2(\mathbb{R})$ on E^3 and E^2. It is also equivariant with respect to the action of $\mathcal{S}_3$ which, on E^2, is given by $(1, 2)(Q_1, Q_2) = (Q_2, Q_1)$ and $(1, 3)(Q_1, Q_2) = (Q_1 - Q_2, -Q_2)$. One easily sees that π_1^* identifies $\mathbb{R}[E^2]$ with the invariants $\mathbb{R}[E^3]^T \subseteq \mathbb{R}[E^3]$.

(b) The map $\pi_2\colon E_2 \to \mathbb{R}^3$ is a quotient with respect to $\mathrm{O}_2(\mathbb{R})$, and π_2^* identifies $\mathbb{R}[a, b, c]$ with the invariants $\mathbb{R}[E^2]^{\mathrm{O}_2(\mathbb{R})}$. The first statement is easy whereas the second needs some work (see Exercise 1.4.3). Moreover, π_2 is equivariant with respect to the $\mathcal{S}_3$-action on E^2 and $\mathbb{R}^3$.

(c) The map $\pi_3\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a quotient map with respect to $\mathcal{S}_3$, and π_3^* identifies $\mathbb{R}[x, y, z]$ with the symmetric polynomials $\mathbb{R}[a, b, c]^{\mathcal{S}_3}$. This will be discussed in the following paragraph (see Proposition 2.2.1).

Clearly, (a), (b) and (c) imply that the composition $\pi = \pi_3 \circ \pi_2 \circ \pi_1$ is a quotient with respect to G, and that π^* identifies $\mathbb{R}[x, y, z]$ with the invariants $\mathbb{R}[E^3]^G$. In particular, $\mathbb{R}[E^3]^G$ is a polynomial ring generated by s_1, s_2, s_3. □

Remark 1.4.2. In the notation of the proof above we see that π_1 is a linear surjective map, hence open, whereas the composition $\pi_3 \circ \pi_2\colon E^2 \to \mathbb{R}^3$ is given by invariants under the compact group $\mathcal{S}_3 \times \mathrm{O}_2(\mathbb{R})$. It follows that this map is proper and that the image carries the quotient topology ([**Sch75**]; cf. [**PS85**, Proposition 0.4]). As already mentioned above this implies that every *continuous* invariant

$f\colon \mathfrak{T} \to \mathbb{R}$ is a continuous function in s_1, s_2, s_3. In particular, we did not loose any information by restricting to polynomial functions!

EXERCISE 1.4.3. Let $E := \mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space with the standard scalar product $(x, y) := x_1 y_1 + \cdots + x_n y_n$, and denote by $\mathbb{R}[E]$ and $\mathbb{R}[E^2]$ the real polynomial functions on E and $E^2 := E \times E$ respectively. We want to determine the invariant polynomials under $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$.

(1) We have $\mathbb{R}[E]^{\mathrm{SO}_n} = \mathbb{R}[E]^{\mathrm{O}_n} = \mathbb{R}[s]$ where $s(x) := |x|^2 := (x, x)$.
(2) We have $\mathbb{R}[E^2]^{\mathrm{O}_n} = \mathbb{R}[s_1, s_2, s_{12}]$ where $s_1(x, y) := |x|^2$, $s_2(x, y) := |y|^2$ and $s_{12}(x, y) := (x, y)$.
(3) For $n \geq 3$ we have $\mathbb{R}[E^2]^{\mathrm{SO}_n} = \mathbb{R}[E^2]^{\mathrm{O}_n}$. What happens for $n = 2$?

(Hint: Restricting an invariant polynomial $f \in \mathbb{R}[E^2]^{\mathrm{O}_n(\mathbb{R})}$ to $D := \mathbb{R}e_1 \times \mathbb{R}e_2$ we get $f|_D = f(x_1, y_2) = q(x_1^2, y_2^2)$, and so the invariant $f_1 := f - q(s_1, s_2)$ vanishes on D and thus on $\mathrm{O}_n(\mathbb{R}) \cdot D$ which is the zeros set of s_{12} in E^2. In order to see that f_1 is divisible by s_{12} one complexifies the spaces and shows that $\mathrm{O}_n(\mathbb{R}) \cdot D$ is ZARISKI dense in the complex zeros set $\mathcal{V}_{\mathbb{C}}(s_{12}) \subseteq E_{\mathbb{C}}^2$.)

1.5. Summary. This example is typical for many classification problems. We start with certain mathematical objects M which we want to classify up to a given equivalence. For example, we want to classify algebras, modules, representations etc. up to isomorphisms. The objects are described by a set X, i.e. to every $x \in X$ there is an object $M(x)$ associated to x. In addition, there is a group G acting on X such that $M(x)$ is equivalent to $M(x')$ if and only if there is a $g \in G$ such that $gx = x'$. As a consequence, the set of equivalence classes of these objects equals the orbit space X/G. Moreover, there is a natural map $\pi_X\colon X \to X/G$, the *quotient map*, which sends any $x \in X$ to its orbit Gx.

In general, there might be several different descriptions of our objects, so that the set X is not uniquely determined. It is an important task to find a description suitable and adapted to the given situation. Moreover, the set X usually has some additional structure, e.g. a topology which allows to say if two objects are close to each other, and to define continuous families of objects. Then the set of equivalence classes inherits, via the quotient map $\pi\colon X \to X/G$, a topology, the so-called *quotient topology.*

In order to distinguish non-equivalent objects we are looking for *invariant functions* $f\colon X \to \mathbb{R}$, i.e. $\mathbb{R}$-valued functions which the property that $f(x) = f(x')$ whenever x and x' belong to the same orbit. If we can even find enough invariants $\{f_1, f_2, \ldots, f_n\}$ so that they *separate the orbits*, i.e. for x, x' not in the same orbit there is an f_j such that $f_j(x) \neq f_j(x')$, then we can define the map $\pi\colon X \to \mathbb{R}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$, whose image can be identified with the orbits space X/G. If X is a topological space and all f_j are continuous, then π is continuous, and one can hope that quotient topology on $\pi(X) \simeq X/G$ coincides with the induced topology from $\mathbb{R}^n$. If this is the case and if, in addition, the image is closed, then we know from Tietzes extension theorem that every continuous invariant function f is a continuous function in $f_1, \ldots, f_n$, i.e., $f(x) = q(f_1(x), \ldots, f_n(x))$ for a continuous function $q\colon \mathbb{R}^n \to \mathbb{R}$.

In our situation, the set X will be an algebraic variety, and the equivalence relation on X will be given by the action of an algebraic group on X. So an important question is whether the orbit space X/G also carries the structure of an algebraic variety. In general, this is not the case, as we will see in some later examples in this Chapter. One of the main difficulties is that even continuous functions cannot separate all orbits, because there exist non-closed orbits. (This did not happen in the example above since the action of the translations T was free and the group $\bar{G} := G/T$ was compact.) Therefore, we have to study very carefully the invariant regular functions. They will help us to get around some of these difficulties.

2. Symmetric Product and Symmetric Functions

2.1. Symmetric product. Given a set X we denote by X^n the n-fold cartesian product, i.e., the set of ordered n-tuples of elements from X:

$$X^n := X \times X \times \cdots \times X = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X\}.$$

However, there are many examples where the ordering does not make sense, e.g. the vertices of triangle, the roots of a polynomial, etc. This leads to the definition of the symmetric product. Denote by $\mathcal{S}_n$ the symmetric group on n letters.

DEFINITION 2.1.1. Let X be a set. On X^n define the following equivalence relation

$$(x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n) \iff \exists \sigma \in \mathcal{S}_n : y_i = x_{\sigma(i)} \text{ for } i = 1, \ldots, n.$$

The *n-th symmetric product* is then defined by $X^{(n)} := X^n/\sim$. It is the set of *unordered n-tuples* of elements from X.

There is an action of the *symmetric group* $\mathcal{S}_n$ on X^n, defined by

$$\sigma(x_1, x_2, \ldots, x_n) := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}),$$

whose orbits are the equivalence classes, and so the symmetric product equals the orbit space: $S^n X = X/\mathcal{S}_n$. (Note that we have to use σ^{-1} in order to get a *left* action of $\mathcal{S}_n$.)

EXAMPLE 2.1.2. The unordered pairs of real numbers $\mathbb{R}^{(2)}$ can be described by using the symmetric functions $x+y$ and xy. In fact, the fibers of the map

$$\pi\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x+y, xy).$$

are the equivalence classes and the image is given by $\{(u, v) \mid u^2 - 4v \geq 0\}$ which is homeomorphic to $\mathbb{R} \times \mathbb{R}_{\geq 0}$. Thus $\pi\colon \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}_{\geq 0}$, $(x, y) \mapsto (x+y, (x-y)^2)$ is the quotient map (see Exercise 2.1.4 below).

EXAMPLE 2.1.3. We have $\mathbb{C}^{(2)} \simeq \mathbb{C}^2$ where the quotient map $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2$ is given by $(x, y) \mapsto (x+y, xy)$. In fact, this morphism is surjective and the fibers are exactly the orbits under $\mathcal{S}_2$. If $(a, b) \in \mathbb{C}^2$ and x_1, x_2 are the roots of $t^2 - at + b$, then $\pi^{-1}(a, b) = \{(x_1, x_2), (x_2, x_1)\}$. In particular, the fiber of (a, b) contains only one element if and only if $a^2 - 4b = 0$.

EXERCISE 2.1.4. Show that the map $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2$ of Example 2.1.3 is proper and open in the standard $\mathbb{C}$-topology. Deduce from this that the same assertions hold for the map $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\sim \subseteq \mathbb{R}^2$ given in Example 2.1.2.

2.2. Symmetric functions. The symmetric group $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$ has a natural representation on $\mathbb{C}^n$ which is given by $\sigma(e_i) := e_{\sigma(i)}$, $i = 1, 2, \ldots, n$. In terms of coordinates we have

$$\sigma(a_1, a_2, \ldots, a_n) := (a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)}).$$

The corresponding matrices P_σ are called permutation matrices (see [**Art91**, Chap. 1.4]). The induced action on the polynomial functions $\mathbb{C}[x_1, \ldots, x_n]$ on $\mathbb{C}^n$ is given by $\sigma x_i = x_{\sigma(i)}$, $i = 1, \ldots, n$, and the invariant polynomial function $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ are the *symmetric polynomials*. The *elementary symmetric functions* $s_1, s_2, \ldots, s_n$ are defined by

$$s_k(x_1, \ldots, s_n) := \sum_{1_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In particular,

$$s_1(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n \quad \text{and} \quad s_n(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n.$$

PROPOSITION 2.2.1. *The elementary symmetric functions $s_1, s_2, \ldots, s_n$ are algebraically independent and generate the algebra of of symmetric polynomials:*

$$\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n} = \mathbb{C}[s_1, s_2, \ldots, s_n].$$

In particular, every symmetric polynomial can be uniquely written as a polynomial in the s_i.

PROOF. (1) We prove this by induction on n, the case $n = 1$ being obvious. Define $s_i' := s_i|_{x_n=0}$. Then $s_1', \ldots, s_{n-1}'$ are the elementary symmetric functions in the variables $x_1, \ldots, x_{n-1}$ and $s_n' = 0$. Assume that $p(s_1, \ldots, s_n) = 0$ where $p = p(t_1, \ldots, t_n)$ is a polynomial of minimal degree > 0 with this property. Then $0 = p(s_1, \ldots, s_n)|_{x_n=0} = p(s_1', \ldots, s_{n-1}', 0)$. Hence, by induction, $p(t_1, \ldots, t_{n-1}, 0) = 0$. Thus p is divisible by t_n which contradicts the minimality of p.

Let $f \in \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ be symmetric. Then $f(x_1, \ldots, x_{n-1}, 0)$ is symmetric in $x_1, \ldots, x_{n-1}$ and so, by induction, $f(x_1, \ldots, x_{n-1}, 0) = q(s_1', \ldots, s_{n-1}')$ with a suitable polynomial q. Define $g := f - q(s_1, \ldots, s_{n-1})$. Then g is symmetric and $g|_{x_n=0} = 0$. It follows that g is divisible by x_n, hence by all x_i's, and so $g = s_n f_0$ where f_0 is symmetric and of smaller degree than g. The claim follows by induction on $\deg f$. □

EXERCISE 2.2.2. Consider the symmetric polynomials $\psi_j := x_1^j + x_2^j + \cdots + x_n^j$ which are called *power sums* or NEWTON *functions*. Prove the following formulas due to NEWTON,

$$(-1)^{j+1} j s_j = \psi_j - s_1\psi_{j-1} + s_2\psi_{j-2} - \cdots + (-1)^{j-1} s_{j-1}\psi_1 \text{ for } j = 1, \ldots, n,$$

and deduce that the power sums $\psi_1, \psi_2, \ldots, \psi_n$ generate the algebra of symmetric functions $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$.
(Hint: The case $j = n$ is easy: Consider $f(t) := \prod_i (t - x_i)$ and calculate $\sum_i f(x_i)$ which is equal to 0. For $j < n$, the right hand side is a symmetric function of degree $\leq j$, hence can be expressed as a polynomial in $s_1, \ldots, s_j$. Now put $x_{j+1} = \ldots = x_n = 0$ and use induction on n. Another proof can be found in [**Wey97**, Chap. II A.3].)

EXERCISE 2.2.3. Show that every rational symmetric function is a rational function in $s_1, \ldots, s_n$. In particular, the field extension $\mathbb{C}(x_1, \ldots, x_n)/\mathbb{C}(s_1, \ldots, s_n)$ is a finite Galois extension of degree $n!$ with Galois group $\mathcal{S}_n$.

EXERCISE 2.2.4. Show that

$$\begin{aligned} \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_{n-1}} &= \mathbb{C}[x_1, \ldots, x_{n-1}]^{\mathcal{S}_{n-1}}[x_n] = \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}[x_n] \\ &= \bigoplus_{j=0}^{n-1} \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n} x_n^j \end{aligned}$$

(Hint: If $f \in \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ and $f = \sum_j f_j x_n^j$, then $f_j \in \mathbb{C}[x_1, \ldots, x_{n-1}]^{\mathcal{S}_{n-1}}$. This gives the second equality. For the last one uses that x_n satisfies an integral equation of degree n, see (1) in the following section 2.3.)

EXERCISE 2.2.5. Show that

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{0 \leq i_k < k} \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n} x_2^{i_2} x_3^{i_3} \cdots x_n^{i_n}.$$

In particular, $\mathbb{C}[x_1, \ldots, x_n]$ is a free module over $\mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ of rank $n!$.
(Hint: By induction, $\mathbb{C}[x_1, \ldots, x_{n-1}] = \bigoplus_{0 \leq i_k < k} \mathbb{C}[x_1, \ldots, x_{n-1}]^{\mathcal{S}_{n-1}} x_2^{i_2} x_3^{i_3} \cdots x_{n-1}^{i_{n-1}}$, hence $\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{0 \leq i_k < k} \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_{n-1}} x_2^{i_2} x_3^{i_3} \cdots x_{n-1}^{i_{n-1}}$. Now use the previous exercise.)

EXERCISE 2.2.6. The *orbit sum* of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is defined by $s_f := \sum_{h \in O_f} h$ where $O_f \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is the orbit of f under $\mathcal{S}_n$. E.g. $s_{x_1 x_2 \cdots x_k} = s_k$ and $s_{x_1^j} = \psi_j$. Show that $\mathbb{C}[x_1, \ldots, x_n]_d^{\mathcal{S}_n}$ has a basis consisting of the orbit sums of the monomials $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where $p = (p_1, p_2, \cdots, p_n)$ runs through the partitions of d into n elements, i.e. $p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$ and $p_1 + p_2 + \cdots + p_n = d$.

2.3. Roots of polynomials. It is well-known that the coefficients of a monic polynomial are the elementary symmetric functions of the roots, up to sign. More precisely, we have the following identity

$$\prod_{i=1}^{n}(t-x_i) = t^n - s_1(x_1,\dots,x_n)t^{n-1} + s_2(x_1,\dots,x_n)t^{n-2} - \cdots + (-1)^n s_n(x_1,\dots,x_n). \tag{1}$$

where both sides are considered as elements of the polynomial ring $\mathbb{C}[x_1,\dots,x_n,t]$.

EXAMPLE 2.3.1. The polynomial $\prod_{i<j}(x_i-x_j)^2$ is symmetric, hence can be written as a polynomial in the elementary symmetric functions s_k:

$$\prod_{i<j}(x_i-x_j)^2 = D(s_1,\dots,s_n).$$

The polynomial D has degree $n(n-1)$ and is called *discriminant*. By definition, $D(b_1,\dots,b_n)=0$ if and only if the polynomial $t^n - b_1t^{n-1} + b_2t^{n-2} - \cdots + (-1)^n b_n$ has multiple roots.

EXERCISE 2.3.2. For any pair $n \geq m$ there is a polynomial $R_{n,m} \in \mathbb{C}[x_{1,\dots},x_n,y_1,\dots,y_m]$ with the following property: Two monic polynomials $f = t^n + a_1t^{n-1} + \cdots + a_n$ and $g = t^m + b_1t^{m-1} + \cdots + b_m$ have a common root if and only if $R_{n,m}(a_1,\dots,a_n,b_1,\dots,b_m) = 0$. (The polynomial $R_{n,m}$ is called the *resultant*.)

Consider the following polynomial map

$$\pi\colon \mathbb{C}^n \to \mathbb{C}^n, (a_1,\dots,a_n) \mapsto (s_1(a_1,\dots,a_n),\dots,s_n(a_1,\dots,a_n)).$$

It follows from the above that π is surjective and that the fibers are the $\mathcal{S}_n$-orbits. In fact, an element $\lambda = (\lambda_1,\dots,\lambda_n)$ of the fiber of $(b_1,\dots,b_n)$ consists of the roots (with multiplicities) of the polynomial $t^n - b_1t^{n-1} + b_2t^{n-2} - \cdots + (-1)^n b_n$. But more is true:

PROPOSITION 2.3.3. (1) *The morphism $\pi\colon \mathbb{C}^n \to \mathbb{C}^n$ is surjective and its fibers are the $\mathcal{S}_n$-orbits.*
(2) *The morphism π is finite and therefore closed in the* ZARISKI *topology.*
(3) *The morphism π is proper and therefore closed in the standard $\mathbb{C}$-topology.*

PROOF. We just proved (1), and (2) follows from Exercise 2.2.5. A proof of the last statement (3) can be found in [**Sch75**]. □

The proposition shows that the polynomial map $\pi\colon \mathbb{C}^n \to \mathbb{C}^n$ is a quotient map with the additional property that the image carries the quotient topology. Therefore, every continuous invariant function f factors through π: $f = \pi^*(\bar{f}) := \bar{f}\circ\pi$ with a continuous function $\bar{f}\colon \mathbb{C}^n \to \mathbb{C}$. And the same holds for the polynomial functions, because

$$\pi^*(\mathbb{C}[y_1,\dots,y_n]) = \mathbb{C}[s_1,\dots,s_n] = \mathbb{C}[x_1,\dots,x_n]^{\mathcal{S}_n}.$$

In particular, the symmetric product $\mathbb{C}^{(n)}$ can be identified with $\mathbb{C}^n$ by using the elementary symmetric functions. It turns out that the more general symmetric products $(\mathbb{C}^k)^{(n)}$ are much more complicated, and they are not really understood so far. Let us look at the first non-trivial case.

EXAMPLE 2.3.4. In order to describe $(\mathbb{C}^2)^{(2)}$ we first calculate the algebra of invariants $\mathbb{C}[x_1,y_1,x_2,y_2]^{\mathcal{S}_2}$ where $\sigma(x_1)=x_2$ and $\sigma(y_1)=y_2$. It is generated by the following five invariants

$$a_1 := x_1+x_2,\ a_2 := y_1+y_2,$$
$$a_3 := (x_1-x_2)^2,\ a_4 := (y_1-y_2)^2,\ a_5 := (x_1-x_2)(y_1-y_2)$$

which satisfy the relation $a_3a_4 = a_5^2$. In fact, using the new generators

$$z_1 := x_1 + x_2,\ z_2 := y_1 + y_2,\ z_3 := x_1 - x_2,\ z_4 := y_1 - y_2$$

we get for a monomial $m = z_1^{r_1} z_2^{r_2} z_3^{r_3} z_4^{r_4}$ that $\sigma(m) = (-1)^{r_3+r_4} m$, and so the invariants are generated by $z_1, z_2, z_3^2, z_4^2, z_3z_4$, as claimed. One can deduce from this that the morphism

$$\pi\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^5$$

given by the five invariants has image $X := \mathcal{V}(y_3y_4 - y_5^2) \subseteq \mathbb{C}^5$ and that the induced map $\pi\colon \mathbb{C}^2 \times \mathbb{C}^2 \to X$ has all the properties of a quotient map, i.e. $(\mathbb{C}^2)^{(2)} \simeq X$. The details will be discussed later, see **??**.

Remark 2.3.5. If we restrict the morphism π to the real points $\mathbb{R}^n \subseteq \mathbb{C}^n$ then the map $\pi|_{\mathbb{R}^n}\colon \mathbb{R}^n \to \mathbb{R}^n$ is not anymore surjective. In fact, the image $\pi(\mathbb{R}^n)$ corresponds to those real polynomials all of whose roots are real. For example, the quadratic polynomial $t^2 - bt + c$ has real roots if and only if $b^2 - 4c \geq 0$. This implies that $\pi(\mathbb{R}^2) = \{(b, c) \mid b^2 - 4c \geq 0\}$.

Is there a similar description for real polynomials of higher degree? In fact, there is a beautiful result due to Sylvester (see [**PS85**]). For a polynomial $p = t^n - b_1t^{n-1} + b_2t^{n-2} - \cdots + (-1)^n b_n$ define the *Bezoutian matrix* by

$$\operatorname{Bez}(p) := (\psi_{i+j-2}(b_1, \ldots, b_n))_{i,j=1,\cdots n}$$

where $\psi_k(b_1, \ldots, b_n)$ are the power sums in the roots of p expressed as polynomials in the coefficients of p (see Exercise 2.2.2). Then we have the following result (see [**Pro78**], cf. [**PS85**]).

Proposition 2.3.6. *Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a real monic polynomial.*

(1) *The roots of p are all real if and only if $\operatorname{Bez}(p)$ is positive semidefinite.*
(2) *The rank of $\operatorname{Bez}(p)$ equals the number of distinct roots, and its signature equals the number of distinct real roots.*

Exercise 2.3.7. Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup. Then the G-invariant polynomials $\mathbb{C}[x_1, \ldots, x_n]^G$ separate the G-orbits on $\mathbb{C}^n$, i.e. for any two orbits O_1, O_2 there is an invariant function which takes different values on O_1 and O_2.
More generally, if X is an affine variety and $G \subseteq \operatorname{Aut}(X)$ a finite group of regular automorphisms of X, then the G-invariant regular functions $\mathcal{O}(X)^G$ separate the G-orbits on X.
(Hint: First construct a function $f \in \mathcal{O}(X)$ which vanishes on one orbit, but is non-zero on any point of the other. Then $\prod_{g\in G} g^*(f)$ is invariant and separates the two orbits. Here $g^*(f) = f \circ g$ is the pull-back.)

3. Quadratic Forms

3.1. Equivalence classes. A (complex) quadratic form q is a homogeneous polynomial function of degree two:

$$q(x_1, \ldots, x_n) = \sum_{i \leq j} c_{ij} x_i x_j,\ c_{ij} \in \mathbb{C}.$$

Two such forms are called *equivalent* it they are obtained from each other by a linear substitution of the variables. The quadratic forms form a vector space $Q_n := \mathbb{C}[x_1, \ldots, x_n]_2$ of dimension $\binom{n+1}{2}$ which can be identified with the complex symmetric $n \times n$-matrices $\operatorname{Sym}_n(\mathbb{C})$: For $A = (a_{ij}) \in \operatorname{Sym}_n(\mathbb{C})$ we set

$$q_A := (x_1, \ldots, x_n) A (x_1, \ldots, x_n)^t = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

The inverse map is given in the following way. Define the associated bilinear form $B_q(x,y) := \frac{1}{2}(q(x+y) - q(x) - q(y))$ and put $A_q := (B(e_i, e_j))$. This identification allows to define the *rank* $\operatorname{rk} q$ and the *discriminant* $\Delta\, q$ of a quadratic form q:

$$\operatorname{rk} q := \operatorname{rk} A_q, \quad \Delta\, q := \det A_q.$$

Clearly, the two forms q_A and q_B are equivalent if and only if there is an invertible $g \in \mathrm{GL}_n(\mathbb{C})$ such that $B = gAg^t$. In particular, the rank is an invariant, but the discriminant is not: $\Delta q_B = (\det g)^2 \Delta q_A$. We have the following well-known classification result.

PROPOSITION 3.1.1. *Two quadratic forms are equivalent if and only if they have the same rank. In particular, a quadratic q form of rank r is a equivalent to*

$$q_r := x_1^2 + \cdots + x_r^2, \quad r = 0, 1, \ldots, n.$$

PROOF. We prove by induction on n that there is a basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ such that $B_q(v_i, v_j) = 0$ except for $i = j$ and $i \leq r = \operatorname{rk} q$. If $q = 0$ there is nothing to prove. Otherwise there is a $v_1 \in \mathbb{C}^n$ such that $q(v_1) = B_q(v_1, v_1) = 1$. Define $V' := \{w \in \mathbb{C}^n \mid B_q(v_1, w) = 0\}$. Then $\mathbb{C}^n = \mathbb{C}v \oplus V'$. By induction, there is a basis $(v_2, \ldots, v_n)$ of V' with the required property, which proves the second part of the proposition. □

3.2. Closures of equivalence classes. We have seen above that there is an action of the general linear group $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{Sym}_n(\mathbb{C})$ given by $(q, A) \mapsto gAg^t$, and that the orbits of this action are precisely the equivalence classes. Considering a quadratic forms q as a function on $\mathbb{C}^n$ there is the following natural linear action of $\mathrm{GL}_n(\mathbb{C})$ on Q_n:

$$(g, q) \mapsto g \cdot q \text{ where } (g \cdot q)(x) := q(g^{-1}x).$$

An easy calculation shows that this action corresponds to the following linear action on the symmetric matrices: $(g, A) \mapsto g^{-t}Ag^{-1}$. Note that this is not an equivalent representation to the previous one given by $(g, A) \mapsto gAg^t$. But it has the same orbits, since it is obtained from the previous one by the outer automorphism $g \mapsto g^{-t}$ of $\mathrm{GL}_n(\mathbb{C})$.

Denote by $C_q \subseteq Q_n$ the equivalence class of the form q and by $C_A \subseteq \mathrm{Sym}_n$ the equivalence class of the symmetric matrix A. We want to study the closures $\overline{C_q}^{\mathbb{C}}$ in the $\mathbb{C}$-topology which turn out to be equal to the closures $\overline{C_q}$ in the ZARISKI topology, see Exercise 3.2.2. Since the equivalence classes C_q are the orbits of the (linear) action of $\mathrm{GL}_n(\mathbb{C})$ and since this action is continuous we see that the closure $\overline{C_q}^{\mathbb{C}}$ is stable under $\mathrm{GL}_n(\mathbb{C})$, hence a union of equivalence classes. In fact, we have the following result.

PROPOSITION 3.2.1. *For the equivalence classes C_q of quadratic forms we have*

$$\{0\} \subseteq \overline{C_{q_1}}^{\mathbb{C}} \subseteq \overline{C_{q_2}}^{\mathbb{C}} \subseteq \cdots \overline{C_{q_n}}^{\mathbb{C}} = Q_n.$$

PROOF. Since the rank function is lower semicontinuous (see Exercise 3.2.3 below) we know that the unions $\bigcup_{i \leq r} C_{q_i}$ are closed and so $\overline{C_{q_r}}^{\mathbb{C}} \subseteq \bigcup_{i \leq r} C_{q_i}$. Moreover, for all $\varepsilon \neq 0$ the form $x_1^2 + \cdots + x_{i-1}^2 + \varepsilon x_r^2$ is equivalent to q_r this implies that $q_{r-1} \in \overline{C_{q_r}}^{\mathbb{C}}$. Since the closure of an equivalence class is a union of equivalence classes the claim follows. □

This observation has the following consequence. *Every continuous invariant function $f\colon Q_n \to \mathbb{C}$ is constant.* In particular, continuous invariant functions cannot separate the orbits.

EXERCISE 3.2.2. Show that $\overline{C_q}^{\mathbb{C}}$ equals the closure $\overline{C_q}$ in the ZARISKI topology.

EXERCISE 3.2.3. The rank function $\operatorname{rk}\colon \mathrm{M}_n(\mathbb{C}) \to \mathbb{R}$ is lower semicontinuous. This means that for all $\alpha \in \mathbb{R}$ the set $\{A \in \mathrm{M}_n(\mathbb{C}) \mid \operatorname{rk} A \le \alpha\}$ is closed in $\mathrm{M}_n(\mathbb{C})$.

3.3. Other equivalence relations. We can study other equivalence relations on Q_n by looking at subgroups $G \subseteq \mathrm{GL}_n(\mathbb{C})$, e.g. the special linear group $\mathrm{SL}_n(\mathbb{C})$ or the orthogonal groups $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{SO}_n(\mathbb{C})$. In the case $G = \mathrm{SL}_n(\mathbb{C})$, the discriminant Δ is an invariant. In fact, we have the following result. (We denote by $\mathbb{C}[Q_n]$ the complex polynomial functions on Q_n.)

PROPOSITION 3.3.1. $\mathbb{C}[Q_n]^{\mathrm{SL}_n(\mathbb{C})} = \mathbb{C}[\Delta]$.

For $q \in Q_n$ we denote by O_q the orbit of q under $\mathrm{SL}_n(\mathbb{C})$.

PROPOSITION 3.3.2. *Consider the morphism $\Delta\colon Q_n \to \mathbb{C}$.*

(1) *For $\lambda \in \mathbb{C} \setminus \{0\}$ the fiber $\Delta^{-1}(\lambda)$ is a single SL_n-orbit, namely the orbit $O_{q^{(\lambda)}}$ of the form $q^{(\lambda)} := \lambda x_1^2 + x_2^2 + \cdots + x_n^2$.*
(2) *$\Delta^{-1}(0) = O_{q_{n-1}} \cup O_{q_{n-2}} \cup \cdots \cup O_{q_1} \cup \{0\}$.*
(3) *For $0 < i < n$ we have $\overline{O_{q_i}}^{\mathbb{C}} \supseteq O_{q_{i-1}}$. In particular, $\Delta^{-1}(0) = \overline{O_{q_{n-1}}}^{\mathbb{C}}$.*
(4) *Every $\mathbb{C}$-continuous invariant function f on Q_n is a $\mathbb{C}$-continuous function of the discriminant Δ.*

PROOF OF PROPOSITION 3.3.1 AND 3.3.2. For a quadratic form q of rank n there is a $g \in \mathrm{GL}_n(\mathbb{C})$ such that $q \cdot q = x_1^2 + \cdots + x_n^2$, by Proposition 3.1.1. Writing g in the form $g = \begin{bmatrix} \mu & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} h$ where $\mu = \det g$ and so $h \in \mathrm{SL}_n(\mathbb{C})$ we see that $h \cdot q = \lambda x_1^2 + \cdots + x_n^2$ where $\lambda = \Delta(q)$. This proves (1). The same argument shows that for $i < n$ the SL_n-orbit of q_i equals the GL_n-orbit which was denoted above by C_{q_i}. Thus (2) and (3) follow from Proposition 3.2.1.

The morphism $s\colon \mathbb{C} \to Q_n$ given by $s(\lambda) := q^{(\lambda)}$ is a section of Δ, i.e. $\Delta \circ s = \mathrm{Id}_{\mathbb{C}}$. Moreover, $U := \mathrm{SL}_n \cdot s(\mathbb{C}) \subseteq Q_n$ contains all orbits except $O_{q_{n-2}}, \ldots, O_{q_1}, \{0\}$. Hence, by (3), U is dense in Q_n. Now let $f\colon Q_n \to \mathbb{C}$ be an invariant polynomial function. Define $\bar{f}(\lambda) := f(s(\lambda))$. Then $\bar{f} \in \mathbb{C}[y]$ and $\bar{f} \circ \Delta$ equals f on $s(\mathbb{C})$. Since both functions are invariant they coincide on U, hence are equal: $f = \bar{f} \circ \Delta$. This proves Proposition 3.3.1, and a similar argument for a $\mathbb{C}$-continuous f gives (4). □

The result shows that invariant polynomial functions can describe equivalence classes “generically”, but for special values there might be several classes with the same polynomial invariants. And this cannot be avoided as long as we work with continuous functions. We will see many other examples below and later in the book.

REMARK 3.3.3. It is well-known that every real quadratic form is equivalent under $\mathrm{O}_n(\mathbb{R})$ to a “diagonal form” $a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$ where the coefficients are unique up to permutations. Equivalently, every real symmetric matrix can be diagonalized with an orthogonal matrix and the diagonal form is unique up to permutations. This does not hold if we consider complex quadratic forms with respect to the complex orthogonal group. In fact, the symmetric matrix $\left[\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right]$ is nilpotent. However, it is true “generically”: There is an open dense set $U \subseteq Q_n$ such that every form in U can be diagonalized with a complex orthogonal transformation. We will discuss this in detail in Chapter IV (see Example IV.3.4.3).

4. Conjugacy Classes of Matrices

4.1. Adjoint representation. Let $\mathrm{M}_n = \mathrm{M}_n(\mathbb{C})$ denote the vector space of complex $n \times n$-matrices. Two matrices $A, B \in \mathrm{M}_n$ are called *conjugate* or *similar* if there is a $g \in \mathrm{GL}_n(\mathbb{C})$ such that $B = gAg^{-1}$. Thus the equivalence relation is given by a group action $(g, A) \mapsto gAg^{-1}$ called the *adjoint representation*, whose orbits are the *conjugacy classes* $C_A := \{gAg^{-1} \mid g \in \mathrm{GL}_n(\mathbb{C})\}$.

Example 4.1.1. Every complex 2×2-matrix A is conjugate to one of the following:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

If A has two different eigenvalues $\lambda \neq \mu$, then its conjugacy class is determined by λ, μ. If A has a twofold eigenvalue λ then it is either conjugate to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ or equal to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$. This occurs if and only if $(\operatorname{tr} A)^2 - 4 \det A = 0$. Looking at the morphism

$$\pi \colon M_2 \to \mathbb{C}^2, \ A \mapsto (\operatorname{tr} A, \det A),$$

this implies that the fibers over $K := \mathcal{V}(x^2 - 4y)$ consist of two conjugacy classes whereas all other fibers form a single class.

Denote by

$$\chi_A(t) := \det(tE - A) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \cdots + (-1)^n a_n$$

the *characteristic polynomial* of A. Its coefficients $a_1, a_2, \ldots, a_n$ are the elementary symmetric functions in the eigenvalues of A. If we consider the matrix $X = (x_{ij})$ with indeterminate entries x_{ij}, then $\chi_X(t) := \det(tE - X)$ is a homogeneous polynomial in the variables t, x_{ij}:

$$\chi_X(t) := \det(tE - X) = t^n - S_1(X)t^{n-1} + S_2(X)t^{n-2} - \cdots + (-1)^n S_n(X).$$

It follows that the coefficients $S_k(X)$ are invariant polynomial functions on $\mathrm{M}_n(\mathbb{C})$, $S_k \in \mathcal{O}(\mathrm{M}_n)^{\mathrm{GL}_n(\mathbb{C})}$, and that they are homogeneous of degree k. In particular,

$$S_1(A) = \operatorname{tr} A \quad \text{and} \quad S_n(A) = \det A.$$

More generally, $S_i(D) = s_i(\lambda_1, \ldots, \lambda_n)$ (see section 2.2) in case D is an upper triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Now define the following morphism

$$\pi \colon \ \mathrm{M}_n \to \mathbb{C}^n, \ A \mapsto (S_1(A), S_2(A), \ldots, S_n(A)),$$

which associates to every matrix A the elementary symmetric functions of the eigenvalues of A.

Proposition 4.1.2. (1) *$\mathcal{O}(\mathrm{M}_n(\mathbb{C}))^{\mathrm{GL}_n(\mathbb{C})} = \mathbb{C}[S_1, S_2, \ldots, S_n]$, and the S_i are algebraically independent.*

(2) *Every $\mathbb{C}$-continuous invariant function f on M_n is a $\mathbb{C}$-continuous function in $S_1, \ldots, S_n$.*

Proof. For a given polynomial $p(t) = t^n + c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$ consider the *companion matrix* R defined by

$$R = R(c_1, c_2, \ldots, c_n) := \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_n \\ 1 & 0 & \cdots & 0 & -c_{n-1} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 0 & -c_2 \\ & & & 1 & -c_1 \end{bmatrix} \in \mathrm{M}_n(\mathbb{C}). \tag{2}$$

Its characteristic polynomial equals $p(t)$ (see Exercise 4.1.3 below) and so the morphism $s\colon \mathbb{C}^n \to \mathrm{M}_n$ given by $s(a_1,\dots,a_n) := R(-a_1,a_2,-a_3,\dots,(-1)^n a_n)$ is a section of π, i.e. $\pi\circ s = \mathrm{Id}_{\mathbb{C}^n}$. Moreover, every matrix A with n different eigenvalues is conjugate to a matrix $R(c_1,\dots,c_n)$, and so the set

$$\mathrm{GL}_n \cdot s(\mathbb{C}^n) = \{gRg^{-1} \mid g \in \mathrm{GL}_n, R \text{ a companion matrix}\} \subseteq M_n$$

contains all matrices with n different eigenvalues and is therefore dense in M_n (Exercise 4.1.4). Now the two claims follow as in the proof of Proposition 3.3.2. □

Exercise 4.1.3. Show that the characteristic polynomial of $R = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_n \\ 1 & 0 & \cdots & 0 & -c_{n-1} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 0 & -c_2 \\ & & & 1 & -c_1 \end{bmatrix}$ equals $t^n + c_1 t^{n-1} + c_2 t^{n-2} + \cdots + c_n$.
(Hint: Calculate $\det(tE - R)$ by expansion on the first column.)

Exercise 4.1.4. Show that in every $\mathbb{C}$-neighborhood of a matrix A there is a matrix with n different eigenvalues.
(Hint: This is clear for upper triangular matrices.)

4.2. The geometry of $\pi\colon \mathrm{M}_n \to \mathbb{C}^n$. Proposition 4.1.2 above shows that every invariant polynomial function $f\colon \mathrm{M}_n \to \mathbb{C}$ factors through π: There is a polynomial $\bar f \in \mathbb{C}[x_1,\dots,x_n]$ such that $f = \bar f \circ \pi$. Let us now study the geometry of the conjugacy classes and of the morphism $\pi\colon \mathrm{M}_n \to \mathbb{C}^n$. Recall that every matrix $A \in M_n$ has a well-defined *Jordan decomposition* $A = A_s + A_n$ where A_s is semisimple (i.e. diagonalizable), A_n is nilpotent and $A_s A_n = A_n A_s$.

Lemma 4.2.1. *Let $A = A_s + A_n$ be the Jordan decomposition of $A \in \mathrm{M}_n$*

(1) *A_s is in the closure of the conjugacy class of A: $A_s \in \overline{C_A}^{\mathbb{C}}$.*
(2) *The conjugacy class of A is closed if and only if A is semisimple.*

Proof. The first statement follows from the fact that $A_s + A_n$ is conjugate to $A_s + \varepsilon A_n$ for all $\varepsilon \neq 0$. This also implies that a closed conjugacy class must be semisimple.

In order to obtain the inverse implication we first remark that every fiber of π contains a unique semisimple conjugacy class. Secondly, one knows that a matrix A is semisimple if and only if its *minimal polynomial* μ_A has no multiple roots. Now let A be semisimple with minimal polynomial μ_A. Clearly, every matrix in C_A has the same minimal polynomial, and so $\mu_A(B) = 0$ for all $B \in \overline{C_A}^{\mathbb{C}}$. As a consequence, B is semisimple, too, and B is conjugate to A because $\pi(B) = \pi(A)$. □

The next proposition is an easy consequence from what we have said so far.

Proposition 4.2.2. *Consider the morphism $\pi\colon \mathrm{M}_n \to \mathbb{C}^n$ given by the coefficients of the characteristic polynomial.*

(1) *π is surjective.*
(2) *The fiber $\pi^{-1}(b)$ is a single conjugacy class $C = C_A$ if and only if $D(b) \neq 0$ where D is the discriminant. In this case A is semisimple with n different eigenvalues.*
(3) *Every fiber F of π contains a unique closed conjugacy class $C = C_H$ where H is semisimple. Moreover,*

$$F = \{A \in \mathrm{M}_n \mid A_s \text{ is conjugate to } H\}.$$

In particular, every fiber contains finitely many conjugacy classes.
(4) *$\pi^{-1}(0)$ equals the set $\mathcal{N}$ of nilpotent matrices.*

One can say much more, but the proofs of the following statements are also more involved and will not be given here.

PROPOSITION 4.2.3. *Every fiber of the morphism* $\pi\colon \mathrm{M}_n \to \mathbb{C}^n$ *is reduced, irreducible and normal of dimension* $n^2 - n$. *Moreover, the ideal* $I(\pi^{-1}(b))$ *of the fiber* $\pi^{-1}(b)$ *equals* $(S_1 - b_1, S_2 - b_2, \cdots, S_n - b_n)$, *and so the fibers are complete intersections (see AI.5.7 and Example AI.5.7.6)..*

4.3. Cyclic matrices. Our Proposition 4.2.2 above has a number of interesting interpretations. Since every fiber contains a single closed conjugacy class which is semisimple, we can say that *the morphism* $\pi\colon \mathrm{M}_n \to \mathbb{C}^n$ *parametrizes the semisimple conjugacy classes.*

On the other hand we have seen that $\pi(A) = \pi(B)$ if and only if A_s is conjugate to B_s which in turn is equivalent to $\overline{C_A} \cap \overline{C_B} \neq \emptyset$. So we can say that *the morphism* $\pi\colon \mathrm{M}_n \to \mathbb{C}^n$ *is the quotient with respect to the equivalence relation*

$$A \sim B \iff \overline{C_A} \cap \overline{C_B} \neq \emptyset.$$

For the last interpretation, we recall that a matrix A is called *cyclic* or *regular* if there is an element $v \in \mathbb{C}^n$ such that $\mathbb{C}^n$ is linearly spanned by the images $\{A^k v \mid k \in \mathbb{N}\}$. It follows that $(v, Av, A^2v, \ldots, A^{n-1}v)$ is a basis of $\mathbb{C}^n$, and so A is conjugate to a companion matrix (2). It is not difficult to see that the cyclic matrices are also characterized by the condition, that in its Jordan normal form there is only one Jordan block for every eigenvalue. From all this one can deduce the following result. The reader ist advised to work out the details, starting with M_2 and M_3 (see Exercise 4.3.2 below).

PROPOSITION 4.3.1. *Let* $A \in \mathrm{M}_n(\mathbb{C})$.

(1) *The map* $\pi\colon M_n \to \mathbb{C}^n$ *is of maximal rank in* A *if and only if* A *is cyclic.*
(2) *If* A *is cyclic, then* C_A *is dense in the fiber* $\pi^{-1}(\pi(A))$.
(3) A *is cyclic if and only if the centralizer* $Z_A := \{g \in \mathrm{GL}_n \mid gA = Ag\}$ *is a commutative algebraic group of dimension* n.
(4) *The cyclic matrices form an open dense set* M_n^{reg}, *and the induced map* $\pi\colon \mathrm{M}_n^{reg} \to \mathbb{C}^n$ *identifies* $\mathbb{C}^n$ *with the orbit space.*

EXERCISE 4.3.2. (1) Every cyclic matrix is conjugate to a companion matrix.
(2) The conjugacy class C_N of a cyclic nilpotent matrix N is dense in $\mathcal{N}$.
(3) Every fiber of π contains a dense conjugacy class which is the class of a cyclic matrix.
(4) The differential $d_A\pi$ of π in a cyclic matrix A is of maximal rank n.

4.4. The nilpotent cone. The set $\mathcal{N}_n \subseteq \mathrm{M}_n$ of nilpotent matrices equals the fiber $\pi^{-1}(0)$ which is defined by the vanishing of the coefficients of the characteristic polynomial, and also defined by the n^2 equations $X^n = 0$ where $X = (x_{ij})_{ij}$ is the $n \times n$-matrix with entries x_{ij}:

$$\mathcal{N}_n = \mathcal{V}(S_1, \ldots, S_n) = \mathcal{V}(X^n).$$

It follows that $\mathcal{N}$ is a closed cone, i.e. closed and stable under multiplication with $\lambda \in \mathbb{C}$. It consists of finitely many conjugacy classes which are represented by the Jordan normal forms. Using the sizes of the Jordan blocks we can be parametrized the nilpotent conjugacy classes by the *partitions of* n, i.e. by the set

$$\mathcal{P}_n := \{(p = (p_1, \ldots, p_n) \in \mathbb{N}^n \mid p_1 \geq p_2 \geq \cdots \geq p_n, \sum_i p_i = n\}.$$

Example 4.4.1. The partitions p of 4 are $(4),(3,1),(2,2),(2,1,1),(1,1,1,1)$ (we leave out the trailing zeroes), and the corresponding classes C_p are those of the matrices $N_{(p)}$ given by

$$\begin{bmatrix}0&1&&\\&0&1&\\&&0&1\\&&&0\end{bmatrix},\begin{bmatrix}0&1&&\\&0&1&\\&&0&\\&&&0\end{bmatrix},\begin{bmatrix}0&1&&\\&0&&\\&&0&1\\&&&0\end{bmatrix},\begin{bmatrix}0&1&&\\&0&&\\&&0&\\&&&0\end{bmatrix},\begin{bmatrix}0&&&\\&0&&\\&&0&\\&&&0\end{bmatrix}.$$

In general, $C_{(n)}$ is the conjugacy class of the regular nilpotent matrix $N_{(n)}$, and $C_{(1,\dots,1)}=\{0\}$. The closure of a nilpotent class $C_{(p)}$ is stable under conjugation, hence is a finite union of nilpotent classes:

$$\overline{C_p}=\bigcup_j C_{q_j}.$$

Exercise 4.4.2. For $n=4$ the inclusion order in $\mathcal{N}_n$ is given by

$$\{0\}=C_{(1,1,1,1)}\subseteq\overline{C_{(2,1,1)}}\subseteq\overline{C_{(2,2)}}\subseteq\overline{C_{(3,1)}}\subseteq\overline{C_{(4)}}.$$

There is a nice combinatorial description of the partitions q appearing in the closure of C_p (see [**Kra78**]). For this define the following partial order on $\mathcal{P}_n$:

$$q=(q_1,\dots,q_n)\prec p=(p_1,\dots,p_n)\iff q_1+\dots+q_k\le p_1+\dots+p_k \text{ for } k=1,\dots,n$$

Proposition 4.4.3. *We have $C_q\subseteq\overline{C_p}$ if and only if $q\prec p$.*

One implication is easy (see the exercises below), for the other we refer to the literatur mentioned above.

Exercise 4.4.4. Define the *dual partition* $\hat{p}$ of p by $\hat{p}_k:=\#\{j\mid p_j\ge k\}$. Then

(1) $\hat{p}_k=\dim\ker(N_p)^k$, and
(2) $q\prec p$ if and only if $\hat{q}\succ\hat{p}$.

Exercise 4.4.5. The function $\operatorname{rk}\colon \mathrm{M}_n\to\mathbb{Z}$, $A\mapsto\operatorname{rk}A$, is lower semicontinuous, i.e. for all $k\in\mathbb{N}$ the subset $\{A\in\mathrm{M}_n\mid \operatorname{rk}A\ge k\}$ is open.

Exercise 4.4.6. Use the previous two exercises to prove one implication of Proposition 4.4.3: If $C_q\subseteq\overline{C_p}$ then $q\prec p$.

Exercise 4.4.7. Work out the inclusions of the closures of the nilpotent classes in $\mathcal{N}_5$ and verify Proposition 4.4.3 for $n=5$.
(Hint: $C_{(3,2)}\subseteq\overline{C_{(4,1)}}$, but $C_{(2,2,1)}\not\subseteq\overline{C_{(3,1,1)}}$ and $C_{(3,1,1)}\not\subseteq\overline{C_{(2,2,1)}}$.)

5. Invariants of Several Vectors

5.1. Pairs of vectors. Let $V=\mathbb{C}^2$ be the two dimensional complex vector space with the usual operation of $\mathrm{GL}_2(\mathbb{C})$ given by

$$gv=\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}:=\begin{bmatrix}\alpha x+\beta y\\\gamma x+\delta y\end{bmatrix}\quad\text{for } g=\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}\in\mathrm{GL}_2(\mathbb{C}),\ v=\begin{bmatrix}x\\y\end{bmatrix}\in V.$$

We consider now *pairs of vectors* from V and define on $V\times V$ the following *equivalence relation*:

$$(v_1,v_2)\sim(w_1,w_2)\quad\iff\quad\text{there is a } g\in\mathrm{SL}_2(\mathbb{C}) \text{ with } gv_i=w_i \text{ for } i=1,2.$$

Clearly, the map

$$\pi=[\ ,\ ]\colon V\times V\to\mathbb{C},\quad (v_1,v_2)\mapsto[v_1,v_2]:=\det\begin{bmatrix}x_1&x_2\\y_1&y_2\end{bmatrix},\ \text{where } v_i=\begin{bmatrix}x_i\\y_i\end{bmatrix},$$

is constant on the equivalence classes. Expressed in terms of coordinates we have

$$\pi=x_1y_2-x_2y_1.$$

One can easily give the *normal forms* for each equivalence class:

(a) If $\lambda := [v_1, v_2] \neq 0$, then $(v_1, v_2) \sim ([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\\lambda\end{smallmatrix}])$ holds. (This follows from the fact that GL_2 operates *transitively* on pairs of linearly independent vectors.)

(b) The fiber $\pi^{-1}(0)$ consists of infinitely many equivalence classes of pairs of linearly dependent vectors. As representatives we could take, for example, $(0,0), (e_1, 0)$ and $(\lambda e_1, e_1)$ with $\lambda \in \mathbb{C}$. (One uses the fact that $\mathrm{SL}_2(\mathbb{C})$ operates transitively on the non-zero vectors.)

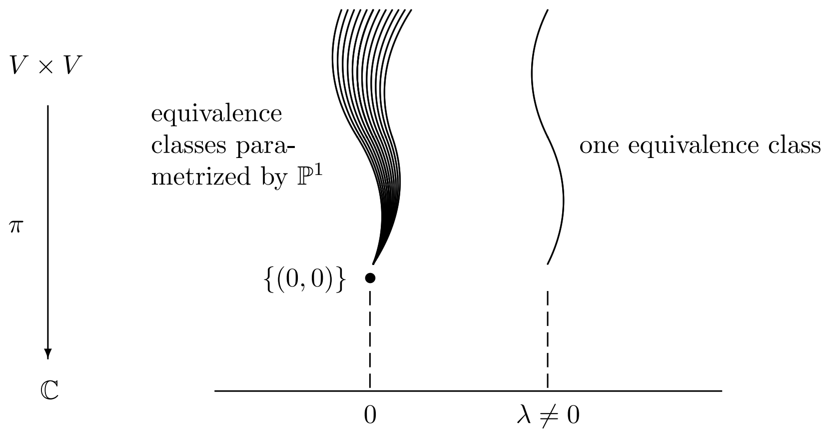


FIGURE 1. The quotient map for pairs of vectors

5.2. The null fiber. Now we would like to look a little closer at the *null fiber* $\mathcal{N} := \pi^{-1}(0)$ as a geometric object. We denote by $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ the *complex projective line*:

$$\mathbb{P}^1 := \{(a,b) \in \mathbb{C}^2 \mid (a,b) \neq (0,0)\}/\sim$$

where $(a,b) \sim (a',b')$ iff the two vectors are linearly dependent. The equivalence class of (a,b) is denoted by $(a:b)$.

PROPOSITION 5.2.1. *There exists a surjective map $\rho\colon \mathcal{N} \setminus \{(0,0)\} \to \mathbb{P}^1$ whose fibers are exactly the equivalence classes.*

PROOF. Set

$$\rho\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) := \begin{cases} (x_1 : x_2) & \text{if } (x_1, x_2) \neq (0,0), \\ (y_1 : y_2) & \text{if } (y_1, y_2) \neq (0,0). \end{cases}$$

Since the two vectors are linearly dependent the map ρ is well-defined and has the desired properties. □

In this way the equivalence classes in $\mathcal{N} \setminus \{(0,0)\}$ can be *parametrized* by the complex projective line $\mathbb{P}^1$, via ρ, see Figure 1 above.

This parametrization can also be explained by the following description of the null fiber $\pi^{-1}(0)$ as a vector bundle over $\mathbb{P}^1$.

PROPOSITION 5.2.2. *There is a vector bundle B of rank two over $\mathbb{P}^1$ and a surjective map $\varphi\colon B \to \mathcal{N}$ such that the following holds:*

(1) *$S_0 := \varphi^{-1}((0,0))$ is the zero section of B, $S_0 \simeq \mathbb{P}^1$;*
(2) *The map φ induces an isomorphism $B \setminus S_0 \xrightarrow{\sim} \mathcal{N} \setminus \{(0,0)\}$;*
(3) *Every fiber of B is isomorphically mapped onto the closure of an equivalence class.*

Proof. We consider the open covering $U_0 \cup U_\infty$ of $\mathbb{P}^1$ defined by $U_0 := \{(\lambda : \mu) \in \mathbb{P}^1 \mid \lambda \neq 0\}$ and $U_\infty := \{(\lambda : \mu) \in \mathbb{P}^1 \mid \mu \neq 0\}$, along with the trivial vector bundles $V \times U_0$ and $V \times U_\infty$. The bundle B is obtained from the following diagram

$$\begin{tikzcd} V \times U_0 \arrow[r, "\subseteq"] \arrow[d] & B \arrow[d] & V \times U_\infty \arrow[l, "\supseteq"'] \arrow[d] \\ U_0 \arrow[r, "\subseteq"] & \mathbb{P}^1 & U_\infty \arrow[l, "\supseteq"'] \end{tikzcd}$$

by "glueing together" the trivial bundles over $U_0 \cap U_\infty$, using the identification over a point $(\lambda : \mu) \in U_0 \cap U_\infty$:

$$(v, (\lambda : \mu))_0 = (w, (\lambda : \mu))_\infty \iff \mu v = \lambda w.$$

Now we define $\varphi \colon B \to \mathcal{N}$ by

$$\varphi((v, (\lambda : \mu))_0) := (v, \frac{\mu}{\lambda} v) \quad \text{and} \quad \varphi((w, (\lambda : \mu))_\infty) := (\frac{\lambda}{\mu} w, w).$$

It is easy to see that φ is well-defined, i.e. that it is compatible with the above identifications, and also that it has the properties which we want. $\square$

Remark 5.2.3. The assertion shows that one gets the null fiber $\mathcal{N}$ from the vector bundle B by *"blowing down" the zero section to a point.* Conversely the bundle is obtained from the null fiber by *"blowing up"* the point $(0, 0)$ to a $\mathbb{P}^1$.

Exercise 5.2.4. Give a direct proof of the third statement of Proposition 5.2.2 that the closure of a non-trivial equivalence class in the null fiber $\mathcal{N}$ is isomorphic to $\mathbb{C}^2$.

5.3. Vector bundles over $\mathbb{P}^1$. The vector bundles on the projective line $\mathbb{P}^1$ are well-known. For each integer $s \in \mathbb{Z}$ there is a line bundle $\mathcal{O}(s)$ and every vector bundle is isomorphic to a direct sum of such line bundles. For the description of such an $\mathcal{O}(s)$ we use, as before, the open covering $\mathbb{P}^1 = U_0 \cup U_\infty$ and consider the trivial bundles $\mathbb{C} \times U_0$ and $\mathbb{C} \times U_\infty$.

$$\begin{tikzcd} \mathbb{C} \times U_0 \arrow[r, "\subseteq"] \arrow[d, "\mathrm{pr}"] & \mathcal{O}(s) \arrow[d, "p"] & \mathbb{C} \times U_\infty \arrow[l, "\supseteq"'] \arrow[d, "\mathrm{pr}"] \\ U_0 \arrow[r, "\subseteq"] & \mathbb{P}^1 & U_\infty \arrow[l, "\supseteq"'] \end{tikzcd}$$

The line bundle $\mathcal{O}(s)$ is now obtained by glueing these two trivial bundles together over $U_0 \cap U_\infty$ using the following identification:

$$(t, (\lambda : \mu))_0 = (u, (\lambda : \mu))_\infty \iff \lambda^s t = \mu^s u.$$

If we compare this with the construction of B above, then we get the following corollary to Proposition 5.2.2.

Corollary 5.3.1. *The vector bundle B from Proposition 5.2.2 above is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

Remark 5.3.2. The bundle $\mathcal{O}(-1)$ is the so-called Hopf-*bundle*. Over the real numbers $\mathbb{R}$ we can visualize this geometrically in the following way. We interpret the real projective line $\mathbb{P}^1(\mathbb{R})$ as the unit circle in the real plane with opposite (antipodal) points identified. If one now takes a copy of $\mathbb{R}$ as the fiber over every point of the unit circle, then one has to identify the fibers which lie over opposite points. Since the coordinates of opposite points differ by a factor of -1, one must glue these fibers together "with a twist". If one applies this process to a half-circle with two end points, then by identifying the two fibers over the two ends one gets a Möbius *band*.

EXERCISE 5.3.3. (1) Show that the line bundle $\mathcal{O}(s)$ can be described in the following way. Let $\eta\colon \mathbb{C}^2\setminus\{0\} \to \mathbb{P}^1$ be the canonical map. Define on $(\mathbb{C}^2\setminus\{0\})\times\mathbb{C}$ the following equivalence relation:

$$((x,y),t) \sim ((x',y'),t') \iff \begin{array}{l}\text{There exists a } \lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\} \text{ with}\\ x = \lambda x',\ y = \lambda y' \text{ and } t = \lambda^s t'\end{array}$$

Then one can identify the set of equivalence classes with $\mathcal{O}(s)$, and the projection onto the first factor $\mathrm{pr}\colon (\mathbb{C}^2\setminus\{0\})\times\mathbb{C} \to \mathbb{C}^2\setminus\{0\}$ induces the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^2\setminus\{0\}\times\mathbb{C} & \xrightarrow{\mathrm{pr}} & \mathcal{O}(s) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow \\ \mathbb{C}^2\setminus\{0\} & \xrightarrow{\eta} & \mathbb{P}^1 \end{array}$$

(2) Give another proof of Proposition 5.2.2 and the corollary by using a similar construction of the bundle B as in the previous exercise for the line bundles $\mathcal{O}(s)$.

(3) A *section* $\sigma\colon \mathbb{P}^1 \to \mathcal{O}(s)$ (i.e. one has $p\circ\sigma = \mathrm{id}_{\mathbb{P}^1}$) induces a map $\overline{\sigma}\colon \mathbb{C}^2\setminus\{0\} \to (\mathbb{C}^2\setminus\{0\})\times\mathbb{C}$ of the form $(x,y)\mapsto((x,y),f(x,y))$. It follows that $((x,y),f(x,y)) \sim ((\lambda x,\lambda y), f(\lambda x,\lambda y))$ for every $\lambda\in\mathbb{C}^*$, i.e. $f(\lambda x,\lambda y) = \lambda^s\cdot f(x,y)$. Nonzero polynomials F with this property only exist for $s\geq 0$, and these then are exactly the homogeneous polynomials $f(x,y)$ of degree s.

The line bundles $\mathcal{O}(n)$ with n negative are thus distinguished by the fact that they do not have any polynomial sections except for the zero section.

5.4. Invariants of several vectors. Instead of pairs we could naturally look at triples or arbitrary n-tuples of vectors in V, considered with the corresponding equivalence relation. Finding a *complete system of invariants* was one of the classical problems of invariant theory. One means by this a system of invariant polynomials $f_1, f_2, \ldots, f_N$ on $V\times V\times\cdots\times V$ (i.e. they are constant on the equivalence classes) with the property that every invariant polynomial can be expressed as a polynomial function in the $f_i's$.

Such a complete system is given, for example, by the functions f_{ij}, $1\leq i<j\leq n$, defined by

$$f_{ij}(v_1,\ldots,v_n) := [v_i,v_j].$$

For a proof of this classical result we refer to the literature, e.g. [**Wey39**], [**Vus76**], [**dCP76**]. We will discuss this in more details in section 6.2.2.

In the case of triples of vectors one is led to study the following map:

$$\pi\colon V\times V\times V\to\mathbb{C}^3,\ (v_1,v_2,v_3)\mapsto([v_1,v_2],[v_1,v_3],[v_2,v_3]).$$

One can easily show that π is surjective and that, with the exception of the null fiber $\mathcal{N} := \pi^{-1}(0)$, every fiber of π is an equivalence class. The null fiber itself is made up of those triples (v_1,v_2,v_3) which span a vector space of dimension ≤ 1.

EXERCISE 5.4.1. Give a proof of these statements.

The equivalence classes in $\mathcal{N}\setminus\{0\}$ can be parametrized by the complex projective plane $\mathbb{P}^2$. Also in this case there is a vector bundle B of rank two over $\mathbb{P}^2$ and a surjective map $\varphi\colon B\to\mathcal{N}$ which maps the zero section S_0 of B onto the origin, and induces an isomorphism $B\setminus S_0 \xrightarrow{\sim} \mathcal{N}\setminus\{0\}$. and which maps every fiber of B isomorphically onto the closure of an equivalence class in $\mathcal{N}$.

$$\begin{array}{ccccc} B & \xrightarrow{\varphi} & \mathcal{N} & \xrightarrow{\subseteq} & V\times V\times V \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^2 & & \{0\} & \xrightarrow{\subseteq} & \mathbb{C}^3 \end{array}$$

Similar to what we saw above one has $B \xrightarrow{\sim} \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ where the line bundles $\mathcal{O}(s)$ on $\mathbb{P}^2$ are defined in the analogous way to what was done at the beginning of section 5.3.

It is recommended that the reader looks carefully at this example and also its generalization to arbitrary n-tuples.

6. Nullforms

Our approach in the last three sections can be described in the following comprehensive way. Given is a complex vector space V and a linear action of a group G on V. We are interested in the *orbits* of the group G in V. In the examples these are the *equivalence classes*. For a description of them we could give a continuous map $\pi\colon V \to \mathbb{C}^r$ which is constant on the orbits and has the property that for almost all $z \in \pi(V) \subseteq \mathbb{C}^r$ the fiber $\pi^{-1}(z)$ is *exactly one orbit*. The fibers over some special points, in particular over the origin, might form a somewhat complicated picture and require particular consideration. In some cases they can be regarded as *"degenerations"* of the general fiber (see e.g. section 3).

Particularly interesting are those orbits whose closures contain zero. For continuity reasons these lie in the *zero fiber* $\pi^{-1}(\pi(0))$, classically called *null fiber*. In all the examples we have considered so far the converse is also true. Namely, the orbits in the zero fiber contain zero in their closures. The exact connection between these will be made clear later (see section IV.2.6).

6.1. Binary forms. Now we would like to study the orbits a little closer in the case of *binary forms*. Using the terminology of Hilbert one calls the forms which arise in this way *nullforms*. We denote by

$$R_n := \left\{ \sum_{i=0}^{n} a_i x^{n-i} y^i \mid a_i \in \mathbb{C} \right\}$$

the space of *binary forms of degree* n, i.e., the $(n+1)$-dimensional complex vector space of homogeneous polynomials of degree n in the two indeterminates x and y. For $f_1, f_2 \in R_n$ an equivalence relation is defined on R_n by

$$f \sim f' \iff \text{there is a } g \in \mathrm{SL}_2(\mathbb{C}),\ g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},\ \text{such that}$$
$$f_2(x,y) = f_1^g(x,y) := f(\alpha x + \beta y, \gamma x + \delta y).$$

We can also express this in another way. The group $G = \mathrm{SL}_2(\mathbb{C})$ *acts* on R_n by "variable substitution"

$$f \mapsto g.f := f^{g^{-1}}, \ \text{i.e.}\ \ g.f(x,y) = f(\delta x - \beta y, -\gamma x + \alpha y)$$

This means that one has $e.f = f$ for the identity matrix $e \in G$ and $(gh).f = g.(h.f)$ for every $g, h \in G$. In particular, we have

$$g.x = \delta x - \beta y \quad \text{and} \quad g.y = -\gamma x + \alpha y$$

For fixed $f \in R_n$ the equivalence class C_f is equal to the G-orbit $G.f$ under this action:

$$C_f := \{f^g \mid g \in G\} = G.f := \{g.f \mid g \in G\}.$$

For $t \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $d = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$ we simply write f_t instead of $t.f$, i.e. $f_t(x,y) = f(t^{-1}x, ty)$.

One gets a rough description of the orbits in R_n from the following lemma.

Lemma 6.1.1. (1) *Every binary form $f \in R_n$ is a product of linear forms.*

(2) *A nonzero form f is equivalent to one of the form $x^r y^s f'$ with $r \geq s \geq 0$ where $f' \in R_{n-r-s}$ is a form which has no linear factor x or y or of multiplicity greater than s.*

PROOF. The assertion (1) follows from the Fundamental Theorem of Algebra. The assertion (2) follows from (1) and the fact that $\mathrm{GL}_2(\mathbb{C})$ operates transitively on pairs of linearly independent linear forms. □

DEFINITION 6.1.2. We call $f \in R_n$ a *nullform* if the origin $0 \in R_n$ lies in the closure of the orbit of f: $0 \in \overline{G.f}$. Denote by $\mathcal{N}_n$ the *set of nullforms in* R_n, classically called the *null fiber* or the *null cone*.

EXAMPLE 6.1.3. The form $f = y^n$ is a nullform. This follows from $f \sim f_t = t^n y^n$ by taking the limit as $t \to 0$. More generally, let $f = x^i y^{n-i}$. Then $f_t = t^{n-2i} x^i y^{n-i}$, and so $x^i y^{n-i}$ is a nullform for $2i < n$.

EXERCISE 6.1.4. If $f \in R_n$ contains a linear factor of multiplicity $> \frac{n}{2}$, then f is a nullform.
(Hint: f is equivalent to $\tilde{f} = y^j f'$ where $2j > n$. It follows that $\tilde{f}_t = t^{2j-n} y^j f''$, and the claim follows by taking the limit $t \to 0$.)

The following Criterion of HILBERT is the central result in the study of nullforms. For a proof we refer the reader to the fifth Chapter, see V.3.

PROPOSITION 6.1.5 (HILBERT's Criterion). *A form $f \in R_n$ is a nullform if and only if there exists a form $\tilde{f} \sim f$ with the property that $\lim_{t\to 0} \tilde{f}_t$ exists and equals 0.*

The criterion asserts that every nullform has a representative in the vector space

$$R_n^+ := \{f \in R_n \mid \lim_{t\to 0} f_t = 0\}.$$

Obviously, for $m = \lfloor \frac{n-1}{2} \rfloor$ one has

$$R_n^+ = \{\sum_{i=0}^m a_i x^i y^{n-i} \mid a_i \in \mathbb{C}\} = \{f \in R_n \mid y^{n-m} \text{ divides } f\}.$$

Since the operation of G does not change the multiplicity of linear forms in f, we get the following result.

PROPOSITION 6.1.6. *An element $f \in R_n$ is a nullform if and only if f contains a linear factor of multiplicity $m > \frac{n}{2}$ or $f = 0$.*

EXAMPLES 6.1.7. (1) For $n = 1$ every form is a nullform. In total there are two equivalence classes in R_1, $C_0 = \{0\}$ and $C_x = R_1 \setminus \{0\}$.
(2) For $n = 2$ we find two equivalence classes of nullforms, C_{y^2} and C_0.
(3) For $n = 3$ and 4 there are three equivalence classes of nullforms, C_{xy^2}, C_{y^2}, C_0, and C_{xy^3}, C_{y^4}, C_0.
(4) For $n \geq 5$ there are always infinitely many equivalence classes of nullforms, with $\lfloor \frac{n-3}{2} \rfloor$ parameters.

If one now notes that the form y^n lies in the closure of every orbit of a nullform which is not zero, then one gets the diagrams of orbits in the null cone $\mathcal{N}_n$ given in Figure 2.

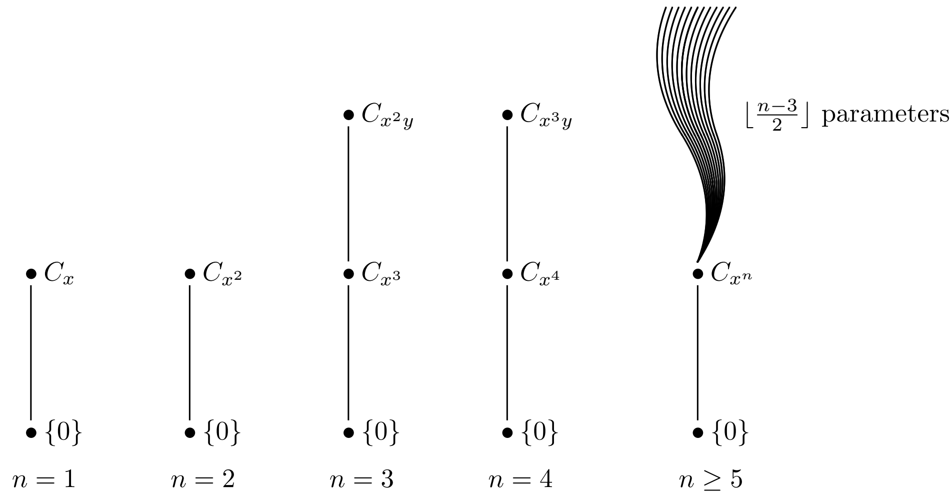


FIGURE 2. The null cone of the binary forms R_n

6.2. The null cone of R_5. In conclusion we would like to describe somewhat more precisely the nullforms $\mathcal{N}_5$ in R_5. We leave it to the reader to show how one can generalize this to forms of arbitrary degree. As above, we let $R_5^+ = \mathbb{C}x^2y^3 \oplus \mathbb{C}xy^4 \oplus \mathbb{C}y^5 \subseteq R_5$. Then there is a surjective map

$$\tilde{\rho}\colon R_5^+ \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{N}_5 \quad \text{given by} \quad (f,g) \mapsto g.f.$$

Let B be the group of *upper triangular matrices* in $\mathrm{SL}_2(\mathbb{C})$:

$$B := \left\{ \begin{bmatrix} t & \beta \\ & t^{-1} \end{bmatrix} \mid t \in \mathbb{C}^*, \beta \in \mathbb{C} \right\}.$$

LEMMA 6.2.1. (a) *One has $R_5^+ \subseteq \mathcal{N}_5$, and every nullform is equivalent to a form in R_5^+.*
(b) *The subspace R_5^+ is B-stable, i.e., for every $b \in B$ and $f \in R_5^+$ one has $b.f \in R_5^+$.*
(c) *If $f \in R_5^+$ with $f \neq 0$ and $g.f \in R_5^+$ for some $g \in \mathrm{SL}_2(\mathbb{C})$, then $g \in B$.*

PROOF. The claims in (a) have already been verified. Next we note that $R_5^+ = \{f \in R_5 \mid y^3 \text{ divides } f\}$. Since y is mapped by elements in B to multiples of y, the claim (b) follows. If f and $g.f$ lie in R_5^+, then $g.f$ has not only y, but also $g.y$ as a factor of at least third order. But then $g.y$ is a multiple of y and, as a consequence, $g \in B$ which proves (c). □

We let B act on $R_5^+ \times \mathrm{SL}_2(\mathbb{C})$ by

$$b(f,g) := (b.f, gb^{-1}),$$

and denote the set of B-orbits by $R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})$. Clearly, the above map $\tilde{\rho}\colon R_5^+ \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{N}_5$ factors through the canonical map $q\colon R_5^+ \times \mathrm{SL}_2(\mathbb{C}) \to R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})$ which assigns to each element its B-orbit. We denote the space of left cosets by $\mathrm{SL}_2(\mathbb{C})/B$, i.e.

$$\mathrm{SL}_2(\mathbb{C})/B := \{gB \mid g \in \mathrm{SL}_2(\mathbb{C})\}.$$

Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbb{C}) \arrow[d] & R_5^+ \times \mathrm{SL}_2(\mathbb{C}) \arrow[l, "\mathrm{pr}"'] \arrow[r, "\tilde{\rho}"] \arrow[d, "q"] & \mathcal{N}_5 \arrow[d, equal] \\
\mathrm{SL}_2(\mathbb{C})/B & R_5^+ \times^B \mathrm{SL}_2(\mathbb{C}) \arrow[l, "p"'] \arrow[r, "\rho"] & \mathcal{N}_5
\end{tikzcd}$$

where the map p is induced by the projection $\mathrm{pr}\colon R_5^+ \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$.

PROPOSITION 6.2.2. (a) $\mathrm{SL}_2(\mathbb{C})/B \simeq \mathbb{P}^1$.
(b) $p\colon R_5^+ \times^B \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})/B$ *is a vector bundle over* $\mathrm{SL}_2(\mathbb{C})/B \simeq \mathbb{P}^1$ *with typical fiber* R_5^+.

PROOF. (a) The isomorphism is induced by the map

$$\eta\colon \mathrm{SL}_2(\mathbb{C}) \to \mathbb{P}^1, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto (\alpha : \gamma)$$

which is constant exactly on the left cosets gB.

(b) The subset $R_5^+ \times \mathrm{SL}_2(\mathbb{C}) \subseteq R_5 \times \mathrm{SL}_2(\mathbb{C})$ is a B-stable where the action of B on $R_5 \times \mathrm{SL}_2(\mathbb{C})$ is defined analogously. It follows that $R_5^+ \times^B \mathrm{SL}_2(\mathbb{C}) \subseteq R_5 \times^B \mathrm{SL}_2(\mathbb{C})$. The isomorphism $R_5 \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sim} R_5 \times \mathrm{SL}_2(\mathbb{C})$ given by $(f,g) \mapsto (g.f, g)$ shows that $R_5 \times^B \mathrm{SL}_2(\mathbb{C})$ is isomorphic to $R_5 \times (\mathrm{SL}_2(\mathbb{C})/B)$. Thus $R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})$ is a vector subbundle of the trivial vector bundle $R_5 \times^B \mathrm{SL}_2(\mathbb{C})/B$ over $\mathrm{SL}_2(\mathbb{C})/B$. $\square$

REMARK 6.2.3. Clearly, the proof of (b) does not depend on the special situation in the above setting. If V is a vector space with a linear action of the group G, $H \subseteq G$ a subgroup and $W \subseteq V$ an H-stable subspace, then $W \times^H G$ is a vector bundle over G/H, namely a vector subbundle of the trivial vector bundle $V \times^H H \simeq V \times (G/H)$.

It follows that the *H-orbits in W correspond in a unique fashion to the G-orbits in* $W \times^H G$. If $O' \subseteq W$ is an H-orbit, then $O' \times^H G \subseteq W \times^H G$ is a G-orbit, and every G-orbit in $W \times^H G$ is of this form.

Altogether this gives us the following result (cf. section 5.4)

PROPOSITION 6.2.4. (a) *The space* $R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})$ *is a vector bundle over* $\mathrm{SL}_2(\mathbb{C})/B \simeq \mathbb{P}^1$ *with typical fiber* W.
(b) *The set* $\rho^{-1}(0)$ *is the zero section* S_0 *of the vector bundle* $R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})$, *and* ρ *induces a bijection* $(R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})) \setminus S_0 \xrightarrow{\sim} \mathcal{N}_5 \setminus \{0\}$.

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbb{C})/B \arrow[d, equal] & R_5^+ \times^B \mathrm{SL}_2(\mathbb{C}) \arrow[l, "p"'] \arrow[r] & \mathcal{N}_5 \\
\mathrm{SL}_2(\mathbb{C})/B & R_5^+ \times^B \mathrm{SL}_2(\mathbb{C}) \setminus S_0 \arrow[l, "p"'] \arrow[r, "\simeq"] \arrow[u, "\subseteq"'] & \mathcal{N}_5 \setminus \{0\} \arrow[u, "\subseteq"']
\end{tikzcd}$$

(The first assertion has already been noted and, as the proofs of the others should create no difficulties, they are left to the reader.)

6.3. A geometric picture of $\mathcal{N}_5$. Among other things, the proposition says that the $\mathrm{SL}_2(\mathbb{C})$-orbits in $\mathcal{N}_5$ and in $R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})$ can be put into one-to-one correspondence. Moreover, the latter correspond for their part to the B-orbits in R_5^+, cf. the above remark. We would now like to make this geometrically clear. In order to do this let u, v, w be the coordinate functions on R_5^+ relative to the basis $\{x^2y^3, xy^4, y^5\}$. For $f \in R_5^+$ one has $B.f = \{b.f \mid b \in B\} = G.f \cap R_5^+$, see Lemma 6.2.1(c). For $b := \begin{bmatrix} t & 0 \\ c & t^{-1} \end{bmatrix}$ one has

$$b.x = t^{-1}x - \beta y \quad \text{and} \quad b.y = ty.$$

From this one easily gets the following description of the B-orbits in R_5^+:

(a) $B.y^5 = \mathbb{C}y^5 - \{0\} =$ *the w-axis minus the origin.*
(b) $B.xy^4 = \{txy^4 + by^5 \mid t \in \mathbb{C}^*, b \in \mathbb{C}\} =$ *the vw-plane minus the w-axis* $\mathbb{C}y^5$.

(c) $B.x^2y^3 = \{tx^2y^3 + 2t^2bxy^4 + t^3b^2y^5 \mid t \in \mathbb{C}^*, b \in \mathbb{C}\}$ = *the cone with the equation* $4uw - v^2 = 0$ *minus the* w*-axis* $\mathbb{C}y^5$.

(d) $B.xy^3(x+qy)$ = *the surface with the equation* $4uw - v^2 + q^2u^6 = 0$ *minus the* w*-axis* $\mathbb{C}y^5$.

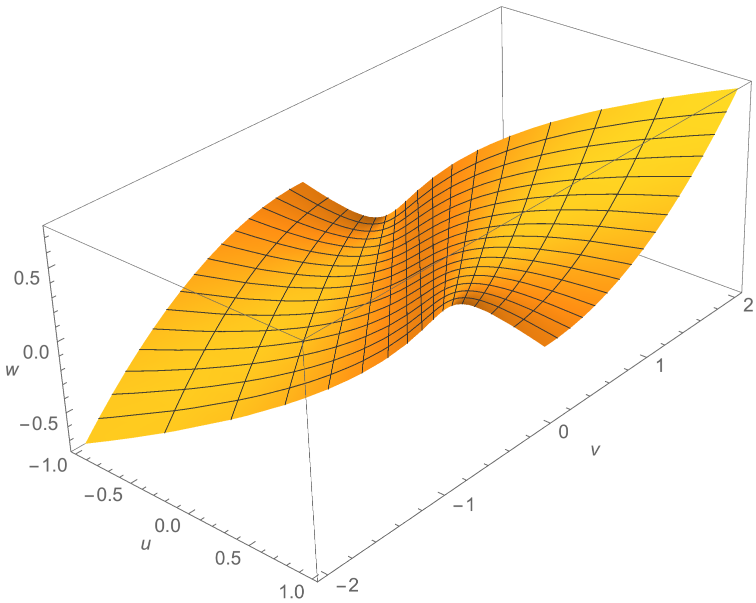


FIGURE 3. The surface $4uw - v^2 + q^2u^6 = 0$

As might be expected, (b) and (c) are "limiting cases" of the family (d), namely for $q \to \infty$ resp. $q \to 0$. All the orbits which occur in (b), (c) and (d) contain the w-axis in their closures. This corresponds to the fact which was noted above that the form y^5 lies in the closure of the orbit of every non-zero nullform.

For the description of the $\mathrm{SL}_2(\mathbb{C})$-orbits $O_f := G.f$ we consider the subspaces $W' := \mathbb{C}xy^4 \oplus \mathbb{C}y^5$ and $W'' := \mathbb{C}y^5$ of R_5^+, and the corresponding subbundles $\mathcal{W}' := W' \times^B \mathrm{SL}_2(\mathbb{C})$ and $\mathcal{W}'' = W'' \times^B \mathrm{SL}_2(\mathbb{C})$ of $\mathcal{W} := R_5^+ \times^B \mathrm{SL}_2(\mathbb{C})$. Then ρ induces *bijections*

$$\mathcal{W} \setminus \mathcal{W}' \xrightarrow{\sim} \bigcup_{q\in\mathbb{C}} O_{xy^3(x+qy)}, \quad \mathcal{W}' \setminus \mathcal{W}'' \xrightarrow{\sim} O_{xy^4}$$
$$\mathcal{W}'' \setminus S_0 \xrightarrow{\sim} O_{y^5}.$$

Corresponding to this we have the following *decompositions into line bundles* (see section 4)

$$\mathcal{W} \simeq \mathcal{O}(-3)^3, \quad \mathcal{W}' \simeq \mathcal{O}(-4)^2, \quad \mathcal{W}'' \simeq \mathcal{O}(-5).$$

One can see this in the following way. We consider the isomorphism $\mathbb{C}y^3 \otimes_{\mathbb{C}} R_2 \to R_5^+$ given by multiplication, $(by^3, a_1x^2 + a_2xy + a_3y^2) \mapsto ba_1x^2y^3 + ba_2xy^4 + ba_3y^5$. Now one notes that the line bundle $\mathbb{C}X^3 \times^B \mathrm{SL}_2(\mathbb{C})$ is isomorphic to $\mathcal{O}(-3)$. Similar to before one sees that $R_2 \times^B \mathrm{SL}_2(\mathbb{C})$ is the trivial bundle $\mathcal{O}^3$ (cf. the proof of b) in the Proposition). From the above isomorphism $\mathbb{C}y^3 \otimes_{\mathbb{C}} R_2 \to W$ one now gets

$$W \times^B \mathrm{SL}_2(\mathbb{C}) \simeq \mathcal{O}(-3) \otimes_{\mathbb{C}} \mathcal{O}^3 = \mathcal{O}(-3)^3.$$

The other cases can be shown in a similar way.

REMARK 6.3.1. The method we gave of describing nullforms using suitable vector bundles was further developed and refined by WIM HESSELINK [**Hes79**]. Here the essential tool is KEMPF's theory of *optimal one-parameter subgroups* ([**Kem78**], cf. 5.5.2).

7. Deformations and Associated Cone

7.1. The associated cone. We consider again the general situation of a complex vector space V on which an algebraic group G operates linearly and rationally, i.e., a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$ is given so that the matrix coefficients of $\rho(g)$ with respect to any (and hence every) basis of V are regular functions on G. Precise definitions are given in the next chapter. Instead of $\rho(g)(v)$ we simply write gv. For $v \in V$ we denote by $O_v := \{gv \mid g \in G\}$ the *orbit* of v under G.

As a generalization of the notion of nullforms from the last section we introduce the following notion.

DEFINITION 7.1.1. A vector $v \in V$ and its orbit O_v, as well, are called *unstable* if zero lies in the closure $\overline{O_v}$ of the orbit of v. Otherwise v and O_v are called *semistable*. We use $\mathcal{N}_V$ to denote the *set of all unstable vectors in* V

If O_v is a semistable orbit in V and $\lambda \in \mathbb{C}^*$, then $\lambda O_v = \{\lambda \cdot (gv) \mid g \in G\}$ is also a semistable orbit. In fact, $\lambda O_v = O_{\lambda v}$ since the action of G is linear, and multiplication with λ is a homeomorphism, hence $\overline{\lambda O_v} = \lambda \overline{O_v} \not\ni 0$.

For an arbitrary orbit $O \subseteq V$ let

$$\mathbb{C}^*\bar{O} := \{\lambda v \mid \lambda \in \mathbb{C}^*, v \in \bar{O}\} = \bigcup_{v\in\bar{O}} \mathbb{C}^* v$$

be the cone spanned by $\bar{O}$. A subset of a vector space is called *homogeneous* or a *cone* if it contains with every v the subset $\mathbb{C}^* v$.

DEFINITION 7.1.2. Let $v \in V$ be semistable. The set of boundary points of the cone $\mathbb{C}^*\overline{O_v}$ is called the *cone associated to* O_v (or associated to $\overline{O_v}$) and will be denoted by $\mathcal{C}O_v$:

$$\mathcal{C}O_v := \partial(\mathbb{C}^*\overline{O_v}).$$

EXAMPLE 7.1.3. Let $V := \mathbb{C}^2$, $G := \mathrm{GL}_1 = \mathbb{C}^*$ and the operation be given by $t(x,y) := (t^{-1}x, ty)$ for $t \in \mathbb{C}^*$, $(x,y) \in V$. The unstable orbits are $O_{(0,0)}, O_{(1,0)} =$ the x-axis minus zero, and $O_{(0,1)} =$ the y-axis minus zero. Thus $\mathcal{N}_V$ is the union of the two coordinate axes in $\mathbb{C}^2$. For $ab \neq 0$ the orbit $O_{(a,b)}$ is the *hyperbola* with the equation $xy - ab = 0$, and $\mathbb{C}^*O_{(a,b)} = \{(u,v) \mid u, v \neq 0\} = \mathbb{C}^2 \setminus \mathcal{N}_V$. It follows that

$$\mathcal{C}O_{(a,b)} = \partial(\mathbb{C}^*O_{(a,b)}) = \mathcal{N}_V\,.$$

EXAMPLE 7.1.4. Let $V := \mathbb{C}^2 \oplus \mathbb{C}^2 \simeq M_2(\mathbb{C})$, $G := \mathrm{SL}_2(\mathbb{C})$, and the operation be given by left multiplication (cf. section **??**). Choose $v, w \in \mathbb{C}^2$ which are linearly independent. This means that the corresponding matrix $[u,v]$ is nonsingular. Then the orbit $O_{(v,w)}$ is closed, hence semistable. In fact, $O_{(v,w)}$ corresponds to the set of matrices in $M_2(\mathbb{C})$ with determinant equal to $\det[u,v]$. It follows that $\mathbb{C}^*O_{(v,w)}$ corresponds to the set of invertible matrices, i.e.

$$\mathbb{C}^*O_{(v,w)} = \text{the set of all pairs of linearly independent vectors in } \mathbb{C}^2,$$

which is open and dense in V. It follows that

$$\mathcal{C}O_{(v,w)} = \{(v,w) \in V \mid v, w \text{ linearly dependen}\} = \mathcal{N}_V\,.$$

One can prove the following general result concerning the associated cone, cf. II.4.2.

PROPOSITION 7.1.5. *Suppose O is a semistable orbit. Then $\mathcal{C}O$ is a closed G-stable cone contained in $\mathcal{N}_V$, and* $\dim \mathcal{C}O = \dim O$. *Moreover, one has $\mathcal{C}O = \overline{\mathbb{C}^* O} \setminus \mathbb{C}^*\overline{O}$.*

The *dimension* is to be understood in the sense of algebraic geometry, see section AI.3.

OUTLINE OF PROOF. Since the G-action is rational and hence continuous, not only $\mathbb{C}^*\overline{O}$, but also $\overline{\mathbb{C}^*O}$ is a G-stable cone. Thus so are the interior of $\mathbb{C}^*\overline{O}$ and also its boundary. Since $\mathbb{C}^*\overline{O}$ consists only of semistable orbits, it follows that $\mathcal{N}_V \cap \mathbb{C}^*O = \emptyset$. From this it is obvious that $\mathcal{C}O \supseteq \overline{\mathbb{C}^*O} \setminus \mathbb{C}^*\overline{O}$. From the first part of the proposition, namely that $\mathcal{C}O \subseteq \mathcal{N}_V$, it follows that $\mathcal{C}O = \overline{\mathbb{C}^*O} \setminus \mathbb{C}^*\overline{O}$. However, the inclusion $\mathcal{C}O \subseteq \mathcal{N}_V$ is not so easy to prove.

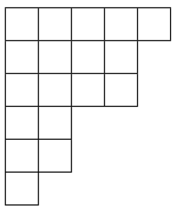
For the statement about the dimension, one first has to convince oneself that $\dim \mathbb{C}^*O = \dim O + 1$. (The map $\mathbb{C}^* \times O \to \mathbb{C}^*O$, $(t, v) \mapsto tv$, has finite fibers: Namely, if it were true that $t_i v_i = \tilde{v}$ for infinitely many different pairs $(t_i, v_i) \in \mathbb{C}^* \times O$, then $v_i = \tilde{v}/t_i \in \mathbb{C}^*\tilde{v}$ for infinitely many v_i, and so the entire line $\mathbb{C}\tilde{v}$ would be in the closure of O.) Since under the boundary map the algebraic dimension decreases by at least one, we find that $\dim \mathcal{C}O \leq \dim O$. For the inequality in the other direction one needs a little more from the general dimension theory. $\square$

7.2. Conjugacy classes of matrices. Here we study the example of matrices. Let $V = \mathrm{M}_n(\mathbb{C})$ and $G = \mathrm{GL}_n(\mathbb{C})$ acting by *conjugation* on V: $A \mapsto gAg^{-1}$ for $A \in M_n(\mathbb{C})$ and $g \in GL_n(\mathbb{C})$ (cf. section 4). We already know that $\mathcal{N}_V$ is equal to the set of *nilpotent matrices*. Now suppose that $A \neq 0$ is semisimple, with eigenvalues $\lambda_1, \ldots, \lambda_s$ with multiplicities $p_1 \geq p_2 \geq \cdots \geq p_s$. They form *partition* $p = (p_1, \ldots, p_s)$ of n. The $\mathrm{GL}_n(\mathbb{C})$-orbit of A is the conjugacy class C_A, and it is closed, by assumption (see 5.5.2).

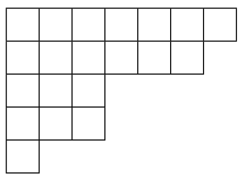
Now we would like to describe the associated cone $\mathcal{C}C_A$. In order to do this we consider the *dual partition* $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_t)$ of the partition p defined by

$$\hat{p}_i := \#\{j \mid p_j \geq i\}.$$

If one describes the partition p by its YOUNG *diagram*, i.e. an arrangement of boxes with p_i boxes in the i-th row, then the dual partition $\hat{p}$ has p_j boxes in the j-th column:



$p = (5, 4, 4, 2, 2, 2, 1)$



$\hat{p} = (7, 6, 3, 3, 1)$

Using this we can now describe the associated cone of a semisimple conjugacy class C_A. For a full proof of the following result one should consult [**Kra78**]. Recall that every partition $q = (q_1, \ldots, q_t)$ of n defines a nilpotent conjugacy class C_q given by the nilpotent matrix with JORDAN blocks of size $q_1, q_2, \ldots, q_t$

THEOREM 7.2.1. *Suppose $A \neq 0$ is a semisimple matrix with eigenvalue multiplicities $p_1 \geq p_2 \geq \cdots \geq p_s$, and suppose $\hat{p}$ is the dual partition to $p = \{p_1, ..., p_s\}$. Then for the cone associated to C_A we have*

$$\mathcal{C}C_A = \overline{C_{\hat{p}}}.$$

Here $C_{\hat{p}}$ denotes the nilpotent conjugacy class of the partition $\hat{p}$.

OUTLINE OF PROOF. If all the eigenvalues of A are different, then $p = (1, \dots, 1)$ and $\hat{p} = (n)$. The assertion then is that $\mathcal{C}C_A$ is the set of all nilpotent matrices. This is easy to check. If $D \in C_A$ is a diagonal matrix and

$$N := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & . & . & \\ & & & . & 1 \\ & & & & 0 \end{bmatrix} \in C_{(n)},$$

then, as is known, $tD + N \in C_{tD} = C_{tA}$ for every $t \in \mathbb{C}^*$. Letting $t \to 0$ gives $N \in \mathcal{C}C_A$ and thus the claim.

By means of an easy extension of this argument one can show that one has the inclusion $C_{\hat{p}} \subseteq \mathcal{C}C_A$. The proof of the opposite inclusion $\overline{C_{\hat{p}}} \supseteq \mathcal{C}C_A$ is more difficult. □

7.3. The case of binary forms of degree five. Let $V := R_5$ be the space of *binary forms of degree* 5 and $G := \mathrm{SL}_2(\mathbb{C})$ with the operation of G on V as in the last section. We know that $f \in V$ is unstable precisely if f contains a linear factor of multiplicity at least three. It follows that we can describe the unstable orbits by means of their representatives

$$0 \quad y^5 \quad xy^4 \quad x^2y^3 \quad xy^3(x+qy) \ (q \in \mathbb{C}^*).$$

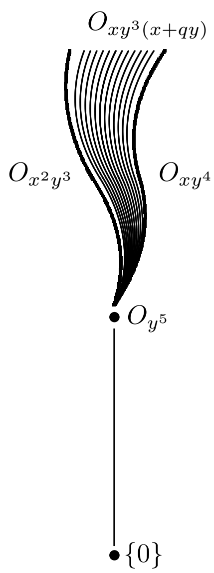


FIGURE 4. The null fiber $\mathcal{N}_5$ of R_5

Since $\dim \mathrm{SL}_2(\mathbb{C}) = 3$, every orbit has dimension less than or equal to three. The stabilizer of y^5 consists of the matrices $\begin{bmatrix} \zeta & \beta \\ & \zeta^{-1} \end{bmatrix}$ with $\zeta^5 = 1$ and $\beta \in \mathbb{C}$. The stabilizers of the other non-zero representatives are finite. Thus it follows that

$$\dim O_{y^5} = 2 \quad \text{and} \quad \dim O_{xy^4} = \dim O_{x^2y^3} = \dim O_{xy^3(x+qy)} = 3.$$

Since $\mathcal{N}_5$ contains a *one parameter family* of 3-dimensional orbits, we find that $\dim \mathcal{N}_5 = 4$. It is not difficult to see that every semistable orbit O is also three-dimensional. From our Proposition 7.1.5 above it follows that

$$\mathcal{C}O \subseteq \mathcal{N}_5 \quad \text{and} \quad \dim \mathcal{C}O = \dim O = 3.$$

Hence $\mathcal{C}O$ is a *finite union of homogeneous orbits* and thus one has

$$\mathcal{C}O \subseteq \{0\} \cup O_{y^5} \cup O_{xy^4} \cup O_{x^2y^3} = \overline{O_{xy^4}} \cup \overline{O_{x^2y^3}},$$

and $\mathcal{C}O$ contains either xy^4 or x^2y^3 or both. (One should note that the orbit of $xy^3(x+qy)$ for $q \in \mathbb{C}^*$ is not a cone.)

EXERCISE 7.3.1. Show using an appropriate limiting process that

$$\mathcal{C}O_{x^2y^2(x+y)} \quad = \quad \overline{O_{xy^4}} \cup \overline{O_{x^2y^3}}.$$

The general result is the following.

PROPOSITION 7.3.2. *Suppose $f \in R_5$ is a semistable form.*

(1) *The cone of O_f contains xy^4.*
(2) *If f has a linear factor of multiplicity 2, then one has*

$$\mathcal{C}O_f = \overline{O_{xy^4}} \cup \overline{O_{x^2y^3}}.$$

(3) *If f is squarefree, then one has*

$$\mathcal{C}O_f = \overline{O_{xy^4}}.$$

PROOF. (1) Since f is semistable, there must be a linear factor of f which occurs with multiplicity one. Without loss of generality we assume that this is x. Then $f = a_1xy^4 + a_2x^2y^3 + a_4x^4y + a_5x^5$ where $a_1 \neq 0$. Then we get $t^3 f_{t^{-1}} = a_1xy^4 + a_2t^2x^2y^3 + \cdots + a_5t^8x^5$. The claim in follows by letting $t \to 0$.

(2) If f has a linear factor of multiplicity two one can show in an analogous way that x^2y^3 lies in $\mathcal{C}O_f$, proving the claim.

(3) We prove this by contradiction. To do so assume that $f = \ell_1 \cdots \ell_5$ with pairwise linearly independent linear factors $l_1, \cdots, l_5$, and that $x^2y^3 \in \mathcal{C}O_f$. Hence there is a sequence $\{g_\nu\}_{\nu\in\mathbb{N}}$ in $\mathrm{GL}_2(\mathbb{C})$ such that $\lim_{\nu\to\infty} g_\nu.f = x^2y^3$.

Now we map the set of nonzero linear forms into the compact space $\mathbb{P}^1$ by letting $\pi(aX + bY) := (a : b) \in \mathbb{P}^1$. Set $P_i := \pi(\ell_i)$ and $P_i^\nu := \pi(g_\nu.\ell_i)$ for $i = 1, \dots, 5$ and $\nu \in \mathbb{N}$. Without loss of generality we may now assume that $\{P_i^\nu\}$ converges to $P_0 := (1 : 0) = \pi(x)$ for $i = 1, 3, 5$, and to $P_\infty := (0 : 1) = \pi(y)$ for $i = 2, 4$ as $\nu \to \infty$. We now use the invariance of the *cross ratio* under linear maps which is defined by the formula

$$\mathrm{CR}(P_1, P_2, P_3, P_4) = \frac{(a_1b_3 - a_3b_1)(a_2b_4 - a_4b_2)}{(a_2b_3 - a_3b_2)(a_1b_4 - a_4b_1)}$$

where $P_i = (a_i : b_i) \in \mathbb{P}^1$. One has $\mathrm{CR}(P_1^\nu, P_2^\nu, P_3^\nu, P_4^\nu) = \mathrm{CR}(P_1, P_2, P_3, P_4)$ for every $\nu \in \mathbb{N}$, and $\mathrm{CR}(P_1, P_2, P_3, P_4) \neq 0$, because all points are different. On the other hand $CR(P_0, P_\infty, P_0, P_\infty) = 0$, and we therefore get a contradiction. ☐

REMARK 7.3.3. Later on we will see that the method we described here concerning the associated cone has far-reaching applications (cf. section IV.5 and also the original work [**BK79**], where this method was introduced). The passage to the associated cone can also be understood as a kind of *deformation*. Roughly speaking this process allows one to carry over *"good properties" of the unstable orbits and their closures to arbitrary orbits.*

8. Ternary Cubics

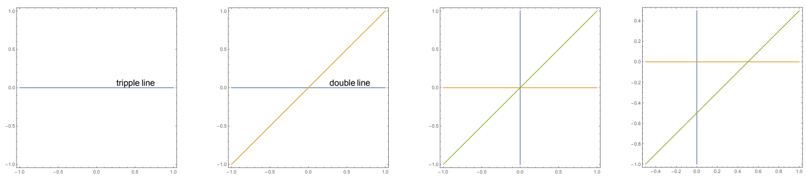
8.1. Normal forms. Let $T := \{f \in \mathbb{C}[x,y,z] \mid f \text{ homogeneous of degree } 3\}$ be the 10-dimensional (complex) vector space of *ternary cubic forms*. The group GL_3 acts on T by linear substitution of variables as in the case of binary forms (see section 5). For the classification of the orbits under GL_3 we associate to each nonzero form $f \in T$ its zero set $\mathcal{V}(f)$ in $\mathbb{P}^2$:

$$\mathcal{V}(f) := \{(x:y:z) \in \mathbb{P}^2 \mid f(x,y,z) = 0\}.$$

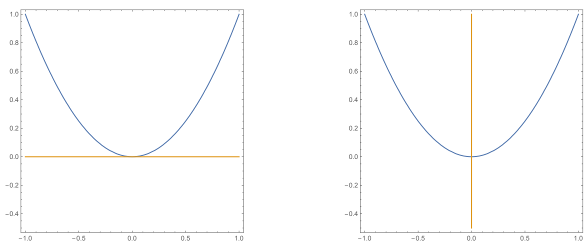
This is a plane projective curve of degree 3. We first classify these curves up to projective equivalence. This corresponds to the classification of the ternary cubics with respect to the action of GL_3. In each case we draw a “real” picture of the curve.

PROPOSITION 8.1.1. *The following is a classification of the ternary cubic forms up to linear substitutions.*

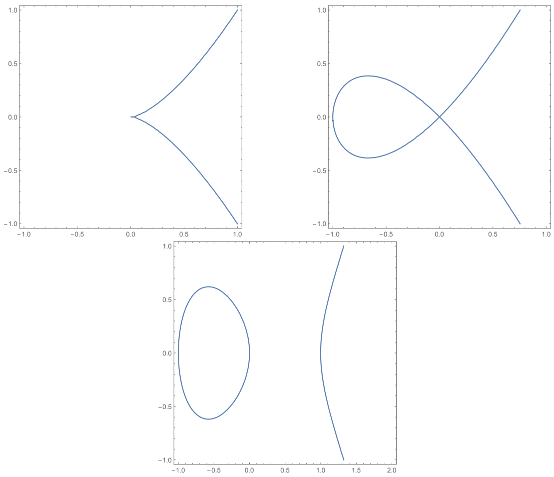
(a) *f is a product of 3 linear factors ℓ_1, ℓ_2, ℓ_3, i.e. $\mathcal{V}(f)$ is a union of 3 lines:*
  (a_1) *$\ell_1 = \ell_2 = \ell_3$: $f = x^3$.*
  (a_2) *ℓ_1 is linearly independent of $\ell_2 = \ell_3$: $f = x^2y$.*
  (a_3) *ℓ_1, ℓ_2, ℓ_3 are linearly dependent, but pairwise linearly independent: $f = xy(x+y)$.*
  (a_4) *ℓ_1, ℓ_2, ℓ_3 are linearly independent: $f = xyz$.*



(b) *f contains an irreducible factor q of degree 2, i.e. $\mathcal{V}(f)$ is a union of a quadric Q and a line L:*
  (b_1) *The line L is tangent to the quadric Q: $f = (x^2 - yz)y$.*
  (b_2) *The line L meets the quadric Q in two points: $f = (x^2 - yz)x$.*



(c) *f is irreducible, i.e. $\mathcal{V}(f)$ is an irreducible cubic C.*
  (c_1) *C has a cusp: $f = y^2z - x^3$*
  (c_2) *C has a double point: $f = y^2z - x^3 - x^2z$*
  (c_3) *C is nonsingular: $f = y^2z - x^3 - ax^2z - bxz^2 - cz^3$, $a, b, c \in \mathbb{C}$. These are the so-called elliptic curves.*



Note that some of these cases can be distinguished by the number of singular points. E.g. the only case with exactly three singular points is (a_4), and the only one with exactly two singular points is (b_2).

Outline of Proof. (a) This is easy and we leave it as an exercise to the reader.

(b) We can assume that $q = x^2 - yz$, hence $f = (x^2 - yz)\ell$. The stabilizer of q is the orthogonal group $\mathrm{O}(q) \subseteq \mathrm{GL}_3$ which acts transitively on the vectors in $\mathbb{C}^3$ of a fixed length. This implies that $\mathrm{O}(q)$ has two orbits in the set of lines in $\mathbb{P}^2$, the tangent vectors to Q and the complement. Therefore, we can assume that $\ell = y$ in case L is tangent to Q, and that $\ell = x$ otherwise, giving (b_1) and (b_2).

(c) If C has a singular point P, then we can assume that $P = (0 : 0 : 1)$. If the tangent cone in P consists of one line, one easily gets (c_1). If the tangent cone consists of two lines, then we get (c_2).

Finally, if C is nonsingular, then one shows by using the Hessian that C has point of inflection. Let this be $(0 : 1 : 0)$ with tangent line $z = 0$. This implies that $f(x, 1, z) = az + bz^2 + cxz + g(x, z)$ where $a \neq 0$ and g is homogenous of degree 3. As a consequence, we get

$$f(x, y, 1) = ay^2 + by + cxy + g(x, 1).$$

Replacing y by $y' := y - \frac{b+cx}{2a}$ we obtain $f(x, y, 1) = ay^2 + h(x)$, and the claim follows easily. $\square$

Exercise 8.1.2. Show that a nonsingular form $f \in T$ is GL_3-equivalent to one of the following forms:

$$y^2z - x^3 - xz^2, \qquad y^2z - x^3 - z^3, \qquad y^2z - x^3 - rxz^2 - rz^3 \text{ with } r \in \mathbb{C}.$$

Remark 8.1.3.

(1) The normal form given in (c_3) is called Weierstrass *normal form*. It contains the special cases (c_1) and (c_2). We also see that this normal form defines a nonsingular cubic if and only if the polynomial $x^3 + ax^2 + bx + c$ has no multiple roots.

(2) There is another normal form for the nonsingular cubics, the Hesse *normal form*

$$h_s := x^3 + y^3 + z^3 - sxyz.$$

This is nonsingular for $s^3 \neq 27$, and for $s^3 = 27$ it is of type (a_4), because the corresponding curve has 3 singular points.

EXERCISE 8.1.4. Show that if ${s'}^3 = s^3$, then $h_{s'}$ is GL_3-equivalent to h_s. However, we will see later that, in general, there are 12 different values of c giving GL_3-equivalent forms h_s.

EXAMPLE 8.1.5. We want to show that every nonsingular cubic $f \in T$ is GL_3-equivalent to a cubic of the form $h_s = x^3 + y^3 + z^3 - sxyz$ with $s \in \mathbb{C}$. It is easy to see that such a cubic is nonsingular if and only if $s^3 \neq 27r^3$. Moreover, the point $S := (1 : -1 : 0) \in C$ is a point of inflection, with tangent line $L\colon 3x + 3y + sz = 0$. This allows to calculate the WEIERSTRASS normal form of h_s by making successively the following substitutions (where we assume $s \neq 0$):

$$\begin{array}{rcl} z & \mapsto & -\frac{3}{s}x - \frac{3}{s}y + \frac{1}{s}z \\ x & \mapsto & x - y \\ y & \mapsto & y + \frac{1}{2}x \end{array}$$

The first two transform the inflection point S to $(1 : 0 : 0)$ and the tangent line L to $z = 0$, and the last eliminates xyz from the resulting expression for h_s:

$$h_s \mapsto y^2 z + (1 - \frac{27}{s^3})x^3 + (-\frac{1}{4} + \frac{27}{s^3})x^2 z - \frac{9}{s^3}xz^2 - \frac{1}{s^3}z^3.$$

We see that this form only depends on s^3, in accordance with Exercise 8.1.2 above. Setting $t := \frac{27}{s^3} - 1$ and using scalar multiplications and affine transformations $x \mapsto \alpha x + \beta$ one can transform the polynomial in x into the form

$$x^3 - \frac{1}{864}(144t + 162)x - \frac{1}{864}(8t^2 + 36t + 27).$$

Now one can use Exercise 8.1.4 to conclude that all GL_3-equivalence classes appear in this way.

8.2. Classification with respect to SL_3. The classification of the ternary cubic forms with respect to the action of SL_3 does not present any fundamental difficulties. One only has to decide whether the multiples of normal forms given in the above list are, with respect to SL_3, equivalent to the original ones or not. Doing this gives rise to an additional parameter in some of the cases. The details of carrying this out are left to the reader.

PROPOSITION 8.2.1. *The SL_3-orbits of the nonzero ternary cubics are represented by the forms from the following list.*

$$(a_1)\ f = x^3 \quad (a_2)\ f = x^2 y \quad (a_3)\ f = xy(x+y) \quad (a_4)\ f = txyz,\ t \in \mathbb{C}^*$$
$$(b_1)\ f = (x^2 - yz)y \quad (b_2)\ f = t(x^2 - yz)x,\ t \in \mathbb{C}^*$$
$$(c_1)\ f = y^2 z - x^3 \quad (c_2)\ f = t(y^2 z - x^3 - x^2 z),\ t \in \mathbb{C}^*$$
$$(c_3)\ f = y^2 z - x^3 - ax^2 z - bxz^2 - cz^3,\ a, b, c \in \mathbb{C}\ \textit{or}$$
$$f = r(x^3 + y^3 + z^3) - sxyz,\ r \in \mathbb{C}^*, s \in \mathbb{C}.$$

In particular, the forms corresponding to nonsingular cubics have 2 parameters, those corresponding to a singular cubic with a double point, to a quadric with a secant line or to the union of 3 lines in general position have one parameter, and the remaining types form just one orbit.

As already mentioned, the forms in (c_3) are nonsingular if and only if the polynomial $x^3 + ax^2 + bx + c$ has no multiple roots, or if and only if $s^3 \neq 27r^3$.

8.3. Nullforms and degenerations. As in the case of binary forms, a cubic form $f \in T$ is called a *nullform* if the closure of the SL_3-orbit O_f contains the origin 0. It is easy to see that the types $(a_1), (a_2), (a_3), (b_1), (c_1)$ are nullforms. It is slightly more difficult to show that these are all nullforms. This will be a consequence of the following considerations.

DEFINITION 8.3.1. A form $h \in T$ is called a *degeneration* of a form $f \in T$ if $h \in \overline{O_f}$. Hence, f is a nullform if 0 is a degeneration of f. We will use the following notation for this relation:

$$\begin{array}{l} \bullet\, O_f \\ \big| \\ \bullet\, O_h \end{array} \qquad \text{or} \qquad \begin{array}{l} \bullet\, f \\ \big| \\ \bullet\, h \end{array}$$

On the following page we describe the degeneration behavior of the ternary forms of degree 3. For this we have given the dimensions of the single orbits from which one can read off the behavior under the transition to the associated cone (one uses Theorem 6.2.2 from section 6). These are easily found by considering the stabilizers of the forms, because the dimension of an orbit is the difference of the dimension of the group and the dimension of the stabilizer, see .

Let us briefly discuss the degenerations claimed in the table.

(1) $$\begin{array}{l} \bullet\, t(y^2z - x^3 - x^2z)x \\ \big| \\ \bullet\, 2t(x^2 - yz)x \end{array} \qquad \text{i.e. } 2t(x^2 - yz)x \in \overline{O}_{t(y^2z - x^3 - x^2z)}$$

This follows by making the substitution

$$\begin{array}{rcl} x & \mapsto & -\sqrt[3]{2}x \\ y & \mapsto & \varepsilon\, y + \sqrt[3]{2}x \\ z & \mapsto & -(\varepsilon\, \sqrt[3]{2})^{-1}z \end{array} \qquad \text{and then letting } \varepsilon \to 0.$$

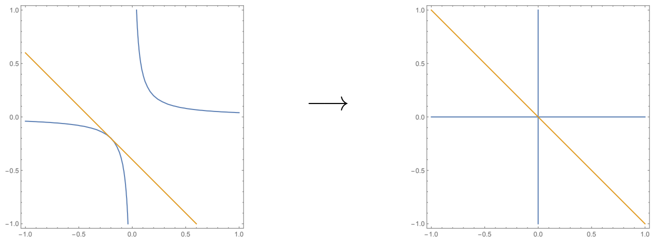
(2) $$\begin{array}{l} \bullet\, y^2z - x^3 \\ \big| \\ \bullet\, (x^2 - yz)y \end{array} \qquad \text{i.e. } (x^2 - yz)y \in \overline{O}_{y^2z - x^3}$$

For $\varepsilon \in \mathbb{C}^*$ the form $f_\varepsilon := (x^2 - yz)y + \varepsilon\, x^3$ is irreducible. Moreover, $\mathcal{V}(f)$ has a singularity at the origin, and it is a vertex. Thus f_ε is of type (c_1). The claim follows by letting $\varepsilon \to 0$.

(3) $$\begin{array}{l} \bullet\, (x^2 - yz)y \\ \big| \\ \bullet\, xy(x + y) \end{array} \qquad \text{i.e. } xy(x + y) \in \overline{O}_{(x^2 - yz)y}$$

One has $xy(x+y) = x(xy+y^2)$. For $\varepsilon \in \mathbb{C}^*$ the quadratic form $xy + y^2 + \varepsilon\, z^2$ is nondegenerate and has $x = 0$ as isotropic line. Thus $x(xy + y^2 + \varepsilon\, z^2) \in O_{(x^2 - yz)y}$, and the claim follows by letting $\varepsilon \to 0$.

Obviously, this deformation can be seen geometrically like this:



The arguments presented above in (1), (2) and (3) are enough to see that the behavior given in the table holds, provided we now prove that the orbits of type (a_4) and (c_3) are closed. In order to do this we use the following stronger form of HILBERT's Criterion from section 6.2.2; a proof for this is given in the third chapter, 6.2.2.

PROPOSITION 8.3.2 (HILBERT's Criterion). *Suppose O_h is a closed orbit in $\overline{O}_f$. Then there is a group homomorphism $\lambda\colon \mathbb{C}^* \to \mathrm{SL}_3$ with the property that $\lim_{t\to 0} f^{\lambda(t)}$ exists and lies in O_h.*

(4) *If $\mathcal{V}(f)$ has no singularities, then O_f is closed. If f is a product of three linearly independent linear forms, then O_f is likewise closed.*

PROOF. If O_f were not closed, then by the Hilbert Criterion there would exist a homomorphism $\lambda\colon \mathbb{C}^* \to \mathrm{SL}_3$ with the property that the limit $\lim_{t\to 0} f^{\lambda(t)}$ exists but does not lie in O_f. By making a change of coordinates we may, without loss of generality, assume that

$$\lambda(t) = \begin{bmatrix} t^\alpha & & \\ & t^\beta & \\ & & t^\gamma \end{bmatrix} \qquad \text{with} \quad \alpha,\beta,\gamma \in \mathbb{Z},\ \ \alpha \geq \beta \geq \gamma \text{ and } \alpha+\beta+\gamma = 0.$$

First we consider the case where $\mathcal{V}(f)$ has no singularities. If $\beta \geq 0$, then the monomials xz^2, yz^2 and z^3 cannot occur in f. For otherwise, as $\alpha+2\gamma < 0, \beta+2\gamma < 0$ and $3\gamma < 0$, the limit $\lim_{t\to 0} f^{\lambda(t)}$ does not exist. But it follows from this that $\mathcal{V}(f)$ has a singularity at the point $(0,0,1)$. In the case $\beta < 0$ a similar argument shows that a linear factor x can be split off f. Thus $\mathcal{V}(f)$ is also singular in this case.

Finally if f is a product of three linearly independent linear factors and if O_f were not closed, then for dimension reasons f would have to be a nullform. Similar arguments to those above now lead to a contradiction. (We are again using the fact that the closure of an orbit contains a unique closed orbit, see Theorem **??**.) $\square$

As we already established above, this completely proves the relationships given in the table.

REMARK 8.3.3. The corresponding, but essentially more difficult investigation of *ternary forms of fourth degree* can be found in a work of G. BRACKLY [**Bra79**].

8.4. Invariants under SL_3. We show now that the SL_3-invariants give some new insight and help to understand some of the behavior above. It is classically known that the invariant ring $\mathcal{O}(T)^{\mathrm{SL}_3}$ of the ternary cubics is generated by two algebraically independent invariants I_4 and I_6 of degree 4 and 6, and that the *discriminant* has the expression

$$\Delta(f) = I_6(f)^2 - \frac{1}{6} I_4(f)^3$$

(see [**Wei23**, §17]). The discriminant tells us whether a form f is nonsingular or not. Therefore, the rational function

$$j(f) := \frac{I_4(f)^3}{\Delta(f)}$$

called *j-invariant*, is well defined for nonsingular forms f and only depends on the GL_3-equivalence class. The main result from classical invariant theory is the following.

Proposition 8.4.1. (1) *If $f \in T$ is nonsingular and $h \in T$ arbitrary, then h is SL_3-equivalent to f if and only if $I_4(h) = I_4(f)$ and $I_6(h) = I_6(f)$.*
(2) *If $f, h \in T$ are nonsingular, then h is GL_3-equivalent to f if and only if $j(h) = j(f)$.*
(3) *$f \in T$ is a nullform if and only if $I_4(f) = I_6(f) = 0$.*

This can be reformulated in more geometric terms. For this consider the map

$$\pi \colon T \to \mathbb{C}^2, \quad f \mapsto (I_4(f), I_6(f)).$$

It is clear that π is constant on SL_3-orbits, because the two coordinate functions I_4 and I_6 are SL_3-invariants. The proposition above can now be reformulated in the following way.

Corollary 8.4.2. (1) *If $f \in T$ is nonsingular, then the fiber of π through f is the SL_3-orbit of f: $\pi^{-1}(\pi(f)) = O_f$.*
(2) *$T_{reg} := \{f \in T \mid \Delta(f) \neq 0\} \subseteq T$ is the open subset of regular forms, and the fibers of $j \colon T_{reg} \to \mathbb{C}$ are the GL_3-orbits.*
(3) *The fiber $\varphi^{-1}(0)$ is the subset of nullforms.*

8.5. Some computations. Using the *symbolic method* one can calculate explicitly these invariants in terms of the coefficients $a_1, a_2, \ldots, a_{10}$ of the form

$$f = a_1x^3 + 3a_2x^2y + 3a_3x^2z + a_4y^3 + 3a_5xy^2 + \cdots + 3a_9yz^2 + 6a_{10}xyz$$

(see section 6.2.2). E.g. one finds

$$\begin{aligned} I_4 = 24(&a_{10}^4 - 2a_3a_6a_{10}^2 - 2a_5a_8a_{10}^2 - 2a_2a_9a_{10}^2 - a_1a_4a_7a_{10} + a_2a_5a_7a_{10} + a_3a_4a_8a_{10} \\ &+ 3a_2a_6a_8a_{10} + 3a_3a_5a_9a_{10} + a_1a_6a_9a_{10} + a_3^2a_6^2 + a_5^2a_8^2 - a_2a_4a_8^2 \\ &+ a_2^2a_9^2 - a_1a_5a_9^2 - a_3a_5^2a_7 + a_2a_3a_4a_7 - a_2^2a_6a_7 + a_1a_5a_6a_7 \\ &- a_1a_6^2a_8 - a_3a_5a_6a_8 - a_3^2a_4a_9 - a_2a_3a_6a_9 + a_1a_4a_8a_9 - a_2a_5a_8a_9) \end{aligned}$$

For the Weierstrass normal form $f = y^2z - x^3 - ax^2z - bxz^2 - cz^3$ the values of these invariants are the following:

$$I_4(\text{Weierstrass}) = \frac{8}{27}(a^2 - 3b), \quad I_6(\text{Weierstrass}) = \frac{8}{243}(2a^3 - 9ab + 27c)$$

$$\Delta(\text{Weierstrass}) = -\frac{64}{2187}(a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2),$$

$$j(\text{Weierstrass}) = \frac{-8(a^2 - 3b)^3}{9(a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2)}.$$

Note that the last expression in Δ, $a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2$, is equal to the discriminant of the polynomial $x^3 + ax^2 + bx + c$.

For the HESSE normal form $r(x^3 + y^3 + z^3) - sxyz$ one finds

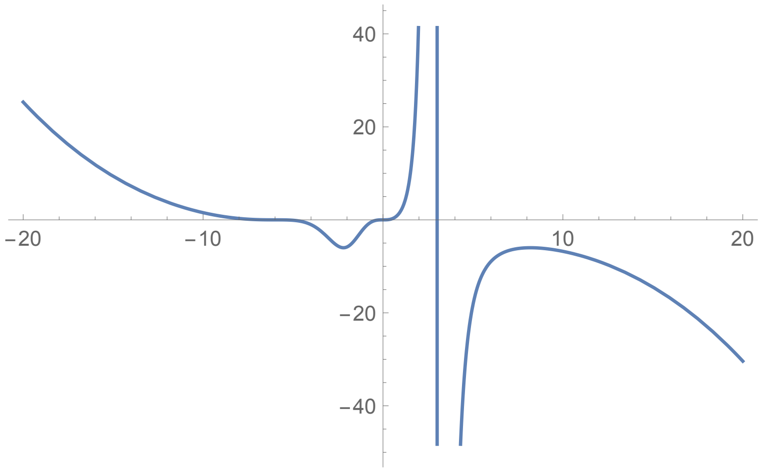
$$I_4(\text{Hesse}) = \frac{1}{54}s(s+6)(s^2 - 6s + 36) = \frac{1}{54}s(s^3 + 216),$$

$$I_6(\text{Hesse}) = \frac{1}{972}(s^2 - 6s - 18)(s^4 + 6s^3 + 54s^2 - 108s + 324) = \frac{1}{972}(s^6 - 540s^3 - 5832)$$

$$\Delta(\text{Hesse}) = \frac{4}{2187}(s-3)^3(s^2 + 3s + 9)^3 = -\frac{4}{2187}(s^3 - 27)^3$$

$$j(\text{Hesse}) = -\frac{s^3(s^3 + 316)^3}{288(s^3 - 27)^3}$$

The form of the j-invariant shows that the dependence of the GL_3-equivalence class from the parameter s is complicated (see the graph below). In general, there are 12 different values of s giving the same GL_3-equivalence class.



REMARK 8.5.1. Without any computation it is clear that the number of different s giving the same GL_3-equivalence class is finite. In fact, the rational function j restricted to the line $H \subseteq T$ of HESSE normal forms is non-constant and thus has finite fibers.

Exercises

For the convenience of the reader we collect here all exercises from Chapter I.

CHAPTER II

Algebraic Groups

Contents

Introduction. Now that we have studied several examples in detail in the first chapter, we would like to turn to the basics. Linear algebraic groups, i.e. closed subgroups of the general linear group GL_n, and homomorphisms between them are the basic notion for what follows. The definitions connected with this and a few simple properties are treated in the first two sections.

As some of our main examples we then describe the classical groups GL_n, SL_n, O_n, SO_n, and Sp_{2m} and give some of there properties. Finally, in the last section, we define the Lie algebra of an algebraic group and give several examples and applications.

There are many exercises included in the text, some of them with hints. The reader is strongly advised to work out the solutions. At the end of each paragraph, we recollect them for the convenience of the reader.

1. Basic Definitions

1.1. Linear algebraic groups. The *general linear group* $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ of invertible complex $n \times n$-matrices has a natural structure of an *affine variety*. It is a special open set (A.1.5) of the vector space $\mathrm{M}_n(\mathbb{C})$ of $n \times n$-matrices, namely

$$\mathrm{GL}_n = \{A \in \mathrm{M}_n(\mathbb{C}) \mid \det A \neq 0\} = \mathrm{M}_n(\mathbb{C})_{\det},$$

with coordinate ring

$$\mathcal{O}(\mathrm{GL}_n) = \mathbb{C}[x_{ij}]_{\det} = \mathbb{C}[x_{ij}, \det^{-1}], \quad \det := \det(x_{ij}) \in \mathbb{C}[x_{ij}]$$

(cf. Example A.1.5.3). This is our basic object.

Exercise 1.1.1.
(1) Show that the multiplication $\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n$ is a morphism of varieties.
(2) Show that left and right multiplication $\lambda_A \colon B \mapsto AB$ and $\rho_A \colon B \mapsto BA$ with a fixed matrix $A \in \mathrm{GL}_n$ are isomorphisms $\mathrm{GL}_n \xrightarrow{\sim} \mathrm{GL}_n$ of varieties.
(3) Show that inversion $A \mapsto A^{-1}$ is an isomorphism $\mathrm{GL}_n \xrightarrow{\sim} \mathrm{GL}_n$ of varieties. (Hint: Use Cramer's rule.)

Given a finite dimensional vector space V every choice of a basis induces an isomorphism $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_n$. Thus $\mathrm{GL}(V)$ carries the structure of an affine variety, too, with coordinate ring

$$\mathcal{O}(\mathrm{GL}(V)) = \mathcal{O}(\mathrm{End}(V))_{\det} \simeq \mathcal{O}(\mathrm{GL}_n).$$

It is easy to see that this structure does not depend on the choice of the basis of V. Subgroups of GL_n are usually called *matrix groups*. Algebraic groups as defined below are special cases of matrix groups. Recall that all topological notions are with respect to the Zariski topology (A.1.2.5) unless otherwise stated.

Definition 1.1.2. A closed subgroup $G \subseteq \mathrm{GL}_n$ is called an *algebraic group* or a *linear algebraic group*. The *identity matrix* in GL_n is denoted by E_n or E, and the *identity element* of an arbitrary group G mostly by e or e_G.

Examples 1.1.3. We start with some well-known examples of matrix groups.

(1) The *special linear group* $\mathrm{SL}_n := \mathcal{V}(\det -1) \subseteq \mathrm{GL}_n$ consists of all matrices with determinant 1. Its coordinate ring is $\mathcal{O}(\mathrm{SL}_n) = \mathbb{C}[x_{ij}]/(\det -1)$. In fact, $\det -1$ is an irreducible polynomial.
(Proof: If $\det -1 = pq$ and if the variable x_{ij} occurs in p, then so do all $x_{ik}, k = 1, \ldots n$, and all $x_{lj}, l = 1, \ldots, n$, and none of these occur in q since no monomial of the determinant contains a product of the form $x_{ij}x_{ik}$ or $x_{ij}x_{lj}$). It follows that all variables occur in p, hence q is a constant.) Similarly, we define $\mathrm{SL}(V) \subseteq \mathrm{GL}(V)$.
(2) The *multiplicative group* $\mathbb{C}^* := \mathrm{GL}_1 = (\mathbb{C} \setminus \{0\}, \cdot)$.
(3) The *additive group* $\mathbb{C}^+ := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{C} \right\} \subseteq \mathrm{GL}_2$ is given by the equations $x_{11} = x_{22} = 1$, $x_{21} = 0$. Its coordinate ring $\mathcal{O}(\mathbb{C}^+) = \mathbb{C}[s]$ is a polynomial ring in one variable where $s := x_{12}|_{\mathbb{C}^+}$. We will identify $\mathbb{C}^+$ with the underlying additive group $(\mathbb{C}, +)$ of the field $\mathbb{C}$.
(4) The group of *upper triangular unipotent matrices* $U_n := \left\{ \begin{bmatrix} 1 & * & \cdots \\ 0 & 1 & \\ \vdots & & \ddots \end{bmatrix} \right\}$ with 1's along the diagonal. Its coordinate ring is the polynomial ring $\mathbb{C}[\bar{x}_{ij} \mid i < j]$ where $\bar{x}_{ij} := x_{ij}|_{U_n}$.

(5) The group of *diagonal matrices* $T_n := \left\{ \begin{bmatrix} * & 0 & \cdots \\ 0 & * & \\ \vdots & & \ddots \end{bmatrix} \right\}$ with nonzero elements along the diagonal. Its coordinate ring is $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ where $t_i := x_{ii}|_{T_n}$

(6) The group of *upper triangular matrices* $B_n := \left\{ \begin{bmatrix} * & * & \cdots \\ 0 & * & \cdots \\ \vdots & & \ddots \end{bmatrix} \right\}$ with nonzero elements along the diagonal. *Then* $B_n = T_n \cdot U_n = U_n \cdot T_n$ *and* T_n *normalizes* U_n.

EXERCISE 1.1.4. Show that the map $\varphi\colon T_n \times U_n \to B_n, (t, u) \mapsto tu$ is an isomorphism of algebraic varieties.

EXAMPLE 1.1.5. The group $\mathcal{P}_n \subseteq \mathrm{GL}_n$ of *permutation matrices* P, i.e., in every row and every column of P there is exactly one nonzero entry which is 1. Thus

$$\mathcal{P}_n = \left\{ P_\sigma := \sum_{i=1}^{n} E_{i\sigma(i)} \mid \sigma \text{ a permutation of } \{1, \ldots, n\} \right\},$$

where E_{ij} is that $n \times n$-matrix which has a 1 as its (i, j)-entry and zeroes otherwise. It is easy to verify that $\sigma \mapsto P_\sigma$ identifies the *symmetric group* $\mathcal{S}_n$ with $\mathcal{P}_n$.

Since, by CAYLEY'S Theorem (cf. [**Art91**, Chap. 6, Theorem 1.3]), any finite group is isomorphic to a subgroup of $\mathcal{S}_n$ for a suitable n, the last example shows that every finite group can be considered as an algebraic group. (Recall that an arbitrary finite set F is an affine variety in a unique way, setting $\mathcal{O}(F) := \mathbb{C}^F$, the algebra of all $\mathbb{C}$-valued functions on F, see Example A.1.4.2.)

Thus the theory of finite groups is part of the theory of algebraic groups. We will see in the sequel that many concepts from finite group theory can be carried over to algebraic groups, some of them easily, some others require more work. We recommend the reader to keep the case of finite groups always in mind.

EXERCISE 1.1.6. (1) Show that every automorphism μ of the line $\mathbb{C}$ is an affine transformation, i.e. $\mu(x) = ax + b$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.
(2) If an automorphism μ of $\mathbb{C}$ has two or more fixed points, then $\mu = \mathrm{id}$.

EXERCISE 1.1.7. (1) The subgroup $\mu_n := \{t \in \mathbb{C}^* \mid t^n = 1\} \subseteq \mathbb{C}^*$ is finite and cyclic of order n, and the μ_n's exhaust all closed subgroups $\neq \mathbb{C}^*$.
(2) The set $\{g \in T_n \mid g \text{ has finite order}\}$ is a ZARISKI dense subgroup of T_n.

EXERCISE 1.1.8. A strict closed subgroup of $\mathbb{C}^+$ is trivial.

EXAMPLE 1.1.9. Let $F \subseteq \mathbb{C}$ be a finite subset of cardinality $|F| \geq 2$. *Then* $\mathrm{Aut}(\mathbb{C} \setminus F)$ *is a finite group.* In fact, an automorphism μ of $\mathbb{C} \setminus F$ is induced by a MÖBIUS transformation $\mu_A \in \mathrm{Aut}(\mathbb{C}(x))$ (see Appendix A.3.7), and μ_A permutes the discrete valuations of $\mathbb{C}(x)$. If $a \in \mathbb{C} \setminus F$ and $\mu(a) = b$, then $\mu_A(\nu_a) = \nu_b$. Thus, we get a homomorphism $\mathrm{Aut}(\mathbb{C} \setminus F) \to \mathrm{Perm}(F \cup \{\infty\})$. An element μ from the kernel fixes ν_∞ and every point of F. It follows that μ is an automorphism of $\mathbb{C}$ with fixed points F, hence $\mu = \mathrm{id}$ by Exercise 1.1.6(2). It follows that $\mathrm{Aut}(\mathbb{C} \setminus F)$ is isomorphic to a subgroup of the symmetric group $S_{|F|+1}$.

EXERCISE 1.1.10. If $F \subseteq \mathbb{C}$ is a finite subset of cardinality $|F| \geq 3$ in "general position", then $\mathrm{Aut}(\mathbb{C} \setminus F)$ is trivial.
(In "general position" means that for every $d \geq 3$ there is a dense open set $U \subseteq \mathbb{C}^d$ such that the claim holds for any F from U.)

PROPOSITION 1.1.11. *Let R be a finite dimensional associative $\mathbb{C}$-algebra with a unit element $1 \in R$. Then the group R^* of invertible elements of R is open in R, and R^* has the structure of an algebraic group given by the closed embedding $R^* \hookrightarrow \mathrm{GL}(R)$, $r \mapsto \lambda_r :=$ left multiplication with r.*

PROOF. Consider the map $\lambda\colon R \to \mathrm{End}(R)$, $r \mapsto \lambda_r$ where $\lambda_r(s) := rs$. This is an injective linear map, hence an isomorphism onto its image $\lambda(R)$, and it satisfies $\lambda(rs) = \lambda(r) \circ \lambda(s)$. It follows that $R^* = \lambda^{-1}(\mathrm{GL}(R))$ and that λ induces a group isomorphism $R^* \xrightarrow{\sim} \lambda(R) \cap \mathrm{GL}(R)$. Thus R^* is open in R and $\lambda\colon R^* \hookrightarrow \mathrm{GL}(R)$ is a closed embedding. □

The proof shows that R^* is a special open set of R. In particular, R^* is irreducible of dimension $\dim R^* = \dim R$.

1.2. Isomorphisms and products. It follows from our definition that an algebraic group G is an affine variety with a group structure. These two structures are related in the usual way. Namely, the multiplication $\mu\colon G \times G \to G$ is a *morphism*, right and left multiplication by a fixed element $g \in G$, $\rho_g\colon h \mapsto hg$ and $\lambda_g\colon g \mapsto gh$ ($h \in G$), are isomorphisms of algebraic varieties, as well as taking inverses $\iota\colon h \mapsto h^{-1}$. In fact, this is clear for GL_n (see Exercise 1.1.1), and follows for arbitrary algebraic groups $G \subseteq \mathrm{GL}_n$ by restriction.

REMARK 1.2.1. We could take a more general point of view, like in the case of topological groups, and define an algebraic group G to be an affine variety with a group structure such that multiplication and inversion are morphisms. It turns out that this leads to the same, i.e. any such "algebraic group object" is isomorphic to a linear algebraic group, see Proposition III.2.4.6.

DEFINITION 1.2.2. Two algebraic groups G and H are *isomorphic* if there is a group homomorphism $\varphi\colon G \to H$ which is an isomorphism of algebraic varieties. Such a φ is shortly called an *isomorphism*, or an *automorphism* of G in case $H = G$. The group of automorphisms of G will be denoted by $\mathrm{Aut}(G)$.

EXAMPLES 1.2.3.
(1) For any $g \in G$ the map $\mathrm{Int}\, g\colon G \xrightarrow{\sim} G$, $h \mapsto ghg^{-1}$, is an automorphism called *conjugation by g*, or *inner automorphism by g*.
(2) If G is commutative, then $\iota\colon G \to G$, $g \mapsto g^{-1}$, is an automorphism.
(3) The map $A \mapsto A^{-t}$ is an automorphism of GL_n and of SL_n.
(4) The groups U_n and $U_n^- := \{A^t \mid A \in U_n\}$ are isomorphic.
(5) The subgroup $T_n' := T_n \cap \mathrm{SL}_n$ is isomorphic to T_{n-1}.

EXERCISE 1.2.4. Give proofs for the claims (1)–(5) in the example above.

EXERCISE 1.2.5. The subgroup $\mathrm{Int}(G) \subseteq \mathrm{Aut}(G)$ of inner automorphisms of a group G is a normal subgroup.

EXERCISE 1.2.6.
(1) For SL_2, the automorphism $A \mapsto A^{-t}$ is inner.
(2) Show that all automorphisms of SL_2 are inner.
(3) For GL_n, $n \geq 2$, and for SL_n, $n \geq 3$, the automorphism $A \mapsto A^{-t}$ it is not inner. (Hint: For GL_n, look at the determinant. For SL_n, if $A \mapsto A^{-t}$ is inner, then the composition of both is an automorphism of GL_n which is the identity on SL_n and the inverse on $\mathbb{C}^* E_n$. This leads to a contradiction as soon as $n > 2$.)

There is an obvious embedding of the product $\mathrm{GL}_n \times \mathrm{GL}_m$ into GL_{n+m} given by $(A, B) \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ which identifies $\mathrm{GL}_n \times \mathrm{GL}_m$ with a closed subgroup of GL_{n+m}. This implies that the *product* $G \times H$ of any two algebraic groups is again an algebraic group in a natural way. More general, the product $G_1 \times G_2 \times \cdots \times G_n$ of a finite number of algebraic groups G_i is an algebraic group.

EXAMPLES 1.2.7. (1) $(\mathbb{C}^*)^n := \mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ is isomorphic to T_n. An algebraic group isomorphic T_n is called an *n-dimensional torus*. We will discuss these groups in detail in section III.3.

(2) The subgroup $\left\{ A = \begin{bmatrix} 1 & 0 & \cdots & 0 & b_1 \\ & 1 & \cdots & 0 & b_2 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & b_n \\ & & & & 1 \end{bmatrix} \right\} \subseteq U_{n+1}$ is isomorphic to $(\mathbb{C}^+)^n$.

EXERCISE 1.2.8. Let V be a finite dimensional complex vector space. Then the underlying additive group V^+ is an algebraic group and $\operatorname{Aut}(V^+) = \mathrm{GL}(V)$.

The following result will be used at several occasions.

LEMMA 1.2.9. *Let $H \subseteq \mathrm{GL}_n$ be an "abstract" subgroup. Then the (*ZARISKI*) closure $\bar{H} \subseteq \mathrm{GL}_n$ is an algebraic group.*

PROOF. We have to show that $\bar{H} \subseteq \mathrm{GL}_n$ is a subgroup. For any $h \in H$ the left multiplication $\lambda_h\colon g \mapsto hg$ induces a morphism $\bar{H} \to \bar{H}$, hence $h\bar{H} \subseteq \bar{H}$ and therefore $H\bar{H} = \bar{H}$ which implies that $\bar{H}\,\bar{H} = \bar{H}$. Similarly, we see that $g \mapsto g^{-1}$ induces an isomorphism $\bar{H} \xrightarrow{\sim} \bar{H}$. □

EXERCISE 1.2.10. Let $H \subseteq \mathrm{GL}_n$ be a commutative subgroup. Then $\bar{H}$ is also commutative. If H is solvable, the so is $\bar{H}$.

EXERCISE 1.2.11. Let G be an algebraic group and $A \subseteq B \subseteq G$ "abstract" subgroups. If A is normal (resp. central) in B, then so is $\bar{A}$ in $\bar{B}$.

1.3. Comultiplication and coinverse. The multiplication $\mu\colon G \times G \to G$ induces in the usual way an algebra homomorphism (A.2.1)

$$\mu^*\colon \mathcal{O}(G) \to \mathcal{O}(G \times G) = \mathcal{O}(G) \otimes \mathcal{O}(G),$$
$$\mu^*(f)(g,h) := f(\mu(g,h)) = f(gh) \text{ for } f \in \mathcal{O}(G),\ g,h \in G$$

which is called *comultiplication*. Similarly, the isomorphism $\iota\colon G \xrightarrow{\sim} G$ taking inverses determines the *coinverse*

$$\iota^*\colon \mathcal{O}(G) \to \mathcal{O}(G), \quad \iota^*(f)(g) := f(g^{-1}).$$

EXAMPLE 1.3.1. For $G = \mathrm{GL}_n$

$$\mu^*\colon \mathbb{C}[x_{ij}, \det^{-1}] \to \mathbb{C}[x_{ij}, \det^{-1}] \otimes \mathbb{C}[x_{ij}, \det^{-1}]$$

is given by

$$x_{ij} \mapsto \sum_{k=1}^{n} x_{ik} \otimes x_{kj},$$

and

$$\iota^*\colon \mathbb{C}[x_{ij}, \det^{-1}] \to \mathbb{C}[x_{ij}, \det^{-1}] \text{ is given by } x_{ij} \mapsto (-1)^{i+j} \det^{-1} \cdot \det X_{ji}$$

where X_{rs} is the $(n-1) \times (n-1)$-submatrix obtained from $X = (x_{ij})$ by removing the rth row and the sth column. In particular, for $G = \mathbb{C}^*$, we have

$$\mu^*\colon \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}] \otimes \mathbb{C}[x, x^{-1}], \quad x \mapsto x \otimes x,$$

and

$$\iota^*\colon \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}], \quad x \mapsto x^{-1},$$

and for $G = \mathbb{C}^+$

$$\mu^*\colon \mathbb{C}[x] \to \mathbb{C}[x] \otimes \mathbb{C}[x], \quad x \mapsto x \otimes 1 + 1 \otimes x,$$

and

$$\iota^*\colon \mathbb{C}[x] \to \mathbb{C}[x], \quad x \mapsto -x.$$

We will see in section 4.3 (Corollary 4.3.5) that $\mathbb{C}^*$ and $\mathbb{C}^+$ are the only one-dimensional connected linear algebraic groups. A first result in this direction is formulated in the following exercise. Another will be given in Example 1.4.8 below.

EXERCISE 1.3.2. (1) The only algebraic group structure on the affine line $\mathbb{C}$ with identity element $e = 0$ is $\mathbb{C}^+$.
(Hint: If $g * h$ is such a multiplication, then $g * z = a(g)z + b(g)$ where $a(g) \in \mathbb{C}^*$ and $b(g) \in \mathbb{C}$. Show that (i) $b(g) = g$, (ii) $a(g)$ is regular, hence a constant, and (iii) $a(g) = 1$.)

(2) The only algebraic group structure on $\mathbb{C} \setminus \{0\}$ with identity $e = 1$ is $\mathbb{C}^*$.
(Hint: Prove and use that every automorphism of $\mathbb{C}^*$ is of the form $z \mapsto \lambda z$ or $z \mapsto \lambda z^{-1}$ where $\lambda \in \mathbb{C}^*$.)

(3) There is no algebraic group structure on $\mathbb{C} \setminus \{z_1, z_2, \dots, z_r\}$ for $r > 1$.
(Hint: Use that $\mathrm{Aut}(\mathbb{C} \setminus \{z_1, z_2, \dots, z_r\})$ is finite for $r > 1$, see Example 1.1.9.)

1.4. Connected component. Next we show that the underlying variety of an algebraic group is nonsingular. More precisely, we have the following result.

PROPOSITION 1.4.1. *An algebraic group G is a smooth variety and the irreducible components of G are its connected components, i.e. they are pairwise disjoint. In particular, G°, the connected component of the identity, is a normal subgroup of G which is both open and closed, the connected components of G are the cosets of G°, and the component group $\pi_0(G) := G/G^\circ$ is finite.*

PROOF. There is an open dense subset U in G which consists only of nonsingular points (Proposition A.4.4.2). Since left multiplication by an element $g \in G$ is an isomorphism, the open set gU also consists of nonsingular points, and the same holds for $\bigcup_{g\in G} gU = G$.

If $h \in G$ lies in exactly one irreducible component of G, then so does gh for every $g \in G$. Thus the irreducible components do not meet and are therefore the connected components.

Since G, as an algebraic variety, has finitely many irreducible components, it follows that G° is open and closed in G. For every $g \in G$ the set $gG^\circ g^{-1}$ is connected and meets G°, and so $gG^\circ g^{-1} = G^\circ$. For every $g \in G^\circ$ the closed subvariety gG° is irreducible and meets G°, hence $gG^\circ = G^\circ$. Therefore, G° is a normal subgroup of G of finite index.

Similarly, if C is an irreducible component of G and $g \in C$, then gG° is irreducible and meets C, and so $gG^\circ = C$. □

EXAMPLE 1.4.2. The groups GL_n and SL_n are both connected. This is clear for GL_n, and follows for SL_n from the fact that $\det -1$ is an irreducible polynomial (see Example 1.1.3(1)).

REMARK 1.4.3. The following is clear.

(1) For algebraic groups the notions *connected* and *irreducible* are equivalent.
(2) All local rings $\mathcal{O}_{G,g}$ (A.1.7.5) of an algebraic group G are isomorphic.
(3) Every closed subgroup $H \subseteq G$ of finite index contains G°. Every connected closed subgroup $H \subseteq G$ is contained in G°.

EXERCISE 1.4.4. Let G be an algebraic group. For $n \in \mathbb{N}$ denote by $G^{(n)} \subseteq G$ the set of elements of order n.

(1) Describe $G^{(2)}$ for $G = \mathrm{GL}_n$.
(2) Show that $G^{(n)}$ is closed in G. (Hint: For $G = \mathrm{GL}_n$ the subsets $G^{(n)}$ are finite unions of closed conjugacy classes.)

PROPOSITION 1.4.5. *Let G be an algebraic group with the property that all elements have finite order. Then G is finite.*

Proof. We can assume that G is connected. By assumption, $G = \bigcup_n G^{(n)}$ where $G^{(n)}$ is the set of elements of order n. By the previous Exercise 1.4.4 these sets are closed and so $G = G^{(n)}$ for some n, because an irreducible complex variety cannot be a countable union of strictly closed subsets (Proposition A.3.2.19). Thus $n = 1$ and $G = \{e\}$. $\square$

Exercise 1.4.6. Let G be a connected algebraic group. Then, for every $n \in \mathbb{Z}, n \neq 0$, the map $G \to G$, $g \mapsto g^n$, is dominant.
(Hint: Show that the fiber of e contains e as an isolated point.)

Exercise 1.4.7. If G is a commutative algebraic group, then $G^{(n)}$ is finite for all n. (Hint: Use the previous exercise to show that the homomorphism $g \mapsto g^n$ has a finite kernel.)

Example 1.4.8. A connected one-dimensional algebraic group G is commutative. In fact, by Proposition 1.4.5 above, the group G contains an element g of infinite order, and so $G = \overline{\langle g \rangle}$ which is a commutative group (Exercise 1.2.10).

The following lemma turns out to be very useful in many applications.

Lemma 1.4.9. *Let G be an algebraic group and let $X \subseteq G$ be a constructible dense subset. Then $G = X \cdot X^{-1} = X \cdot X$.*

Proof. X contains a subset U which is open and dense in G, and the same holds for $V := U \cap U^{-1}$. This implies that $gV \cap V \neq \emptyset$ for any $g \in G$, and so $g \in V \cdot V^{-1} = V \cdot V$. $\square$

Exercise 1.4.10. Let $X \subseteq \mathrm{GL}_n$ be an irreducible constructible subset containing the identity matrix E, and let $H := \langle X \rangle$ be the subgroup generated by X. Then H is closed and connected, and $H = \underbrace{X \cdot X \cdots X}_{N \text{ times}}$ for $N := 2 \dim H$.

Exercise 1.4.11. Let G be an algebraic group and $R \subseteq G$ a subset such that $G = \overline{\langle R \rangle}$. Then there are finitely many element $g_1, g_2, \ldots, g_m \in R$ such that $G = \overline{\langle g_1 \rangle} \cdot \overline{\langle g_2 \rangle} \cdots \overline{\langle g_m \rangle}$.

1.5. Exercises. For the convenience of the reader we collect here all exercises from the first section.

Exercise. (1) Show that the multiplication $\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n$ is a morphism of varieties.
(2) Show that left and right multiplication $\lambda_A \colon B \mapsto AB$ and $\rho_A \colon B \mapsto BA$ with a fixed matrix $A \in \mathrm{GL}_n$ are isomorphisms $\mathrm{GL}_n \xrightarrow{\sim} \mathrm{GL}_n$ of varieties.
(3) Show that inversion $A \mapsto A^{-1}$ is an isomorphism $\mathrm{GL}_n \xrightarrow{\sim} \mathrm{GL}_n$ of varieties. (Hint: Use Cramer's rule.)

Exercise. Show that the map $\varphi \colon T_n \times U_n \to B_n, (t, u) \mapsto tu$ is an isomorphism of algebraic varieties.

Exercise. (1) Show that every automorphism μ of the line $\mathbb{C}$ is an affine transformation, i.e. $\mu(x) = ax + b$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.
(2) If an automorphism μ of $\mathbb{C}$ has two or more fixed points, then $\mu = \mathrm{id}$.

Exercise. (1) The subgroup $\mu_n := \{t \in \mathbb{C}^* \mid t^n = 1\} \subseteq \mathbb{C}^*$ is finite and cyclic of order n, and the μ_n's exhaust all closed subgroups $\neq \mathbb{C}^*$.
(2) The set $\{g \in T_n \mid g \text{ has finite order}\}$ is a Zariski dense subgroup of T_n.

Exercise. A strict closed subgroup of $\mathbb{C}^+$ is trivial.

Exercise. If $F \subseteq \mathbb{C}$ is a finite subset of cardinality $|F| \geq 3$ in "general position", then $\mathrm{Aut}(\mathbb{C} \setminus F)$ is trivial.
(In "general position" means that for every $d \geq 3$ there is a dense open set $U \subseteq \mathbb{C}^d$ such that the claim holds for any F from U.)

Exercise. The subgroup $\mathrm{Int}(G) \subseteq \mathrm{Aut}(G)$ of inner automorphisms of a group G is a normal subgroup.

Exercise.
(1) For SL_2, the automorphism $A \mapsto A^{-t}$ is inner.
(2) Show that all automorphisms of SL_2 are inner.
(3) For GL_n, $n \geq 2$, and for SL_n, $n \geq 3$, the automorphism $A \mapsto A^{-t}$ it is not inner. (Hint: For GL_n, look at the determinant. For SL_n, if $A \mapsto A^{-t}$ is inner, then the composition of both is an automorphism of GL_n which is the identity on SL_n and the inverse on $\mathbb{C}^* E_n$. This leads to a contradiction as soon as $n > 2$.)

Exercise. Let V be a finite dimensional complex vector space. Then the underlying additive group V^+ is an algebraic group and $\mathrm{Aut}(V^+) = \mathrm{GL}(V)$.

Exercise. Let $H \subseteq \mathrm{GL}_n$ be a commutative subgroup. Then $\bar{H}$ is also commutative. If H is solvable, the so is $\bar{H}$.

Exercise. Let G be an algebraic group and $A \subseteq B \subseteq G$ "abstract" subgroups. If A is normal (resp. central) in B, then so is $\bar{A}$ in $\bar{B}$.

Exercise.
(1) The only algebraic group structure on the affine line $\mathbb{C}$ with identity element $e = 0$ is $\mathbb{C}^+$.
(Hint: If $g * h$ is such a multiplication, then $g * z = a(g)z + b(g)$ where $a(g) \in \mathbb{C}^*$ and $b(g) \in \mathbb{C}$. Show that (i) $b(g) = g$, (ii) $a(g)$ is regular, hence a constant, and (iii) $a(g) = 1$.)
(2) The only algebraic group structure on $\mathbb{C} \setminus \{0\}$ with identity $e = 1$ is $\mathbb{C}^*$.
(Hint: Prove and use that every automorphism of $\mathbb{C}^*$ is of the form $z \mapsto \lambda z$ or $z \mapsto \lambda z^{-1}$ where $\lambda \in \mathbb{C}^*$.)
(3) There is no algebraic group structure on $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_r\}$ for $r > 1$.
(Hint: Use that $\mathrm{Aut}(\mathbb{C} \setminus \{z_1, z_2, \ldots, z_r\})$ is finite for $r > 1$.)

Exercise. Let $H \subseteq \mathrm{GL}_n$ be a commutative subgroup. Then $\bar{H}$ is commutative. If H is solvable, the so is $\bar{H}$.

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Exercise. Let G be an algebraic group. For $n \in \mathbb{N}$ denote by $G^{(n)} \subseteq G$ the set of elements of order n.
(1) Describe $G^{(2)}$ for $G = \mathrm{GL}_n$.
(2) Show that $G^{(n)}$ is closed in G. (Hint: For $G = \mathrm{GL}_n$ the subsets $G^{(n)}$ are finite unions of closed conjugacy classes.)

Exercise. Let G be a connected algebraic group. Then, for every $n \in \mathbb{Z}, n \neq 0$, the map $G \to G$, $g \mapsto g^n$, is dominant.
(Hint: Show that the fiber of e contains e as an isolated point.)

Exercise. If G is a commutative algebraic group, then $G^{(n)}$ is finite for all n.
(Hint: Use the previous exercise to show that the homomorphism $g \mapsto g^n$ has a finite kernel.)

Exercise. Let $X \subseteq \mathrm{GL}_n$ be an irreducible constructible subset containing the identity matrix E, and let $H := \langle X \rangle$ be the subgroup generated by X. Then H is closed and connected, and $H = \underbrace{X \cdot X \cdots X}_{N \text{ times}}$ for $N := 2 \dim H$.

Exercise. Let G be an algebraic group and $R \subseteq G$ a subset such that $G = \overline{\langle R \rangle}$. Then there are finitely many element $g_1, g_2, \ldots, g_m \in R$ such that $G = \overline{\langle g_1 \rangle} \cdot \overline{\langle g_2 \rangle} \cdots \overline{\langle g_m \rangle}$.

2. Homomorphisms and Exponential Map

2.1. Homomorphisms. Let G, H be algebraic groups.

DEFINITION 2.1.1. A map $\varphi\colon G \to H$ is called a *regular group homomorphism* if φ is group homomorphism and a morphism of algebraic varieties.

In the following a *homomorphism* between algebraic groups always means a *regular group homomorphism* unless otherwise stated. A homomorphism φ is an *isomorphism* (see 1.2.2) if and only if φ is bijective and φ^{-1} is regular. We will see below that the second condition is automatically satisfied.

EXAMPLES 2.1.2.

(1) The determinant $\det\colon \mathrm{GL}_n \to \mathbb{C}^*$ is a surjective homomorphism.
(2) For every $n \in \mathbb{Z}$ the map $\mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^n$, is a homomorphism. For $n \neq 0$ it is surjective with kernel $\mu_n := \{z \in \mathbb{C}^* \mid z^n = 1\}$ which is a cyclic subgroup of order n.
(3) For any $a \in \mathbb{C} \setminus \{0\}$ the multiplication $a \cdot \mathrm{id}\colon \mathbb{C}^+ \to \mathbb{C}^+$ is an isomorphism.
(4) If $N \in \mathrm{M}_n$ is a nonzero nilpotent matrix, then the exponential map
$$s \mapsto \exp(sN) := \sum_{k=0}^{n-1} \frac{s^k}{k!} N^k$$
is an injective homomorphism $\mathbb{C}^+ \to \mathrm{GL}_n$. We will discuss this in more detail in Section 2.5.
(5) The canonical map $\begin{bmatrix} t_1 & * & * \\ & \ddots & * \\ & & t_n \end{bmatrix} \mapsto \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}$ is a surjective homomorphism $B_n \to T_n$ with kernel U_n.

EXERCISE 2.1.3. Every homomorphism $\alpha\colon \mathbb{C}^* \to \mathbb{C}^+$ is trivial, and the same holds for every homomorphism $\beta\colon \mathbb{C}^+ \to \mathbb{C}^*$.

PROPOSITION 2.1.4. *Suppose G and H are algebraic groups and $\varphi\colon G \to H$ is a homomorphism. Then the kernel $\ker\varphi$ is a closed subgroup of G and the image $\operatorname{im}\varphi = \varphi(G)$ is a closed subgroup of H. Moreover, if φ is bijective, then φ is an isomorphism.*

PROOF. Clearly, the kernel $\ker\varphi = \varphi^{-1}(e)$ is a closed subgroup. For the image we first remark that not only is $\varphi(G)$ a subgroup of H, but $\overline{\varphi(G)}$ is as well (Lemma 1.2.9). Also, $\varphi(G)$ contains an open dense subset U of $\overline{\varphi(G)}$, cf. A.3.4. For any $h \in \overline{\varphi(G)}$ the sets U and hU are open and dense in $\overline{\varphi(G)}$. Hence $U \cap hU \neq \emptyset$. Therefore there are elements $u, v \in U$ with $u = hv$. But then $h = uv^{-1} \in \varphi(G)$ and so $\varphi(G) = \overline{\varphi(G)}$.

Now assume that φ is bijective. Then the induced homomorphism $\varphi^\circ\colon G^\circ \to H^\circ$ is bijective as well, hence birational (Appendix A, Proposition A.2.3.4). Thus there are open sets $U \subseteq G^\circ$ and $V := \varphi(U) \subseteq H^\circ$ such that $\varphi|_U\colon U \xrightarrow{\sim} V$ is an isomorphism. But then, for every $g \in G$, $\varphi|_{gU}\colon gU \to \varphi(g)V$ is an isomorphism as well, and the claim follows because $G = \bigcup_{g\in G} gU$. $\square$

EXERCISE 2.1.5. For every $m \in \mathbb{Z}$, $m \neq 0$, the map $t \mapsto t^m\colon T_n \to T_n$ is a surjective homomorphism and a finite morphism.

EXERCISE 2.1.6. Let G be an algebraic group and let $N, H \subseteq G$ be closed subgroups where N is normal. If $N \cap H = \{e\}$ and if G is generated by $N \cup H$, then $G = NH = HN$, and the multiplication $N \times H \to G$ is an isomorphism of varieties. If, in addition, H is also normal, then N and H commute, and G is isomorphic to the product $N \times H$.

EXAMPLE 2.1.7. **The affine group** Aff_n**.** An automorphism φ of $\mathbb{C}^n$ is called an *affine transformation* if it is of the form

$$\varphi(x) = Ax + b \text{ where } A \in \mathrm{GL}_n \text{ and } b \in \mathbb{C}^n.$$

The group of affine transformations,

$$\mathrm{Aff}_n := \mathrm{Aff}_n(\mathbb{C}) := \{\beta(x) = Ax + b \mid A \in \mathrm{GL}_n, b \in \mathbb{C}^n\},$$

has a natural structure of an algebraic group. In fact, the matrix group

$$\left\{ \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_{n+1} \mid A \in \mathrm{GL}_n,\ b \in \mathbb{C}^n \right\} \subseteq \mathrm{GL}_{n+1}$$

is canonically isomorphic to Aff_n, by the map $\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \mapsto \beta\colon x \mapsto Ax + b$. Similarly, we define the group of affine transformations $\mathrm{Aff}(V)$ for any finite dimensional vector space V.

The *translations* $t_b\colon x \mapsto x + b$ form a closed normal subgroup isomorphic to $(\mathbb{C}^+)^n$, and there is a split exact sequence

$$0 \longrightarrow (\mathbb{C}^+)^n \xrightarrow{\iota} \mathrm{Aff}_n \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \mathrm{GL}_n \longrightarrow 1$$

where $\iota(b) := t_b$, $\pi(\beta) := A$ if $\beta(x) = Ax + b$, and $\sigma(A)(x) := Ax$.

Every automorphism of $\mathbb{C}$ is an affine transformation (see Exercise 1.1.6(1)) whereas for $\mathbb{C}^n, n > 1$, this is not the case. E.g. for every polynomial $f \in \mathbb{C}[x]$ the morphism $(x, y) \mapsto (x, y + f(x))$ is an automorphism of $\mathbb{C}^2$.

EXAMPLE 2.1.8. **The projective linear group.** Let $\varphi\colon \mathrm{GL}_n \to \mathrm{GL}(M_n)$ be the group homomorphism defined by $g \mapsto \mathrm{Int}(g)$. It is not difficult to see that φ is a morphism of varieties, hence a homomorphism of algebraic groups. For this, one calculates

$$gE_{ij}g^{-1} = \sum_{k,\ell} f_{ijk\ell}E_{k\ell}$$

and shows that $f_{ijk\ell} \in \mathcal{O}(\mathrm{GL}_n)$.

Moreover, $\ker\varphi = \mathbb{C}^*E$ and so $\mathrm{im}\,\varphi \simeq \mathrm{GL}_n / \mathbb{C}^*$. This group is the *projective linear group* and is denoted by $\mathrm{PGL}_n = \mathrm{PGL}_n(\mathbb{C})$.

EXERCISE 2.1.9. (1) Show that every morphism $\varphi\colon \mathbb{C}^* \to \mathbb{C}^*$ such that $\varphi(1) = 1$ is a group homomorphism. Determine the automorphism group of $\mathbb{C}^*$ (as an algebraic group).
(2) Show that every nontrivial group homomorphism $\mathbb{C}^+ \to \mathbb{C}^+$ is an isomorphism and determine the automorphism group of $\mathbb{C}^+$.

The proposition above has the following consequence which is the well-known *mapping property* in standard group theory.

PROPOSITION 2.1.10. *Let G, G', H be algebraic groups, $\varphi\colon G \to G'$ a surjective homomorphism and $\mu\colon G \to H$ a homomorphism such that* $\ker\mu \supseteq \ker\varphi$*. Then there is a unique homomorphism $\mu'\colon G' \to H$ such that $\mu = \mu' \circ \varphi$:*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "\mu"'] & G' \arrow[d, "\mu'"] \\ & H \end{tikzcd}$$

PROOF. Everything is clear except that μ' is a morphism. Consider the homomorphism $\psi\colon G \to G' \times H$, $\psi(g) := (\varphi(g), \mu(g))$. Then the image $\psi(G) \subseteq G' \times H$ is a closed subgroup, and $\ker\psi = \ker\varphi$. Therefore, the projection $\mathrm{pr}_{G'}\colon G' \times H \to \bar{G}$ induces a bijection $p\colon \psi(G) \to G'$. Hence, p is an isomorphism, and so $\mu' = \mathrm{pr}_H \circ p^{-1}$. The claim follows. $\square$

At this point we do not know how to form *quotient groups* G/N for a closed normal subgroup $N \subseteq G$, except if we can find a homomorphism $\varphi\colon G \to G'$ with kernel N, as in Example 2.1.8 above. But we will see later in chapter IV (see section IV.3.3) the for a so-called *linearly reductive* subgroup $H \subseteq G$ the left and the right cosets are affine varieties with the usual properties.

EXERCISE 2.1.11. Let H be an algebraic group, and let $\varphi\colon \mathrm{SL}_n \to H$ and $\lambda\colon \mathbb{C}^* \to H$ be homomorphisms. Assume that the images $\lambda(\mathbb{C}^*)$ and $\varphi(\mathrm{SL}_n)$ in H commute and that $\lambda(\zeta) = \varphi(\zeta E_n)$ for all $\zeta \in \mathbb{C}^*$ such that $\zeta^n = 1$. Then there exists a homomorphism $\tilde{\varphi}\colon \mathrm{GL}_n \to H$ such that $\tilde{\varphi}|_{\mathrm{SL}_n} = \varphi$ and $\tilde{\varphi}|_{\mathbb{C}^*} = \lambda$ where we identify $\mathbb{C}^*$ with $\mathbb{C}^* E_n \subseteq \mathrm{GL}_n$.

2.2. Characters and the character group. Let G be an algebraic group.

DEFINITION 2.2.1. A homomorphism $\chi\colon G \to \mathbb{C}^*$ is called a *character* of G. The set of characters is denoted by $\mathcal{X}(G)$:

$$\mathcal{X}(G) := \{\chi\colon G \to \mathbb{C}^* \mid \chi \text{ is a homomorphism}\}.$$

Characters can be multiplied, and so $\mathcal{X}(G)$ is a commutative group, the *character group* of G. It is usually written additively: $\chi_1 + \chi_2 \colon g \mapsto \chi_1(g)\chi_2(g)$. Every character is an invertible regular function on G and so $\mathcal{X}(G)$ is a subgroup of the group $\mathcal{O}(G)^*$ of invertible functions on G.

EXAMPLES 2.2.2.

(1) $\mathcal{X}(T_n) = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$ where the characters ε_i are defined by

$$\varepsilon_i(\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}) = t_i.$$

This follows from the fact that $\mathcal{O}(T_n) = \mathbb{C}[\varepsilon_1, \varepsilon_1^{-1}, \ldots, \varepsilon_n, \varepsilon_n^{-1}]$ which implies that the invertible function of $\mathcal{O}(T_n)$ are of the form $c\varepsilon_1^{a_1}\cdots\varepsilon_n^{a_n}$ where $c \in \mathbb{C}^*$ and $a_1, \ldots, a_n \in \mathbb{Z}$.

(2) $\mathcal{X}(U_n)$ is trivial, because the coordinate ring $\mathcal{O}(U_n)$ is a polynomial ring.

(3) $X(\mathrm{GL}_n) = \langle\det\rangle$. In fact, if $f \in \mathbb{C}[x_1, \ldots, x_n]$ is irreducible and non-constant, then $(\mathbb{C}[x_1, \ldots, x_n]_f)^* = \mathbb{C}^*\langle f\rangle$. Another argument will be given in Exercise 2.2.5 below.

EXERCISE 2.2.3. Show that $\mathcal{X}(T_n) \subseteq \mathcal{O}(T_n)$ is a $\mathbb{C}$-basis of the vector space $\mathcal{O}(T_n)$.

EXERCISE 2.2.4. Show that SL_2 is generated by U_2 and U_2^-. In particular, the character group $\mathcal{X}(\mathrm{SL}_2)$ is trivial.
(Hint: $U_2^- U_2 \subseteq \mathrm{SL}_2$ is closed and irreducible of dimension 2, $U_2 U_2^- U_2$ is strictly larger than $U_2^- U_2$, and therefore dense in SL_2. Now use Lemma 1.4.9.)

EXERCISE 2.2.5. Show that $\mathcal{X}(\mathrm{SL}_n)$ is trivial and deduce that $\mathcal{X}(\mathrm{GL}_n) \simeq \mathbb{Z}$ where $\mathcal{X}(\mathrm{GL}_n)$ is generated by $\det\colon \mathrm{GL}_n \to \mathbb{C}^*$.
(Hint: Use the previous Exercise 2.2.4 to show that $T_n' := T_n \cap \mathrm{SL}_n$ is contained in $\langle U_n^-, U_n\rangle$. Since $U_n^- T_n' U_n \subseteq \mathrm{SL}_n$ is dense, we get that $\mathcal{X}(\mathrm{SL}_n)$ is trivial. It follows that every character of GL_n vanishes on SL_n, and thus factors through $\det\colon \mathrm{GL}_n \to \mathbb{C}^*$.)

We have seen in Example 2.2.2(1) that the characters $\mathcal{X}(T_n)$ form a basis of $\mathcal{O}(T_n)$. The linear independence is a well-known general fact, see the following lemma. The fact that they linearly generate the coordinate ring characterizes the so-called *diagonalizable groups*, see Section III.3.3.

LEMMA 2.2.6. *The subset $\mathcal{X}(G) \subseteq \mathcal{O}(G)$ is linearly independent.*

PROOF. Let $\sum_{i=1}^n a_i\chi_i = 0$ be a nontrivial linear dependence relation of minimal length $n > 1$. Then $0 = \sum_{i=1}^n a_i\chi_i(hg) = \sum_{i=1}^n a_i\chi_i(h)\chi_i(g)$ for all $h, g \in G$, and so $\sum_{i=1}^n a_i\chi_i(h)\chi_i = 0$ for all $h \in G$. Thus

$$0 = \chi_1(h)\sum_{i=1}^n a_i\chi_i - \sum_{i=1}^n a_i\chi_i(h)\chi_i = \sum_{i=2}^n a_i(\chi_1(h) - \chi_i(h))\chi_i,$$

and this is, for a suitable $h \in G$, a nontrivial linear dependence relation of length $< n$, contradicting the assumption. $\square$

Every homomorphism $\varphi\colon G \to H$ of algebraic groups induces a homomorphism $\mathcal{X}(\varphi)\colon \mathcal{X}(H) \to \mathcal{X}(G)$ of the character groups: $\mathcal{X}(\varphi) := \varphi^*\colon \chi \mapsto \chi \circ \varphi$. More precisely, $\mathcal{X}$ is a *contravariant functor* from algebraic groups to abelian groups which means that $\mathcal{X}(\mathrm{id}_G) = \mathrm{id}_{\mathcal{X}(G)}$ and that $\mathcal{X}(\varphi \circ \psi) = \mathcal{X}(\psi) \circ \mathcal{X}(\varphi)$.

PROPOSITION 2.2.7. *The functor $G \mapsto \mathcal{X}(G)$ is left exact, i.e. for every exact sequence $K \xrightarrow{\psi} G \xrightarrow{\varphi} H \longrightarrow 1$ of algebraic groups the corresponding sequence of the character groups $0 \longrightarrow \mathcal{X}(H) \xrightarrow{\mathcal{X}(\varphi)} \mathcal{X}(G) \xrightarrow{\mathcal{X}(\psi)} \mathcal{X}(K)$ is exact.*

Recall that a sequence $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$ of groups and homomorphisms is called *exact* if $\ker \varphi = \operatorname{im} \psi$.

PROOF. Clearly, if $\varphi\colon G \to H$ is surjective, then $\mathcal{X}(\varphi)\colon \mathcal{X}(H) \to \mathcal{X}(G)$ is injective. Moreover, if $\chi \in \mathcal{X}(G)$ belongs to the kernel of $\mathcal{X}(\psi)$, then χ vanishes on $\operatorname{im} \psi = \ker \varphi$, and so the claim follows from the mapping property (Proposition 2.1.10). $\square$

EXERCISE 2.2.8. For two algebraic groups H, G we have $\mathcal{X}(H \times G) = \mathcal{X}(H) \oplus \mathcal{X}(G)$ in a canonical way.

EXERCISE 2.2.9. If G is a finite commutative group, then $\mathcal{X}(G) \simeq G$.
(Hint: Prove this first for a finite cyclic group G, and then use the previous exercise.)

REMARK 2.2.10. Let G be an algebraic group and $\mathcal{X}(G) \subseteq \mathcal{O}(G)^*$ the subgroup of characters. Then the linear span $\mathbb{C}\,\mathcal{X}(G) \subseteq \mathcal{O}(G)$ is a subalgebra, namely the *group algebra* of $\mathcal{X}(G)$. We will see later that the character group $\mathcal{X}(G)$ is always finitely generated, hence a free abelian if G is connected. Another interesting result is that for a connected algebraic group G every invertible $f \in \mathcal{O}(G)^*$ such that $f(e) = 1$ is a character. Both results are due to ROSENLICHT, and will be proved in III.5.7.

2.3. Normalizer, centralizer, and center. Let $H \subseteq G$ be a closed subgroup.

DEFINITION 2.3.1. The *normalizer* and the *centralizer* of H in G are defined by

$$\mathrm{N}_G(H) := \{g \in G \mid gHg^{-1} = H\},$$
$$\mathrm{C}_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\},$$

and the *centralizer* or *stabilizer* of an element $h \in G$ by

$$\mathrm{C}_G(h) := \{g \in G \mid gh = hg\}.$$

All three are closed subgroups of G, and H is normal in $\mathrm{N}_G(H)$.
(In fact, for any $h \in H$ define the morphism $\varphi_h\colon G \to G$ by $\varphi_h(g) := ghg^{-1}$. Then $\mathrm{C}_G(h) = \varphi_h^{-1}(h)$, $\mathrm{C}_G(H) = \bigcap_{h \in H} \mathrm{C}_G(h)$, and $\mathrm{N}_G(H) = N \cap N^{-1}$ where $N := \bigcap_{h \in H} \varphi_h^{-1}(H) = \{g \in G \mid gHg^{-1} \subseteq H\}$.)

REMARK 2.3.2. The centralizer $\mathrm{C}_G(H)$ is closed for any subgroup $H \subseteq G$. This does not hold for the normalizer as one sees from the example $\mathrm{N}_{\mathrm{GL}_2(\mathbb{C})}(U_2(\mathbb{Q}))$. Moreover, for a closed subgroup $H \subseteq G$ we have $\mathrm{N}_G(H) = \{g \in G \mid gHg^{-1} \subseteq H\}$ where again this might fail for arbitrary subgroups, e.g. for $U_2(\mathbb{Z}) \subseteq \mathrm{GL}_2(\mathbb{C})$.

EXAMPLE 2.3.3. We have $\mathrm{N}_{\mathrm{GL}_n}(U_n) = B_n$ and $\mathrm{N}_{\mathrm{GL}_n}(B_n) = B_n$.

PROOF. Using the row operations induced by left multiplication with matrices from U_n and the column operations induced by right multiplication with matrices from B_n, we see that any $g \in \mathrm{GL}_n$ can be reduced to a permutation matrix P_σ, i.e. $U_n g B_n = U_n P_\sigma B_n$ for a suitable $\sigma \in S_n$. On the other hand, $P_\sigma E_{ij} P_\sigma^{-1} = E_{\sigma i\, \sigma j}$, and so P_σ normalizes U_n or B_n if and only if $\sigma = \mathrm{id}$. □

EXERCISE 2.3.4. Show that the normalizer N_n of $T_n \subseteq \mathrm{GL}_n$ is generated by T_n and the permutation matrices $\mathcal{P}_n$: $\mathrm{N}_{\mathrm{GL}_n}(T_n) = \mathcal{P}_n \cdot T_n = T_n \cdot \mathcal{P}_n$, and this is a semidirect product. In particular, $N_n^\circ = T_n$ and $N_n/N_n^\circ \xrightarrow{\sim} S_n$.

EXERCISE 2.3.5. Describe the normalizer N of $T_n' := T_n \cap \mathrm{SL}_n$ in SL_n. Show that $N^\circ = T_n'$ and that $N/N^\circ \simeq S_n$. In this case, N is not a semidirect product, i.e. the exact sequence $1 \to N^\circ \to N \to \mathcal{S}_n \to 1$ does not split.

EXERCISE 2.3.6. Let G be an algebraic group. If $h \in G$ and $H := \overline{\langle h \rangle} \subseteq G$, then $\mathrm{C}_G(H) = \mathrm{C}_G(h)$.

EXERCISE 2.3.7. Show that the centralizer of T_n in GL_n is equal to T_n.

As is standard we define the *center* of a group G to be

$$\mathrm{Z}(G) := \{g \in G \mid gh = hg \text{ for every } h \in G\} = \mathrm{C}_G(G).$$

The center $\mathrm{Z}(G)$ of an algebraic group G is a *closed characteristic* subgroup of G. (Recall that a subgroup $H \subseteq G$ is called *characteristic* if H is stable under all automorphisms of G.)

EXAMPLE 2.3.8. Consider the group

$$\mathrm{O}_2(\mathbb{C}) = \mathrm{O}_2 := \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2 \mid A^t A = E \right\} \subseteq \mathrm{GL}_2 .$$

Obviously, $\det(\mathrm{O}_2) = \{\pm 1\}$, and so O_2 is not connected. It is easy to determine the center: $\mathrm{Z}(\mathrm{O}_2) = \{\pm E\}$. Let

$$\mathrm{SO}_2(\mathbb{C}) = \mathrm{SO}_2 := \{A \in \mathrm{O}_2 \mid \det A = 1\} = \mathrm{O}_2 \cap \mathrm{SL}_2 = \{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a^2 + b^2 = 1\}.$$

Then $\mathrm{O}_2 / \mathrm{SO}_2 \simeq \mathbb{Z}/2\mathbb{Z}$ *and* $\mathrm{SO}_2 \simeq \mathbb{C}^*$ (see the exercise below)*. In particular,* SO_2 *is connected.* Therefore we see that

$$(\mathrm{O}_2)^\circ = \mathrm{SO}_2,\ \mathrm{O}_2 = \mathrm{SO}_2 \cup \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathrm{SO}_2, \text{ and}$$

$$\pi_\circ(\mathrm{O}_2) = \mathrm{O}_2 / \mathrm{SO}_2 \simeq \mathbb{Z}/2\mathbb{Z}.$$

We will see in section 3.2 that similar results hold for all orthogonal groups.

EXERCISE 2.3.9. Show that the map $\mathrm{SO}_2 \to \mathbb{C}^*\colon \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a + ib$, defines an isomorphism of algebraic groups.

2.4. Commutator subgroup. The subgroup of G generated by all *commutators* $(g,h) := ghg^{-1}h^{-1}$, $g,h \in G$, is called the *commutator subgroup* (or *derived group*) of G and will be denoted by (G,G). It is well-known that (G,G) is the (unique) smallest normal subgroup of G such that the quotient group $G/(G,G)$ is commutative.

PROPOSITION 2.4.1. *The commutator subgroup* (G,G) *of an algebraic group* G *is a closed subgroup of* G.

PROOF. Let (G, G°) denote the subgroup of G which is generated by commutators of the form $(g,h) := ghg^{-1}h^{-1}$ with $g \in G$ and $h \in G^\circ$. This subgroup is normal in G and has finite index in (G,G): Because the image of G° in $G' := G/(G,G^\circ)$ is central and of finite index, it follow that the commutator subgroup (G', G') is finite. (This is a general group theoretic fact, see Exercise 2.4.2(1) below.) Thus it suffices to show that (G, G°) is closed in G.

For any $2n$-tuple $(g_1, ..., g_{2n})$ of elements of G we define the following subset of (G, G°):

$$\begin{aligned} K &= K(g_1, ..., g_{2n}) \\ &:= \{(g_1, h_1)(g_2, h_2)^{-1}(g_3, h_3)\cdots(g_{2n}, h_{2n})^{-1} \mid h_i \in G^\circ\}. \end{aligned}$$

This subset K is the image of a morphism $(G^\circ)^{2n} \to G$, and so its closure $\bar{K}$ is irreducible, and K contains an open dense subset of $\bar{K}$ (A.3.4). Choose a $K = K(g_1, ..., g_{2n})$ with $\dim \bar{K}$ maximal. For an arbitrary tuple $(g'_1, ..., g'_{2m})$ one has

$$K(g'_1, ...g'_{2m}) \subseteq K(g'_1, ..., g'_{2m}, g_1, ..., g_{2n})$$

and thus $K(g'_1, ..., g'_{2m}) \subseteq \bar{K}$. Since $(G, G^\circ) = \bigcup K(g'_1, ..., g'_{2m})$ it follows that $\bar{K} = \overline{(G, G^\circ)}$. In particular, $\bar{K}$ is a closed subgroup (Lemma 1.2.9). If $g \in \overline{(G, G^\circ)}$, then $gK \cap K \neq \emptyset$ since K contains an open dense subset of $\overline{(G, G^\circ)}$. This implies that $g \in KK^{-1} \subseteq (G, G^\circ)$ and thus $(G, G^\circ) = \overline{(G, G^\circ)}$. □

EXERCISE 2.4.2.
(1) Let H be a (abstract) group and assume that the center of H has finite index in H. Then the commutator subgroup (H, H) is finite. (See [**Hum75**, VII.17.1 Lemma A].)
(2) Show that $(\mathrm{GL}_2, \mathrm{GL}_2) = (\mathrm{SL}_2, \mathrm{SL}_2) = \mathrm{SL}_2$ and that $(B_2, B_2) = U_2$.

2.5. Exponential map. For every complex matrix A the exponential series $\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!}A^k$ is a well-defined invertible matrix, and the map $A \mapsto \exp(A)$ has a number of nice properties, e.g. $\exp(A+B) = \exp(A)\exp(B)$ in case A and B commute (cf. [**Art91**, Chap. 4, Sec. 8]). If N is nilpotent, then $\exp(N)$ is a finite sum of at most n terms, and so the map $\exp\colon \mathcal{N}_n \to \mathrm{GL}_n$ is a morphism where $\mathcal{N}_n \subseteq \mathrm{M}_n$ denotes the closed subset of nilpotent matrices.

A matrix A is called *unipotent* if $A - E$ is nilpotent, or equivalently, if all eigenvalues are equal to 1. Thus the set $\mathcal{U}_n = E + \mathcal{N}_n \subseteq \mathrm{GL}_n$ of unipotent matrices is a closed subset, and the image of $\mathcal{N}_n$ under exp is contained in $\mathcal{U}_n$.

PROPOSITION 2.5.1. *The exponential map $N \mapsto \sum_{k=0}^{n-1} \frac{1}{k!}N^k$ induces an isomorphism*

$$\exp\colon \mathcal{N}_n \xrightarrow{\sim} \mathcal{U}_n$$

which commutes with conjugation: $\exp(gNg^{-1}) = g\exp(N)g^{-1}$.

The proof needs some preparation. Let R be a finite dimensional associative $\mathbb{C}$-algebra. Every polynomial $p(x) \in \mathbb{C}[x]$ defines a morphism of varieties $p\colon R \to R$, $r \mapsto p(r)$. Now take $R := \mathbb{C}[t]/(t^n)$, and let $\mathfrak{m} := (\bar{t}) \subseteq R$ be the maximal ideal where $\bar{t} := t + (t^n) \in R$. Then $1 + \mathfrak{m}$ is a closed subgroup of the group R^* of invertible elements of R, and the polynomial $e(x) := \sum_{k=0}^{n-1} \frac{1}{k!}x^k \in \mathbb{C}[x]$ induces a morphism $e\colon \mathfrak{m} \to 1 + \mathfrak{m}$, $a \mapsto \sum_{k=0}^{n-1} \frac{1}{k!}a^k$.

LEMMA 2.5.2. *The map $e\colon \mathfrak{m} \to 1 + \mathfrak{m}$ is an isomorphism of algebraic groups. Its inverse $e^{-1}\colon 1 + \mathfrak{m} \to \mathfrak{m}$ is of the form $1 + a \mapsto l(a)$ with a polynomial $l(x)$ of degree $< n$. In particular, $e(l(x)) = 1 + x \mod x^n$ and $l(e(x) - 1) = x \mod x^n$.*

PROOF. It is easy to see that $e(a+b) = e(a)e(b)$ for $a, b \in \mathfrak{m}$, and so $e\colon \mathfrak{m} \to 1 + \mathfrak{m}$ is a homomorphism of algebraic groups. Moreover, e is injective, because

$e(a_k\bar{t}^k + \text{higher degree}) = 1 + a_k\bar{t}^k + \text{higher degree}$. Since $\dim(1+\mathfrak{m}) = \dim\mathfrak{m}$ we see that e is also surjective, hence an isomorphism.

Set $l := e^{-1}(1+\bar{t}) \in \mathfrak{m}$. Then $e(l) = 1+\bar{t}$. If we consider l as a polynomial in x of degree $< n$ this means that $e(l(x)) = 1 + x \mod x^n$. Therefore, for any $h \in (x) \subseteq \mathbb{C}[x]$, we have $e(l(h)) = 1 + h \mod x^n$ which implies that $e(l(a)) = 1+a$ for all $a \in \mathfrak{m}$. This shows that the map $1 + a \mapsto l(a)$ is the inverse morphism to e. Thus $l(e(h)-1) = h$ for all $h \in \mathfrak{m}$ and so $l(e(x)-1) = x \mod x^n$. □

Proof of Proposition 2.5.1. The lemma above shows that $e(l(x)) = 1+x \mod x^n$ and $l(e(x)-1) = x \mod x^n$. This implies that the maps $\exp\colon \mathcal{N}_n \to \mathcal{U}_n$, $N \mapsto e(N)$, and $\log\colon \mathcal{U}_n \to \mathcal{N}$, $1 + M \mapsto l(M)$ are inverse to each other, and the claim follows. □

Remarks 2.5.3.
(1) It is well-known that the inverse function log of e^x is given by $\log(1+y) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} y^k$. It follows that $\log\colon \mathcal{U}_n \to \mathcal{N}_n$ is given by
$$U \mapsto \log(U) := \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k}(U-1)^k.$$
Since we will not need this explicit formula we leave the proof to the reader (see Exercise 2.5.4 below).
(2) Denote by $\mathfrak{n}_n \subseteq \mathrm{M}_n$ the subspace of upper triangular nilpotent matrices. It follows from the above that exp induces an isomorphism $\mathfrak{n}_n \xrightarrow{\sim} U_n$. (In fact, since exp and log are given by polynomials e and l it is obvious that the image of an upper triangular matrix under both maps is again upper triangular.)

Exercise 2.5.4. Define the polynomials
$$E_n(x) := \sum_{k=0}^{n} \frac{1}{k!}x^k \quad \text{and} \quad L_n(x) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$$
and show that $E(L(x)) = x \mod x^{n+1}$ and $L(E(x)) = x \mod x^{n+1}$.
(Hint: For all $z \in \mathbb{C}$ we have $e^z = L_n(z) + z^{n+1}h(z)$ with a holomorphic function h, and for all y in a neighborhood U of $1 \in \mathbb{C}$ we have $\ln(y) = L_n(y) + (y-1)^{n+1}g(y)$ with g holomorphic in U. Now use that $e^{\ln y} = y$ in U and $\ln(e^z) = z$ in $U' := \ln(U)$.)

2.6. Unipotent elements. Let $u \in \mathrm{GL}_n$ be a unipotent matrix $\neq E_n$. It is clear that $\langle u\rangle$ and the closure $\overline{\langle u\rangle}$ consist of unipotent matrices. In fact, the latter is isomorphic to $\mathbb{C}^+$ as we will see now.

Proposition 2.6.1. *For any nilpotent $N \in \mathrm{M}_n$ the map $\alpha_N\colon \mathbb{C}^+ \to \mathrm{GL}_n$ given by $s \mapsto \exp(sN)$ is a homomorphism with image $\overline{\langle u\rangle}$ where $u := \exp(N)$, and $d\alpha_N(1) = N$. For $N \neq 0$ it is an isomorphism onto its image. In addition, every homomorphism $\alpha\colon \mathbb{C}^+ \to \mathrm{GL}_n$ is of the form α_N where $N = \frac{\alpha(s)-E}{s}|_{s=0}$.*

Proof. Clearly, α_N is a homomorphism. By definition, $\alpha_N(s) = E + sN + \frac{1}{2}s^2N^2 + \cdots$, and so $\alpha_N\colon \mathbb{C} \to \mathrm{M}_n$ is a closed immersion in case $N \neq 0$. Moreover, $d\alpha_N(1) = \frac{\alpha_N(s)-\alpha_N(0)}{s}|_{s=0} = N$. The image contains $u := \alpha_N(1) = \exp(N) \neq E$, hence $\operatorname{im}\alpha_N = \overline{\langle u\rangle}$. This proves the first part of the proposition.

Now let $\alpha\colon \mathbb{C}^+ \to \mathrm{GL}_n$ be an arbitrary homomorphism. The underlying morphism $\alpha\colon \mathbb{C} \to \mathrm{M}_n$ has the form $\alpha(s) = A_0 + sA_1 + s^2A_2 + \cdots + s^mA_m$ with suitable matrices $A_0, A_1, \ldots, A_m \in \mathrm{M}_n$. Since $\alpha(0) = E$ we get $A_0 = E$, and $\alpha(2s) = \alpha(s)^2$ gives the following relations:
$$\sum_{0\le j\le k} A_jA_{k-j} = 2^kA_k \text{ for all } k \ge 0$$

where $A_j = 0$ for $j > m$. It is not hard to see that this implies, by induction, that $A_j = \frac{1}{j!}A_1^j$, hence $\alpha = \alpha_{A_1}$. $\square$

COROLLARY 2.6.2. *An element $u \in \mathrm{GL}_n$, $u \neq E$, is unipotent if and only if $\overline{\langle u \rangle} \simeq \mathbb{C}^+$.*

This allows to define unipotent elements of an arbitrary algebraic group.

DEFINITION 2.6.3. An element u of an algebraic group G is called *unipotent* if either $u = e$ or $\overline{\langle u \rangle} \simeq \mathbb{C}^+$. If all elements of G are unipotent, then G is called a *unipotent group*.

EXAMPLE 2.6.4. Clearly, the groups U_n are unipotent, as well as every closed subgroup of them. The *vector groups* V^+ where V is a finite dimensional $\mathbb{C}$-vector space are examples of commutative unipotent groups (see Exercise 1.2.8). We will see later that every commutative unipotent group is isomorphic to a vector group (Proposition III.4.3.2).

If $\varphi\colon G \to H$ is a homomorphism and if $u \in G$ is unipotent, then $\varphi(u) \in H$ is unipotent. Embedding G into GL_n we also see that the set $G_u \subseteq G$ of unipotent elements of G is a closed subset. If G is commutative, then G_u is even a closed subgroup, because the product of two commuting unipotent elements is again unipotent (see Exercise 2.6.6 below).

EXERCISE 2.6.5. Let U be a unipotent group. Then the power maps $p_m\colon U \to U$ for $m \neq 0$ are isomorphisms of varieties .
(Hint: This is clear for $U \simeq \mathbb{C}^+$. From that one can deduce that p_m is bijective, and the claim follows, e.g. from IGUSA's Lemma A.5.6.5.)

EXERCISE 2.6.6. Let G be an algebraic group, and let $u, v \in G$ be two commuting unipotent elements. Then uv is unipotent.
(Hint: It suffices to prove this for $G = \mathrm{GL}_n$. Then $u = E + N$ and $v = E + M$ with commuting nilpotent matrices N, M.)

EXERCISE 2.6.7. Let $N \in \mathrm{M}_n$ be nilpotent. Then the matrix $N' := \exp(N) - E$ is conjugate to N.
(Hint: $N' = Ng = gN$ with an invertible $g \in \mathrm{GL}_n$. Since g commutes with N it follows that gN is conjugate to N.)

EXERCISE 2.6.8.
(1) For GL_n the power map $p_m\colon g \mapsto g^m$ is surjective for $m \neq 0$.
(Hint: One can assume that g is in Jordan normal form, $g = tu$, where t is diagonal, u unipotent, and $tu = ut$. Then there is a subtorus $T \subseteq T_n$, $T \xrightarrow{\sim} \mathbb{C}^{*r}$, which commutes with u and contains t. Hence $g \in T \times \overline{\langle u \rangle} \xrightarrow{\sim} \mathbb{C}^{*r} \times \mathbb{C}^+$, and the claim follows from the Exercises 2.6.5 and 2.1.5.)
(2) Let G be an abstract group and $Z \subseteq G$ its center. If p_m is surjective for G, then so is for G/Z. If p_m is surjective for Z and G/Z, then so is for G.
(3) Study the power maps p_m for SL_2. Are they surjective for $m \neq 0$?

2.7. Exercises. For the convenience of the reader we collect here all exercises from the second section.

EXERCISE. Every homomorphism $\alpha\colon \mathbb{C}^* \to \mathbb{C}^+$ is trivial, and the same holds for every homomorphism $\beta\colon \mathbb{C}^+ \to \mathbb{C}^*$.

EXERCISE. For every $m \in \mathbb{Z}$, $m \neq 0$, the map $t \mapsto t^m\colon T_n \to T_n$ is a surjective homomorphism.

EXERCISE. Let G be an algebraic group and let $N, H \subseteq G$ be closed subgroups where N is normal. If $N \cap H = \{e\}$ and if G is generated by $N \cup H$, then $G = NH = HN$, and the multiplication $N \times H \to G$ is an isomorphism of varieties. If, in addition, H is also normal, then G is isomorphic to the product $N \times H$.

Exercise. Show that $\mathrm{Aut}(\mathbb{A}^1) = \mathrm{Aff}_1$.

Exercise. (1) Show that every morphism $\varphi\colon \mathbb{C}^* \to \mathbb{C}^*$ such that $\varphi(1) = 1$ is a group homomorphism. Determine the automorphism group of $\mathbb{C}^*$ (as an algebraic group).
(2) Show that every nontrivial group homomorphism $\mathbb{C}^+ \to \mathbb{C}^+$ is an isomorphism and determine the automorphism group of $\mathbb{C}^+$.

Exercise. Let H be an algebraic group, and let $\varphi\colon \mathrm{SL}_n \to H$ and $\lambda\colon \mathbb{C}^* \to H$ be homomorphisms. Assume that the images $\lambda(\mathbb{C}^*)$ and $\varphi(\mathrm{SL}_n)$ in H commute and that $\lambda(\zeta) = \varphi(\zeta E_n)$ for all $\zeta \in \mathbb{C}^*$ such that $\zeta^n = 1$. Then there exists a homomorphism $\tilde{\varphi}\colon \mathrm{GL}_n \to H$ such that $\tilde{\varphi}|_{\mathrm{SL}_n} = \varphi$ and $\tilde{\varphi}|_{\mathbb{C}^*} = \lambda$ where we identify $\mathbb{C}^*$ with $\mathbb{C}^* E_n \subseteq \mathrm{GL}_n$.

Exercise. Show that $\mathcal{X}(T_n) \subseteq \mathcal{O}(T_n)$ is a $\mathbb{C}$-basis of the vector space $\mathcal{O}(T_n)$.

Exercise. Show that SL_2 is generated by U_2 and U_2^-. In particular, the character group $\mathcal{X}(\mathrm{SL}_2)$ is trivial.
(Hint: $U_2^- U_2 \subseteq \mathrm{SL}_2$ is closed and irreducible of dimension 2, $U_2 U_2^- U_2$ is strictly larger than $U_2^- U_2$, and therefore dense in SL_2. Now use Lemma 1.4.9.)

Exercise. Show that $\mathcal{X}(\mathrm{SL}_n)$ is trivial and deduce that $\mathcal{X}(\mathrm{GL}_n) \simeq \mathbb{Z}$ where $\mathcal{X}(\mathrm{GL}_n)$ is generated by $\det\colon \mathrm{GL}_n \to \mathbb{C}^*$.
(Hint: Use the previous Exercise 2.2.4 to show that $T_n' := T_n \cap \mathrm{SL}_n$ is contained in $\langle U_n^-, U_n\rangle$. Since $U_n^- T_n' U_n \subseteq \mathrm{SL}_n$ is dense, we get that $\mathcal{X}(\mathrm{SL}_n)$ is trivial. It follows that every character of GL_n vanishes on SL_n, and thus factors through $\det\colon \mathrm{GL}_n \to \mathbb{C}^*$.)

Exercise. For two algebraic groups H, G we have $\mathcal{X}(H \times G) = \mathcal{X}(H) \oplus \mathcal{X}(G)$ in a canonical way.

Exercise. If G is a finite commutative group, then $\mathcal{X}(G) \simeq G$.
(Hint: Prove this first for a finite cyclic group G, and then use the previous exercise.)

Exercise. Show that the normalizer N_n of $T_n \subseteq \mathrm{GL}_n$ is generated by T_n and the permutation matrices $\mathcal{P}_n$: $\mathrm{N}_{\mathrm{GL}_n}(T_n) = \mathcal{P}_n \cdot T_n = T_n \cdot \mathcal{P}_n$, and this is a semidirect product. In particular, $N_n^\circ = T_n$ and $N_n/N_n^\circ \xrightarrow{\sim} S_n$.

Exercise. Describe the normalizer N of $T_n' := T_n \cap \mathrm{SL}_n$ in SL_n. Show that $N^\circ = T_n'$ and that $N/N^\circ \simeq S_n$. In this case, N is not a semidirect product, i.e. the exact sequence $1 \to N^\circ \to N \to \mathcal{S}_n \to 1$ does not split.

Exercise. Let G be an algebraic group. If $g \in G$ and $H := \overline{\langle g\rangle} \subseteq G$, then $\mathrm{C}_G(H) = \mathrm{C}_G(g)$.

Exercise. Show that the centralizer of T_n in GL_n is equal to T_n.

Exercise. Show that the map $\mathrm{SO}_2 \to \mathbb{C}^*\colon \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a+ib$, defines an isomorphism of algebraic groups.

Exercise. (1) Let H be a (abstract) group and assume that the center of H has finite index in H. Then the commutator subgroup (H, H) is finite. (See [**Hum75**, VII.17.1 Lemma A].)
(2) Show that $(\mathrm{GL}_2, \mathrm{GL}_2) = (\mathrm{SL}_2, \mathrm{SL}_2) = \mathrm{SL}_2$ and that $(B_2, B_2) = U_2$.

Exercise. Define the polynomials

$$E_n(x) := \sum_{k=0}^{n} \frac{1}{k!} x^k \quad \text{and} \quad L_n(x) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (x-1)^k$$

and show that $E(L(x)) = x \mod x^{n+1}$ and $L(E(x)) = x \mod x^{n+1}$.
(Hint: For all $z \in \mathbb{C}$ we have $e^z = L_n(z) + z^{n+1} h(z)$ with a holomorphic function h, and for all y in a neighborhood U of $1 \in \mathbb{C}$ we have $\ln(y) = L_n(y) + (y-1)^{n+1} g(y)$ with g holomorphic in U. Now use that $e^{\ln y} = y$ in U and $\ln(e^z) = z$ in $U' := \ln(U)$.)

EXERCISE. Let U be a unipotent group. Then the power maps $p_m \colon U \to U$ for $m \neq 0$ are isomorphisms of varieties .
(Hint: This is clear for $U \simeq \mathbb{C}^+$. From that one can deduce that p_m is bijective, and the claim follows, e.g. from IGUSA's Lemma A.5.6.5.)

EXERCISE. Let G be an algebraic group, and let $u, v \in G$ be two commuting unipotent elements. Then uv is unipotent.
(Hint: It suffices to prove this for $G = \mathrm{GL}_n$. Then $u = E + N$ and $v = E + M$ with commuting nilpotent matrices N, M.)

EXERCISE. Let $N \in \mathrm{M}_n$ be nilpotent. Then the matrix $N' := \exp(N) - E$ is conjugate to N.
(Hint: $N' = Ng = gN$ with an invertible $g \in \mathrm{GL}_n$. Since g commutes with N it follows that gN is conjugate to N.)

EXERCISE.
1. For GL_n the power map $p_m \colon g \mapsto g^m$ is surjective for $m \neq 0$. (Hint: One can assume that g is in Jordan normal form, $g = tu$, where t is diagonal, u unipotent, and $tu = ut$. Then there is a subtorus $T \subseteq T_n$, $T \xrightarrow{\sim} \mathbb{C}^{*r}$, which commutes with u and contains t. Hence $g \in T \times \overline{\langle u \rangle} \xrightarrow{\sim} \mathbb{C}^{*r} \times \mathbb{C}^+$, and the claim follows from the Exercises 2.6.5 and 2.1.5.)
2. Let G be an abstract group and $Z \subseteq G$ its center. If p_m is surjective for G, then so is for G/Z. If p_m is surjective for Z and G/Z, then so is for G.
3. Study the power maps p_m for SL_2. Are they surjective for $m \neq 0$?

3. The Classical Groups

3.1. General and special linear groups. In order to describe a set of generators for GL_n and SL_n we consider the following matrices:

$$\begin{aligned} u_{ij}(s) &:= E + sE_{ij} \in U_n^- \cup U_n \subseteq \mathrm{SL}_n, \quad 1 \leq i, j \leq n, i \neq j, \\ t_i(t) &:= E + (t-1)E_{ii} \in T_n \subseteq \mathrm{GL}_n, \quad 1 \leq i \leq n, \\ t_{ij}(t) &:= t_i(t) t_j(t^{-1}) \in T_n' \subseteq \mathrm{SL}_n, \quad 1 \leq i, j \leq n, i \neq j, \end{aligned}$$

$$u_{ij}(s) = \begin{bmatrix} 1 & & & \\ & \ddots & s & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad t_i(t) = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & t & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix},$$

Then we have

$$\begin{aligned} \mathrm{GL}_n &= \langle u_{ij}(s), t_k(t) \mid 1 \leq i, j, k \leq n, i \neq j, s, t \in \mathbb{C}, t \neq 0 \rangle, \\ \mathrm{SL}_n &= \langle u_{ij}(s), t_{kl}(t) \mid 1 \leq i, j, k, l \leq n, i \neq j, k \neq l, s, t \in \mathbb{C}, t \neq 0 \rangle, \\ U_n &= \langle u_{ij}(s) \mid i < j, s \in \mathbb{C} \rangle, \quad U_n^- = \langle u_{ij}(s) \mid i > j, s \in \mathbb{C} \rangle. \end{aligned}$$

PROOF. The matrix $A' := u_{ij}(s)A$ is obtained from A by an elementary row operation, i.e., by adding s times the jth row to the ith row and leaving all others unchanged. Similarly, right multiplication by $u_{ij}(s)$ corresponds to an elementary column operation. This implies that

$$U_n = \langle u_{ij}(s) \mid i < j, s \in \mathbb{C} \rangle \quad \text{and} \quad U_n^- = \langle u_{ij}(s) \mid i > j, s \in \mathbb{C} \rangle.$$

The map $U_n^- \times T_n \times U_n \to \mathrm{GL}_n$, $(u, t, v) \mapsto utv$, is injective, hence has a dense image $X := U_n^- \cdot T_n \cdot U_n \subseteq \mathrm{GL}_n$. By Lemma 1.4.9 we get $X \cdot X = \mathrm{GL}_n$, and the statement for GL_n follows. Replacing T_n by T_n' we get the claim for SL_n. □

REMARK 3.1.1. We know that $\mathrm{SL}_2 = \langle u_{12}(s), u_{21}(s) \mid s \in \mathbb{C}\rangle$ (Exercise 2.2.4). This implies that $t_{ij}(t) \in \langle u_{ij}(s), u_{ji}(s) \mid s \in \mathbb{C}\rangle$, hence $T_n' \subseteq \langle U_n, U_n^-\rangle$. It follows that

$$\mathrm{SL}_n = \langle u_{ij}(s) \mid 1 \leq i, j \leq n, i \neq j, s \in \mathbb{C}\rangle.$$

For the centers of GL_n and SL_n, an easy calculation shows that

$$\mathrm{Z}(\mathrm{GL}_n) = \{\lambda E \mid \lambda \in \mathbb{C}^*\} \simeq \mathbb{C}^* \quad \text{and} \quad \mathrm{Z}(\mathrm{SL}_n) = \{\lambda E \mid \lambda^n = 1\} \simeq \mathbb{Z}/n\mathbb{Z}.$$

This is also an immediate consequence of the Lemma of SCHUR (see Lemma 3.1.4 below). We call a subgroup $G \subseteq \mathrm{GL}(V)$ *irreducible* if no nontrivial proper subspace $W \subsetneq V$ is stable under G (i.e. satisfies $gW \subseteq W$ for all $g \in G$).

EXAMPLE 3.1.2. If $G \subseteq \mathrm{GL}_2$ is a noncommutative finite subgroup, then G is irreducible. Otherwise, there is a one-dimensional G-stable subspace $U = \mathbb{C}v \subseteq \mathbb{C}^2$, which admits a G-stable complement $V = \mathbb{C}v \subseteq \mathbb{C}^2$, because G is finite. With respect to the new basis (u, v) the elements of G are diagonal matrices, contradicting the assumption.

Other examples of irreducible subgroups $G \subseteq \mathrm{GL}(V)$ are those where G acts transitively on $V \setminus \{0\}$.

EXERCISE 3.1.3. Let $G \subseteq \mathrm{GL}_n$ be irreducible. Then $G^t := \{g^t \mid g \in G \subseteq \mathrm{GL}_n$ is irreducible.
(Hint: If $U \subseteq \mathbb{C}^n$ is stable under G^t, then $U^\perp := \{v \in \mathbb{C}^n \mid u^t\, v = 0 \text{ for all } u \in U\}$ is G-stable.)

LEMMA 3.1.4 (Lemma of SCHUR). *Let $G \subseteq \mathrm{GL}(V)$ be an irreducible subgroup. Then every linear map $\varphi\colon V \to V$ commuting with G is a scalar multiplication. In particular, $\mathrm{C}_{\mathrm{GL}(V)}(G) = \mathbb{C}^* \mathrm{id}_V$ and $\mathrm{Z}(G) = G \cap \mathbb{C}^* \mathrm{id}_V$.*

PROOF. Let $\varphi \in \mathrm{End}(V)$ commuting with G, and let $W \subseteq V$ be the eigenspace of φ corresponding to an eigenvalue λ. Since φ commutes with G the subspace W is stable under all $g \in G$. In fact, if $w \in W$ and $g \in G$, then $\varphi(gw) = g\varphi(w) = g(\lambda w) = \lambda(gw)$, and so $gw \in W$. Hence $W = V$ and so $\varphi = \lambda \, \mathrm{id}_V$. □

An interesting application is given in the following lemma. It will follow again later in the context of representation theory (see Corollary III.1.2.5). For a subset $X \subseteq \mathrm{End}(V)$ we denote by $\langle X\rangle \subseteq \mathrm{End}(V)$ the *linear span of* X, i.e.

$$\langle X\rangle := \{\sum_i \lambda_i x_i \mid \lambda_i \in \mathbb{C},\ x_i \in X\}.$$

If $G \subseteq \mathrm{GL}(V)$ is a subgroup, then $\langle G\rangle \subseteq \mathrm{End}(V)$ is a subalgebra which is stable under left- and right multiplication by G.

LEMMA 3.1.5. *A subgroup $G \subseteq \mathrm{GL}(V)$ is irreducible if and only if $\langle G\rangle = \mathrm{End}(V)$.*

PROOF. We can assume that $V = \mathbb{C}^n$, and so $G \subseteq \mathrm{GL}_n$.

(1) If $A \in \mathrm{M}_n$ is a matrix of rank 1, then $\langle GAG\rangle = \mathrm{M}_n$. In fact, $A = uv^t$ for some nonzero vectors $u, v \in \mathbb{C}^n$, and so $\langle GAG\rangle \supseteq \langle Gu\rangle \cdot \langle (G^t v)^t\rangle$. Since $\langle Gu\rangle = \mathbb{C}^n = \langle G^t v\rangle$, this shows that $\langle GAG\rangle$ contains all matrices of rank 1, and the claim follows.

(2) Denote by $p_i\colon \mathrm{M}_n \to \mathbb{C}^n$ the projection onto the i-th column. If $U \subseteq \mathrm{M}_n$ is stable under left multiplication by G, then the same holds for the image $p_i(U)$ and for the kernel of $p_i|_U$. It follows that $p_i(U) = \mathbb{C}^n$ or $\{0\}$. If $U \neq (0)$ this implies that U contains a subspace V which is stable under left multiplication with G such that, for all i, $p_i|_V : V \to \mathbb{C}^n$ is either an isomorphism or the zero map.

We claim that every nonzero matrix $A \in V$ has rank 1. To prove this write $A = [a^{(1)}, \dots a^{(n)}]$ as a matrix of column vectors $a^{(i)}$. If $p_i|_V$ and $p_j|_V$ are both isomorphisms, then the isomorphism $p_j|_V \circ (p_i|_V)^{-1} : \mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^n$ is given by $a^{(i)} \mapsto a^{(j)}$, and it commutes with G. Thus, by the Lemma of Schur, the nonzero columns of A are multiples of each other, and the claim follows.

(3) The subspace $\langle G \rangle$ is stable under left- and right-multiplication. Hence, by (2), it contains a matrix A of rank one, and so $\langle G \rangle = \mathrm{M}_n$, by (1). □

3.2. Orthogonal groups. Suppose that $q: V \to \mathbb{C}$ is a *nondegenerate quadratic form* on the finite dimensional vector space V where $\dim V \geq 2$. We define the *orthogonal group* of the form q to be

$$\mathrm{O}(V) := \mathrm{O}(V, q) := \{g \in \mathrm{GL}(V) \mid q(gv) = q(v) \text{ for every } v \in V\}.$$

Denote by $q(\ ,\ )$ the corresponding *symmetric bilinear form*, i.e.

$$q(v, w) := \frac{1}{2}(q(v + w) - q(v) - q(w)).$$

Thus, for $V = \mathbb{C}^n$, $q(v, w) = v^t Q w$ where Q is the symmetric matrix $(q(e_i, e_j))_{i,j}$. In general, there is always a basis of V such that the form q is given by $q(v) = x_1^2 + x_2^2 + \ldots + x_n^2$ where $(x_1, \ldots, x_n)$ are the coordinates of v (cf. Proposition I.3.1.1). Such a basis $(v_1, v_2, \ldots, v_n)$ is called an *orthonormal bases* of V with respect to q, i.e., we have $q(v_i, v_j) = \delta_{ij}$. It follows that $\mathrm{O}(V, q)$ is isomorphic to the *classical orthogonal group*

$$\mathrm{O}_n := \mathrm{O}_n(\mathbb{C}) := \{g \in \mathrm{GL}_n \mid g^t g = E_n\},$$

and that any two orthogonal groups $\mathrm{O}(V, q)$, $\mathrm{O}(V, q')$ are conjugate in $\mathrm{GL}(V)$.

Furthermore, the *special orthogonal group* is defined in the following way:

$$\begin{aligned} \mathrm{SO}_n &:= \mathrm{SO}_n(\mathbb{C}) := \mathrm{O}_n \cap \mathrm{SL}_n, \\ \mathrm{SO}(V) &:= \mathrm{SO}(V, q) := \mathrm{O}(V, q) \cap \mathrm{SL}(V). \end{aligned}$$

We have $\mathrm{O}_n = \mathrm{SO}_n \cup \begin{bmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{bmatrix} \mathrm{SO}_n$, and so $\mathrm{O}_n / \mathrm{SO}_n \simeq \mathbb{Z}/2\mathbb{Z}$.

Exercise 3.2.1. Describe $\mathrm{O}(V, q)$ and $\mathrm{SO}(V, q)$ for $V := \mathbb{C}^2$ and $q(x, y) := xy$.

Proposition 3.2.2. *$\mathrm{O}(V)$ is an irreducible subgroup of $\mathrm{GL}(V)$, and $\mathrm{SO}(V)$ is irreducible for $\dim V > 2$.*

Proof. Let $v_1, v_2 \in V$ such that $q(v_1) = q(v_2) \neq 0$ and put $V_i := (\mathbb{C}v_i)^\perp := \{w \in V \mid q(v_i, w) = 0\}$, $i = 1, 2$. Then $V = \mathbb{C}v_i \oplus V_i$ and $q|_{V_i}$ is nondegenerate. It follows that the linear map $g: V \to V$ which sends an orthogonal basis of V_1 to an orthogonal basis of V_2 and v_1 to v_2 belongs to $\mathrm{O}(V)$. We can even arrange that $g \in \mathrm{SO}(V)$. Thus $\mathrm{SO}(V)$ acts transitively on the vectors of a fixed length $\neq 0$.

The vectors of length $\neq 0$ form a dense subset of V, namely the complement of the closed set $\mathcal{V}(q) = \{v \in V \mid q(v) = 0\}$). Therefore, any $\mathrm{SO}(V)$-stable subspace $U \subsetneqq V$ must be contained in $\mathcal{V}(q)$.

For every nonzero $v \in \mathcal{V}(q)$ we can find a $w \in V$ such that $q(v, w) \neq 0$. Then $U := \mathbb{C}v \oplus \mathbb{C}w$ is nondegenerate, i.e. $q|_U$ is nondegenerate, and we get an inclusion $\mathrm{O}(U) \subseteq \mathrm{O}(V)$ by extending an $h \in \mathrm{O}(U)$ with the identity on $U^\perp$. (In case $\dim V > 2$ there is an element $h' \in \mathrm{O}(U^\perp)$ such that $\mathrm{O}(U)h' \subseteq \mathrm{SO}(V)$.) Since O_2 has no stable lines in $\mathbb{C}^2$, we see that $\mathrm{O}(U)$ is irreducible in $\mathrm{GL}(U)$. Hence, every $\mathrm{O}(V)$-stable subspace V' (resp. $\mathrm{SO}(V)$-stable subspace V' in case $\dim V > 2$) such that $v \in V'$ has to contain U. In particular, there is a vector of length $\neq 0$ in V', and so $V' = V$ by the first part of the proof. □

REMARK 3.2.3. The proof above shows that $\mathrm{SO}(V)$ acts transitively on the vectors v of a fixed length $q(v) \neq 0$. A vector v is called *isotropic* if $q(v) = 0$. We leave it as an exercise to prove that $\mathrm{O}(V)$ acts transitively on the nonzero isotropic vectors.

It follows from SCHUR'S Lemma 3.1.4 and the proposition above that we get the following description of the centers of O_n and SO_n:

$$\mathrm{Z}(\mathrm{O}_n) = \mathrm{O}_n \cap \mathbb{C}^* E_n = \{\pm E_n\} \quad \text{and}$$

$$\mathrm{Z}(\mathrm{SO}_n) = \mathrm{SO}_n \cap \mathbb{C}^* E_n = \begin{cases} \{\pm E_n\} & \text{for } n \text{ even, } n > 2, \\ \{E_n\} & \text{for } n \text{ odd.} \end{cases}$$

We remark that that $\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} \in \mathrm{O}_n \setminus \mathrm{SO}_n$ for all n, and that $-E_n \in \mathrm{O}_n \setminus \mathrm{SO}_n$ for odd n, but not for even n. In particular, $\mathrm{O}_n = \{\pm E_n\} \times \mathrm{SO}_n$ for odd n.

EXERCISE 3.2.4. Let $g \in \mathrm{O}_n \setminus \mathrm{SO}_n$. Show that the automorphism of SO_n defined by conjugation with g is inner for n odd, but not inner for n even.
(Hint: The kernel of the homomorphism $\mathrm{Int}\colon \mathrm{O}_n \to \mathrm{Aut}(\mathrm{SO}_n)$ is equal to $\pm E_n$, and so it belongs to SO_n for even n, but not for odd n.)

Recall that a subspace $W \subseteq V$ is called *isotropic* if $q|_W$ is trivial, or, equivalently, if $W \subseteq W^\perp$.

LEMMA 3.2.5. *Let V be of even dimension $n = 2m$. If $W \subseteq V$ is a maximal isotropic subspace, then $\dim W = m$, and for every basis $(w_1, \dots, w_m)$ of W there exist $w_1', \dots, w_m' \in V$ such that $W' := \langle w_1', \dots, w_m' \rangle$ is isotropic of dimension m, and $q(w_i, w_j') = \delta_{ij}$.*

PROOF. Let $(w_1, \dots, w_r)$ be a basis of W. Since W is maximal isotropic there exists a $w_1' \in \langle w_2, \dots, w_r \rangle^\perp$ such that $q(w_1, w_1') = 1$. Replacing w_1' by $w_1' + a w_1$ for a suitable $a \in \mathbb{C}$ we can assume that w_1' is isotropic. Then $U := \mathbb{C}w_1 \oplus \mathbb{C}w_1'$ is nondegenerate with $q|_U$ given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $V = U \oplus U^\perp$, and $W_1 := W \cap U^\perp = \langle w_2, \dots, w_r \rangle$ is maximal isotropic in $U^\perp$. Now the claim follows by induction. □

The lemma says that any basis $(w_1, \dots, w_m)$ of a maximal isotropic subspace W can be extended to a basis $(w_1, \dots, w_m, w_1', \dots, w_m')$ of V such that the form q is given by the matrix $I := \begin{bmatrix} 0 & E_m \\ E_m & 0 \end{bmatrix}$. This implies that all maximal isotropic subspaces are equivalent under $\mathrm{O}(V)$, and that we get a closed embedding $\mathrm{GL}(W) \hookrightarrow \mathrm{SO}(V)$ by $g \mapsto \begin{bmatrix} g & 0 \\ 0 & g^{-t} \end{bmatrix}$. Another way to see this is the following. The form q defines an isomorphism $W' \xrightarrow{\sim} W^*$ by $w' \mapsto q(w', ?)$ which allows to identify V with $W \oplus W^*$ where the quadratic form on the latter is given by $q(w, \ell) := 2\ell(w)$. Thus, we obtain a closed embedding $\mathrm{GL}(W) \hookrightarrow \mathrm{SO}(V)$ by $g \mapsto (g, (g^*)^{-1})$.

EXERCISE 3.2.6. Let $W \subseteq V$ be an maximal isotropic subspace and define $H := \{g \in \mathrm{O}(V) \mid g|_W = \mathrm{id}_W\}$. Show that H is isomorphic to a vector group U^+. In particular, $H \subseteq \mathrm{SO}(V)$.
(Hint: In the notation above the subgroup H consists of matrices of the form $\begin{bmatrix} E_m & B \\ 0 & E_m \end{bmatrix}$ with suitable matrices B.)

EXERCISE 3.2.7. For even n there are two equivalence classes of maximal isotropic subspaces with respect to SO_n.
(Hint: Let $W \subseteq V$ be a maximal isotropic subspace, and assume that for a given $g \in \mathrm{O}(V)$ there is an $h \in \mathrm{SO}(V)$ such that $gW = hW$. Then $h^{-1}gW = W$, hence there is a

$m \in \mathrm{GL}(W) \subseteq \mathrm{SO}(V)$ such that $mh^{-1}g$ is the identity on W. Now the previous exercise implies that $mh^{-1}g \in \mathrm{SO}(V)$, and so $g \in \mathrm{SO}(V)$.)

PROPOSITION 3.2.8. SO_n *is connected. Hence* $\mathrm{O}_n^\circ = \mathrm{SO}_n$ *and* $\mathrm{O}_n / \mathrm{O}_n^\circ \simeq \mathbb{Z}/2\mathbb{Z}$.

PROOF. This is clear for SO_2 since $\mathrm{SO}_2 \simeq \mathbb{C}^*$ (Example 2.3.8). So we can assume that $n \geq 3$. Let $g \in \mathrm{SO}_n$. We show that there is an irreducible closed subset $X \subseteq \mathrm{SO}_n$ containing g and E_n. Put $v := ge_1$.

(a) If $(v, e_1) = 0$, then $U := \mathbb{C}e_1 \oplus \mathbb{C}v$ is nondegenerate, and there is an $h \in \mathrm{SO}(U)$ such that $hv = e_1$. It follows that hg belongs to the stabilizer of e_1, $hg \in (\mathrm{SO}_n)_{e_1} \simeq \mathrm{SO}_{n-1}$, and so $g \in X := \overline{\mathrm{SO}(U) \cdot (\mathrm{SO}_n)_{e_1}}$ which is, by induction, an irreducible closed subset of SO_n containing g and E_n.

(b) If $(v, e_1) \neq 0$, then there is a $w \in \mathbb{C}^n$ such that $|w| = 1$ and that both spaces $U_1 := \mathbb{C}e_1 \oplus \mathbb{C}w$ and $U_2 := \mathbb{C}u \oplus \mathbb{C}w$ are nondegenerate (see Exercise 3.2.9 below). Now, similarly as in case (a), $g \in X := \overline{\mathrm{SO}(U_1) \cdot \mathrm{SO}(U_2) \cdot (\mathrm{SO}_n)_{e_1}}$, and the claim follows. □

EXERCISE 3.2.9. Let V be a finite dimensional $\mathbb{C}$-vector space with a nondegenerate quadratic form q. For any pair $u, v \in V \setminus \{0\}$ there is a $w \in V$ such that the subspaces $\langle u, w\rangle$ and $\langle v, w\rangle$ are nondegenerate, and one can even assume that $q(w) = 1$.

EXERCISE 3.2.10. $\mathrm{O}(V)$ acts transitively on the set of isotropic vectors $\neq 0$, and the same holds for $\mathrm{SO}(V)$ for $\dim V > 2$.
(Hint: This is clear for O_2 and can be reduced to this case as in the second part of the proof of Proposition 3.2.2, using the previous exercise. The claim for $\mathrm{SO}(V)$ follows since $\mathcal{V}(q)$ is irreducible for $\dim V > 2$.)

3.3. Symplectic groups. Suppose $\beta\colon V \times V \to \mathbb{C}$ is *a nondegenerate alternating bilinear form*, i.e. $\beta(u, v) = -\beta(v, u)$. Such a form exists only if $\dim V$ is even: $n := \dim V = 2m$. The *symplectic group* with respect to β is then defined by

$$\mathrm{Sp}(V) := \mathrm{Sp}(V, \beta) := \{g \in \mathrm{GL}(V) \mid \beta(gu, gv) = \beta(u, v) \text{ for } u, v \in V\}.$$

We will see below that all such forms are equivalent under $\mathrm{GL}(V)$. Therefore, with respect to a suitable basis of V, the form β can be written as

$$\beta(x, y) = \sum_{i=1}^{m} (x_i y_{m+i} - x_{m+i} y_i) = x^t J y$$

with corresponding matrix $J := \begin{bmatrix} 0 & E_m \\ -E_m & 0 \end{bmatrix}$. Thus $\mathrm{Sp}(V, \beta)$ is isomorphic to the *classical symplectic group* defined by

$$\mathrm{Sp}_{2m} := \mathrm{Sp}_{2m}(\mathbb{C}) := \{F \in \mathrm{M}_{2m} \mid F^t J F = J\}.$$

If we write $F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A, B, C, D \in \mathrm{M}_m(\mathbb{C})$, then one has

$$F \in \mathrm{Sp}_{2m} \iff \begin{cases} A^t D - C^t B = E, \\ A^t C, B^t D \quad \text{are symmetric.} \end{cases}$$

There are other "standard" forms which appear in the literature, e.g.

$$\sum_{i=1}^{m} (x_{2i-1} y_{2i} - x_{2i} y_{2i-1}) = x^t J' y \quad \text{and} \quad \sum_{i=1}^{m} (x_i y_{2m+1-i} - x_{2m+1-i} y_i) = x^t J'' y,$$

with corresponding matrices

$$(*) \qquad J' := \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} \qquad \text{or} \qquad J'' := \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & ⋰ & & \\ & -1 & & & \\ -1 & & & & \end{bmatrix}$$

PROPOSITION 3.3.1. (1) *There is a basis of V such that $\beta(x,y) = x^t J y$, and so $\mathrm{Sp}(V,\beta)$ is isomorphic to Sp_{2m}.*

(2) *$\mathrm{Sp}(V)$ acts transitively on $V \setminus \{0\}$. In particular, $\mathrm{Sp}_{2m} \subseteq \mathrm{GL}_{2m}$ is an irreducible subgroup, and $\mathrm{Z}(\mathrm{Sp}_{2m}) = \mathrm{Sp}_{2m} \cap \mathbb{C}^* E_{2m} = \{\pm E_{2m}\}$.*

PROOF. (1) Since β is nondegenerate we can find two vectors $v, w \in V$ such that $\beta(v,w) = 1$. Then β restricted to $U := \mathbb{C}v \oplus \mathbb{C}w$ is given by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, hence is nondegenerate, and so $V = U \oplus U^\perp$. Now the claim follows easily by induction on $\dim V$.

(2) If $v, w \in V$ are as in (1), then there is a $g \in \mathrm{Sp}(V)$ such that $gv = w$. In fact, $\mathrm{Sp}(U) \times \mathrm{Sp}(U^\perp) \subseteq \mathrm{Sp}(V)$ and $\mathrm{Sp}(U) \simeq \mathrm{Sp}_2 = \mathrm{SL}_2$ (see the exercise below). The same argument works whenever $\beta(v,w) \neq 0$. In case $\beta(v,w) = 0$ one easily shows that there is a $u \in V$ such that $\beta(v,u) \neq 0$ and $\beta(w,u) \neq 0$, and the claim follows. □

EXERCISE 3.3.2. Show that $\mathrm{Sp}_2 = \mathrm{SL}_2$.

PROPOSITION 3.3.3. *Sp_{2m} is connected and is contained in SL_{2m}.*

PROOF. This is clear for $2m = 2$ (see the exercise above). By definition, $\det g = \pm 1$ for $g \in \mathrm{Sp}_{2m}$, and so the second claim follows from the first.

Now we claim that $\mathrm{Sp}(V,\beta)$ acts transitively on the set

$$Y := \{(v,w) \in V \times V \mid \beta(v,w) = 1\}.$$

In fact, if $\beta(v,w) = 1$, then $U := \mathbb{C}v \oplus \mathbb{C}w$ is nondegenerate and so $V = U \oplus U^\perp$. Therefore, there is a basis $(v_1 := v, v_2 := w, v_3, \ldots, v_{2m})$ of V with corresponding matrix J', see $(*)$. The same construction applied to another $(v', w') \in Y$ yields a basis $(v_1' := v', v_2' := w', v_3', \ldots, v_{2m}')$ of V with corresponding matrix J'. Hence the map $g\colon v_i \mapsto v_i'$ belongs to $\mathrm{Sp}(V)$, and the claim follows.

Since the function $\beta - 1$ is irreducible, the subset $P \subseteq V \times V$ is an irreducible subvariety. It follows that $\mathrm{Sp}(V)^\circ$ acts transitively on P, too (use Exercise 3.3.4 below applied to $G := \mathrm{Sp}(V) \subseteq \mathrm{GL}(V \oplus V)$). This implies that for every $g \in \mathrm{Sp}(V)$, there is an $h \in \mathrm{Sp}(V)^\circ$ such that $hg(v) = v$ and $hg(w) = w$. Therefore, hg belongs to the intersection if the two stabilizers of v and w. But $\mathrm{Sp}(V)_v \cap \mathrm{Sp}(V)_w = \mathrm{Sp}(U^\perp)$ where $U := \mathbb{C}v \oplus \mathbb{C}w$, hence $g \in \overline{\mathrm{Sp}(V)^\circ \cdot \mathrm{Sp}(U^\perp)}$ which is, by induction, an irreducible closed subset of $\mathrm{Sp}(V)$ containing g and id_V, and the proposition follows. □

EXERCISE 3.3.4. Let $G \subseteq \mathrm{GL}(V)$ be an algebraic group and let $Y \subseteq V$ be a closed subset which is stable under G, i.e. $gy \in Y$ for all $g \in G, y \in Y$. If Y is irreducible and if G acts transitively on Y, then so does G°.
(Hint: Choose a $v \in Y$ and consider the morphism $\varphi\colon G \to Y$ given by $g \mapsto gy$. Then show that every connected component G_i of G has a dense image in Y from which the claim follows immediately.)

3.4. Exercises. For the convenience of the reader we collect here all exercises from the third section.

EXERCISE. Let $G \subseteq \mathrm{GL}_n$ be irreducible. Then $G^t := \{g^t \mid g \in G \subseteq \mathrm{GL}_n$ is irreducible. (Hint: If $U \subseteq \mathbb{C}^n$ is stable under G^t, then $U^\perp := \{v \in \mathbb{C}^n \mid u^t\, v = 0 \text{ for all } u \in U\}$ is G-stable.)

EXERCISE. Describe $\mathrm{O}(V, q)$ and $\mathrm{SO}(V, q)$ for $V := \mathbb{C}^2$ and $q(x, y) := xy$.

EXERCISE. Let $W \subseteq V$ be an maximal isotropic subspace and define $H := \{g \in \mathrm{O}(V) \mid g|_W = \mathrm{id}_W\}$. Show that H is isomorphic to a vector group U^+. In particular, $H \subseteq \mathrm{SO}(V)$.
(Hint: Choosing a suitable basis of V the subgroup H consists of matrices of the form $\begin{bmatrix} E_m & B \\ 0 & E_m \end{bmatrix}$.)

EXERCISE. For even n there are two equivalence classes of maximal isotropic subspaces with respect to SO_n.
(Hint: Let $W \subseteq V$ be a maximal isotropic subspace, and assume that for a given $g \in \mathrm{O}(V)$ there is an $h \in \mathrm{SO}(V)$ such that $gW = hW$. Then $h^{-1}gW = W$, hence there is a $m \in \mathrm{GL}(W) \subseteq \mathrm{SO}(V)$ such that $mh^{-1}g$ is the identity on W. Now the previous exercise implies that $mh^{-1}g \in \mathrm{SO}(V)$, and so $g \in \mathrm{SO}(V)$.)

EXERCISE. Let V be a finite dimensional $\mathbb{C}$-vector space with a nondegenerate quadratic form q. For any pair $u, v \in V \setminus \{0\}$ there is a $w \in V$ such that the subspaces $\langle u, w\rangle$ and $\langle v, w\rangle$ are nondegenerate, and one can even assume that $q(w) = 1$.

EXERCISE. $\mathrm{O}(V)$ acts transitively on the set of isotropic vectors $\neq 0$, and the same holds for $\mathrm{SO}(V)$ for $\dim V > 2$.
(Hint: This is clear for O_2 and can be reduced to this case as in the second part of the proof of Proposition 3.2.2, using the previous exercise. The claim for $\mathrm{SO}(V)$ follows since the space $\mathcal{V}(q)$ of isotropic vectors is irreducible for $\dim V > 2$.)

EXERCISE. Show that $\mathrm{Sp}_2 = \mathrm{SL}_2$.

EXERCISE. Let $G \subseteq \mathrm{GL}(V)$ be an algebraic group and let $Y \subseteq V$ be a closed subset which is stable under G, i.e. $gy \in Y$ for all $g \in G, y \in Y$. If Y is irreducible and if G acts transitively on Y, then so does G°.
(Hint: Choose a $v \in Y$ and consider the morphism $\varphi\colon G \to Y$ given by $g \mapsto gy$. Then show that every connected component G_i of G has a dense image in Y from which the claim follows immediately.)

4. The Lie Algebra of an Algebraic Group

4.1. Lie algebras. The aim of this paragraph is to show that the tangent space $T_e(G)$ of an algebraic group G at the identity element $e \in G$ is a *Lie algebra* in a natural way. This Lie algebra allows to "linearize" many questions concerning the structure of G, the representation theory of G, and actions of G on varieties. We will see a number of applications in the next chapter (see e.g. Section III.5).

DEFINITION 4.1.1. A *Lie algebra* is a vector space L together with an alternating bilinear map $[\ ,\ ]\colon L \times L \to L$, called the *Lie bracket*, which satisfies the JACOBI *identity*:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \quad \text{for all } a, b, c \in L.$$

The identity means that $\operatorname{ad} a\colon L \to L$, $\operatorname{ad} a(b) := [a, b]$, is a derivation of L.

The standard example is an *associative algebra* A with Lie bracket defined by $[a, b] := ab - ba$. We leave it to the reader to check the JACOBI-identity. A Lie algebra is called *commutative* if $[a, b] = 0$ for all a, b. For an associative algebra A this means that A is commutative as an algebra.

EXAMPLE 4.1.2. The *vector fields* $\mathrm{Vec}(X)$ on a variety X (see A.4.5.1) form a Lie algebra. In fact, let $\alpha, \beta \in \mathrm{Vec}(X) = \mathrm{Der}(\mathcal{O}(X))$ be two derivations. Then one easily checks that $[\alpha, \beta] := \alpha \circ \beta - \beta \circ \alpha$ is again a derivation. E.g., for $X = \mathbb{C}^2$ one gets

$$\begin{aligned}[x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}] &= x\frac{\partial}{\partial y}y\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}x\frac{\partial}{\partial y} \\ &= x\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial x\partial y} - y\frac{\partial}{\partial y} - xy\frac{\partial^2}{\partial x\partial y} \\ &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.\end{aligned}$$

It should be clear what we mean by a *Lie subalgebra* of L, an *ideal* of L, and a *homomorphism* of Lie algebras. For instance, a homomorphism $A \to B$ of associative algebras is also a Lie algebra homomorphism with respect to the Lie bracket $[\ ,\ ]$ defined above.

EXERCISE 4.1.3. Suppose $[\ ,\ ]\colon L \times L \to L$ is an alternating bilinear map where L is a two-dimensional vector space. Show that

(a) L is a Lie algebra, i.e., the JACOBI-identity is satisfied.

(b) If $[\ ,\ ] \neq 0$, then there is a basis $\{u, v\}$ of L with $[u, v] = v$.

Thus all noncommutative two-dimensional Lie algebras are isomorphic.

EXERCISE 4.1.4. For a general $\mathbb{C}$-algebra A a *derivation* is a linear map $\delta\colon A \to A$ such that $\delta(ab) = a\,\delta(b) + \delta(a)\,b$ for $a, b \in A$. Show that the vector space of derivations $\mathrm{Der}(A) \subseteq \mathrm{End}_{\mathbb{C}}(A)$ is a Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$. (This generalizes Example 4.1.2.)

4.2. The Lie algebra of GL_n. The tangent space of GL_n at the identity matrix E is given by $T_E\mathrm{GL}_n = \mathrm{M}_n$, and similarly, $T_e\,\mathrm{GL}(V) = \mathrm{End}(V)$ (see Example A.4.1.3). Both are associative algebras, and so these tangent spaces carry a natural structure of a Lie algebra. This leads to the following definition of the Lie algebra of GL_n and of $\mathrm{GL}(V)$:

$$\left.\begin{array}{lcl}\mathrm{Lie}\,\mathrm{GL}_n & := & \mathrm{M}_n \\ \mathrm{Lie}\,\mathrm{GL}(V) & := & \mathrm{End}(V)\end{array}\right\} \text{ with Lie bracket } [A, B] := AB - BA.$$

We also use the notation $\mathfrak{gl}_n$ and $\mathfrak{gl}(V)$.

Suppose $G \subseteq \mathrm{GL}_n$ is a closed subgroup. Then T_eG is a subspace of M_n. We will show in section 4.4 that it is a *Lie subalgebra* of M_n, and we will denote it by $\mathrm{Lie}\,G$ or simply by the corresponding Gothic letter $\mathfrak{g}$. For the classical groups this is easy (see section 4.3 below).

EXERCISE 4.2.1. Describe the tangent spaces in E_n of the subgroups $B_n, T_n, U_n \subseteq \mathrm{GL}_n$ and show that they are Lie subalgebras of M_n.

In the following we use the technique of "epsilonization" for calculations with Lie algebras (cf. Appendix A.4.3). Let $\mathbb{C}[\varepsilon] := \mathbb{C} \oplus \mathbb{C}\varepsilon, \varepsilon^2 = 0$, be the *algebra of dual numbers*, and let $\mathrm{GL}_n(\mathbb{C}[\varepsilon])$ be the group of invertible $n \times n$-matrices with coefficients in $\mathbb{C}[\varepsilon]$. For a closed subgroup $G \subseteq \mathrm{GL}_n$ define $G(\mathbb{C}[\varepsilon]) \subseteq \mathrm{GL}_n(\mathbb{C}[\varepsilon])$ to be the subgroup consisting of those elements in $\mathrm{GL}_n(\mathbb{C}[\varepsilon])$ which satisfy the same polynomial equations as the elements of G, i.e. all polynomial equations from the ideal $I(G) \subseteq \mathcal{O}(\mathrm{GL}_n)$. Then one has

$$\mathrm{Lie}\,G = \{A \in \mathrm{M}_n(\mathbb{C}) \mid E + \varepsilon A \in G(\mathbb{C}[\varepsilon])\}.$$

In addition, if $\mu\colon G \to H$ is a homomorphism of algebraic groups, then μ induces a group homomorphism $\mu\colon G(\mathbb{C}[\varepsilon]) \to H(\mathbb{C}[\varepsilon])$ which has the following description (see Appendix A.4.6):

$$\mu(e + \varepsilon A) = e + \varepsilon d\mu_e(A).$$

EXAMPLE 4.2.2. Consider the multiplication $\mu\colon G \times G \to G$, $(g,h) \mapsto gh$. Its differential $d\mu_{(e,e)}\colon \operatorname{Lie} G \oplus \operatorname{Lie} G \to \operatorname{Lie} G$ is given by addition $(A,B) \mapsto A+B$. (One has $(e+\varepsilon A)(e+\varepsilon B) = e + \varepsilon(A+B)$ in $\mathrm{M}_n(\mathbb{C}[\varepsilon])$.)

Similarly, for the inverse $\iota\colon G \to G$, $g \mapsto g^{-1}$, we get $d\iota_e(A) = -A$. It follows that the differential of the power map $p_m\colon g \mapsto g^m$ at e is multiplication by m, and so p_m is dominant in case G is connected and $m \neq 0$ (cf. Exercise 1.4.6).

4.3. The classical Lie algebras. Next we describe the Lie algebras of the classical groups.

(1) The *special linear group* $\mathrm{SL}_n \subseteq \mathrm{GL}_n$ (3.1) is defined by $\det = 1$. For the matrices in $\mathrm{M}(\mathbb{C}[\varepsilon])$ we have
$$\det(E+\varepsilon A) = 1 + \varepsilon \operatorname{tr} A.$$
Thus the tangent space $T_E \mathrm{SL}_n$ is a subspace of $\{A \in \mathrm{M}_n \mid \operatorname{tr} A = 0\}$. Since SL_n is of codimension 1 in GL_n we get
$$\begin{aligned} n^2 - 1 = \dim \mathrm{SL}_n &\leq \dim T_E \mathrm{SL}_n \\ &\leq \dim\{A \in \mathrm{M}_n \mid \operatorname{tr} A = 0\} = n^2 - 1. \end{aligned}$$
Thus we have equality everywhere, and so
$$_n := \operatorname{Lie} \mathrm{SL}_n = \{A \in \mathrm{M}_n \mid \operatorname{tr} A = 0\}$$
which is a Lie subalgebra of $\mathfrak{gl}_n$, i.e. closed under the bracket $[A,B] = AB - BA$. Similarly, $(V) := \operatorname{Lie}\mathrm{SL}(V) \subseteq \mathfrak{gl}(V)$ is the subalgebra of traceless endomorphisms.

(2) The *orthogonal group* $\mathrm{O}_n \subseteq \mathrm{GL}_n$ is given by $A^tA = E$ (see 3.2). Since $(E+\varepsilon A)^t(E+\varepsilon A) = E + \varepsilon(A^t + A)$ we see that $T_E \mathrm{O}_n$ is a subspace of $\{A \in \mathrm{M}_n \mid A \text{ skew-symmetric}\}$ which is of dimension $\binom{n}{2}$. On the other hand, the condition $A^tA = E$ corresponds to $\binom{n+1}{2}$ polynomial equations in the entries of $A \in \mathrm{M}_n$ and so, by KRULL'S Principal Ideal Theorem (Proposition A.3.3.5), we get $\dim \mathrm{O}_n \geq n^2 - \binom{n+1}{2} = \binom{n}{2}$. Thus
$$\operatorname{Lie}\mathrm{O}_n = \operatorname{Lie}\mathrm{SO}_n = \{A \in \mathrm{M}_n \mid A \text{ skew-symmetric}\}$$
which is a Lie subalgebra of $\mathfrak{gl}_n$, i.e. closed under the bracket $[A,B] = AB - BA$. We will also use the notation $\mathfrak{so}_n$, $\mathfrak{so}(V,q)$ or $\mathfrak{so}(V)$.

(3) The *symplectic group* Sp_{2m} is defined by $F^tJF = J$ where $J = \left[\begin{smallmatrix} 0 & E_m \\ -E_m & 0 \end{smallmatrix}\right]$ (3.3). Since $(E+\varepsilon F)^t J (E + \varepsilon F) = J + \varepsilon(F^tJ + JF)$ we see that $\operatorname{Lie}\mathrm{Sp}_n$ is a subspace of $\{F \in \mathrm{M}_{2m} \mid F^tJ + JF = 0\}$. The dimension of this space is $\binom{2m+1}{2}$, because $J^t = -J$ and so the equation means that JF is symmetric. On the other hand, the condition $F^tJF = J$ corresponds to $\binom{2m}{2}$ polynomial equations (both sides are skew symmetric), hence, as above,
$$\operatorname{Lie}\mathrm{Sp}_{2m} = \{F \in \mathrm{M}_{2m} \mid F^tJ + JF = 0\}$$
which again is a Lie subalgebra of $\mathfrak{gl}_{2m}$, i.e. it is closed under the bracket $[A,B] = AB - BA$. The Lie algebra will also be denoted by $\mathfrak{sp}_{2m}$, $\mathfrak{sp}(V,\beta)$ or $\mathfrak{sp}(V)$. Using the block form $F = \left[\begin{smallmatrix} U & V \\ W & Z \end{smallmatrix}\right]$ one finds
$$\operatorname{Lie}\mathrm{Sp}_{2m} = \{\begin{bmatrix} U & V \\ W & -U^t \end{bmatrix} \in \mathrm{M}_{2m} \mid V, W \text{ symmetric}\}.$$

REMARK 4.3.1. The considerations above imply that the polynomial equations given by the conditions $A^tA = E$ for SO_n, respectively $F^tJF = J$ for Sp_{2m}, are not only defining equations for the corresponding classical group G, but they even generate the ideal of functions vanishing on G. For that, using Proposition A.4.8.3, we have to show that these equations define the tangent space $T_gG \subseteq \mathrm{M}_n$ for every

$g \in G$. But this is clear, since the conditions in $g \in G$ are $g^t A + A^t g = 0$, resp. $g^t JA + A^t Jg = 0$, which are both equivalent to $g^{-1}A \in \operatorname{Lie} G$.

In addition, we have calculated the dimensions of the groups:

$$\begin{aligned}
\dim \mathfrak{gl}_n &= \dim \mathrm{GL}_n = n^2, \quad \dim_n = \dim \mathrm{SL}_n = n^2 - 1, \\
\dim \mathfrak{so}_n &= \dim \mathrm{O}_n = \dim \mathrm{SO}_n = \binom{n}{2}, \\
\dim \mathfrak{sp}_{2m} &= \dim \mathrm{Sp}_{2m} = \binom{2m+1}{2} = m(2m+1)
\end{aligned}$$

We remark that these are precisely the dimensions (over $\mathbb{R}$) of the corresponding real groups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, and $\mathrm{Sp}_{2m}(\mathbb{R})$ (cf. Appendix B.2.1).

Exercise 4.3.2. Let $\beta\colon V \times V \to \mathbb{C}$ be a symmetric or alternating bilinear form which might be degenerate. Define

$$G(\beta) := \{g \in \mathrm{GL}_n \mid \beta(gv, gw) = \beta(v, w) \text{ for } v, w \in V\}.$$

Then

$$\operatorname{Lie} G(\beta) = \{A \in \operatorname{End}(V) \mid \beta(Av, w) + \beta(v, Aw) = 0 \text{ for all } v, w \in V\},$$

and this is a Lie subalgebra of $\mathfrak{gl}(V)$.

Exercise 4.3.3. Consider the quadratic form $q(x, y, z) := xz$ on $\mathbb{C}^3$ and define

$$G(q) := \{g \in \mathrm{GL}_3 \mid q(ga) = q(a) \text{ for all } a \in \mathbb{C}^3\}$$

as in the previous exercise. Describe $G(q)$ and its Lie algebra $\operatorname{Lie} G(q)$. Is $G(q)$ connected? And what is $\dim G(q)$?

4.4. The adjoint representation. We turn back to the general case of an arbitrary algebraic group G. For any $g \in G$ we denote by $\operatorname{Int} g\colon G \to G$ the *inner automorphism* $h \mapsto ghg^{-1}$ and by $\operatorname{Ad} g$ its differential at $e \in G$:

$$\operatorname{Ad} g := (d \operatorname{Int} g)_e \colon T_e G \to T_e G.$$

For $G = \mathrm{GL}_n$ we have $\operatorname{Ad} g(A) = gAg^{-1}$ ($g \in \mathrm{GL}_n, A \in \mathrm{M}_n$) since $\operatorname{Int} g$ is a linear map $\mathrm{M}_n \to \mathrm{M}_n$. Moreover, $g \mapsto \operatorname{Ad} g$ is a regular homomorphism $\operatorname{Ad}\colon \mathrm{GL}_n \to \mathrm{GL}(\mathrm{M}_n)$, because the entries of gAg^{-1} are regular functions on GL_n. By restriction the same holds for any closed subgroup $G \subseteq \mathrm{GL}_n$:

$$\operatorname{Ad} g(A) = gAg^{-1} \text{ for } g \in G \subseteq \mathrm{GL}_n \text{ and } A \in T_e G \subseteq \mathrm{M}_n$$

and $\operatorname{Ad}\colon G \to \mathrm{GL}(T_e G)$, $g \mapsto \operatorname{Ad} g$, is a homomorphism of algebraic groups. This already shows that $\operatorname{Lie} G \subseteq \mathrm{M}_n$ is closed under conjugation with elements $g \in G$.

The homomorphism Ad is called the *adjoint representation of* G. Its differential will be denoted by ad:

$$\operatorname{ad} := (d \operatorname{Ad})_e \colon T_e G \to \operatorname{End}(T_e G).$$

Proposition 4.4.1. *For a closed subgroup $G \subseteq \mathrm{GL}_n$ we have*

$$\operatorname{ad} A(B) = [A, B] \quad \textit{for } A, B \in T_e G \subseteq \mathrm{M}_n\,.$$

In particular, $T_e G$ is a Lie subalgebra of M_n.

Proof. By definition, one has $\operatorname{Ad}(E + \varepsilon A) = \mathrm{id} + \varepsilon \operatorname{ad} A$. On the other hand,

$$\begin{aligned}
\operatorname{Ad}(E + \varepsilon A)B &= (E + \varepsilon A)B(E + \varepsilon A)^{-1} = (E + \varepsilon A)B(E - \varepsilon A) \\
&= B + \varepsilon(AB - BA) = B + \varepsilon[A, B] \\
&= (\mathrm{id} + \varepsilon[A, -])B,
\end{aligned}$$

and the claim follows. □

The proposition shows that for any algebraic group $G \subseteq \mathrm{GL}_n$ the tangent space $T_e G$ carries the structure of a Lie algebra with bracket $[A, B] := \operatorname{ad} A(B)$. In particular, this structure is independent of an embedding of G into GL_n.

Definition 4.4.2. The tangent space T_eG together with this structure of a Lie algebra is called the *Lie algebra of* G. It will be denoted by $\operatorname{Lie} G$ or by the corresponding Gothic letter $\mathfrak{g}$.

Corollary 4.4.3. *Let $N \subseteq G$ be a closed normal subgroup. Then* $\operatorname{Lie} N \subseteq \operatorname{Lie} G$ *is an ideal, i.e. we have* $[A, B] \in \operatorname{Lie} N$ *for* $A \in \operatorname{Lie} G$ *and* $B \in \operatorname{Lie} N$.

Proof. For every $g \in G$ the inner automorphism $\operatorname{Int} g$ sends N isomorphically onto N, hence $\operatorname{Ad} g(\operatorname{Lie} N) = \operatorname{Lie} N$. This shows that $\operatorname{Lie} N \subseteq \operatorname{Lie} G$ is stable under the adjoint representation $\operatorname{Ad}\colon G \to \operatorname{GL}(\operatorname{Lie} G)$. It follows that its differential $\operatorname{ad} = d\operatorname{Ad}_e$ also stabilizes $\operatorname{Lie} H$, hence $[A, B] = \operatorname{ad} A(B) \in \operatorname{Lie} N$ for $A \in \operatorname{Lie} G$ and $B \in \operatorname{Lie} N$. □

Proposition 4.4.4. *Suppose $\mu\colon G \to H$ is a homomorphism of algebraic groups. Then the differential $d\mu_e\colon \operatorname{Lie} G \to \operatorname{Lie} H$ is a Lie algebra homomorphism, i.e.,*

$$d\mu_e([A, B]) = [d\mu_e(A), d\mu_e(B)].$$

We will simply write $d\mu$ or sometimes $\operatorname{Lie} \mu$ instead of $d\mu_e$.

Proof. The adjoint representation $\operatorname{Ad}\colon G \to \operatorname{GL}(\operatorname{Lie} G)$ determines a morphism $\varphi\colon G \times \operatorname{Lie} G \to \operatorname{Lie} G$ by $(g, A) \mapsto \operatorname{Ad} g(A)$. It is easy to calculate the differential $d\varphi_{(e,B)}\colon \operatorname{Lie} G \times \operatorname{Lie} G \to \operatorname{Lie} G$:

$$d\varphi_{(e,B)}(A, C) = [A, B] + C. \tag{3}$$

In fact, one can reduce to $G = \operatorname{GL}_n$ where $\varphi(g, A) = gAg^{-1}$. Thus

$$\begin{aligned}\varphi(E + \varepsilon A, B + \varepsilon C) &= (E + \varepsilon A)(B + \varepsilon C)(E + \varepsilon A)^{-1} \\ &= B + \varepsilon(AB - BA + C) \\ &= B + \varepsilon\, d\varphi_{(E,B)}(A, C).\end{aligned}$$

Since $\mu \circ \operatorname{Int} g = \operatorname{Int} \mu(g) \circ \mu$ we get $d\mu \circ \operatorname{Ad} g = \operatorname{Ad} \mu(g) \circ d\mu$ for all $g \in G$. This means that the diagram

$$\begin{array}{ccc} G \times \operatorname{Lie} G & \xrightarrow{\varphi_G} & \operatorname{Lie} G \\ {\scriptstyle \mu\times d\mu}\downarrow & & \downarrow{\scriptstyle d\mu} \\ H \times \operatorname{Lie} H & \xrightarrow{\varphi_H} & \operatorname{Lie} H \end{array}$$

is commutative. Calculating the differential at (e, B), using equation (3), we find

$$\begin{aligned} d\mu([A, B]) = d\mu(d\varphi_{(e,B)}(A, 0)) &= d\varphi_{(e,d\mu(B))}(d\mu(A), 0)) \\ &= [d\mu(A), d\mu(B)], \end{aligned}$$

and the claim follows. □

Corollary 4.4.5. *Under the assumptions of the proposition above we have*

$$\operatorname{Lie} \mu(G) = d\mu(\operatorname{Lie} G) \quad \textit{and} \quad \operatorname{Lie}(\ker \mu) = \ker(d\mu).$$

Proof. There is an open dense set $U \subseteq G$ such that the differential $d\mu_g$ of $\mu\colon G \to \mu(G)$ is surjective for all $g \in U$ (Theorem A.4.9.1). By G-equivariance, it is surjective everywhere. In particular, $d\mu\colon \operatorname{Lie} G \to \operatorname{Lie} \mu(G)$ is surjective proving the first claim. The second follows, because $\operatorname{Lie} \ker \mu \subseteq \ker d\mu$ and $\dim \ker \mu = \dim G - \dim \mu(G) = \dim \operatorname{Lie} G - \dim d\mu(\operatorname{Lie} G) = \dim \ker d\mu$. □

Example 4.4.6. For $g \in G$ consider the commutator mapping

$$\gamma_g\colon G \to G, \quad h \mapsto ghg^{-1}h^{-1}.$$

Then one has

$$(d\gamma_g)_e = \operatorname{Ad} g - \operatorname{Id}.$$

To see this one factors γ_g as the composition $G \xrightarrow{\Delta} G \times G \xrightarrow{\operatorname{Int} g \times \iota} G \times G \xrightarrow{\mu} G$, where $\Delta(g) := (g,g)$. The assertion then follows from Example 4.2.2. One could also reduce to $G = \mathrm{GL}_n$ and use epsilonization:

$$g(e+\varepsilon A)g^{-1}(e+\varepsilon A)^{-1} = g(e+\varepsilon A)g^{-1}(e-\varepsilon A) = e+\varepsilon(gAg^{-1}-A).$$

4.5. Invariant vector fields. It is well-known that the vector fields on a manifold form a Lie algebra in a canonical way. This also holds for the algebraic vector fields $\operatorname{Vec}(X)$ on an affine variety X (see Proposition A.4.5.12), and gives us another way to define the Lie algebra structure on the tangent space T_eG of an algebraic group G.

Call a vector field $\delta \in \operatorname{Vec}(G)$ *left-invariant* if it is invariant under left multiplication on G:

$$(d\lambda_g)_h(\delta_h) = \delta_{gh} \text{ for all } g,h \in G.$$

Given any $A \in \operatorname{Lie} G$ one can construct a left-invariant vector field δ_A on G by setting $(\delta_A)_g := (d\lambda_g)_e A$.

Proposition 4.5.1. *Given $A \in \operatorname{Lie} G$ there is a unique left-invariant vector field δ_A such that $(\delta_A)_e = A$. Moreover, $\delta_{[A,B]} = [\delta_A, \delta_B]$.*

Proof. One easily reduces to the case $G = \mathrm{GL}_n$. Then, for $A = (a_{ij}) \in \mathrm{M}_n$ and $g = (g_{k\ell}) \in \mathrm{GL}_n$, we get $(\delta_A)_g = gA = \sum_{i,j}(\sum_k g_{ik}a_{kj})\frac{\partial}{\partial x_{ij}}|_g$, hence

$$\delta_A = \sum_{i,j}(\sum_k x_{ik}a_{kj})\frac{\partial}{\partial x_{ij}} = \sum_k (XA)_{ij}\frac{\partial}{\partial x_{ij}} \quad \text{where } X = (x_{ij}).$$

It follows that δ_A is a regular left invariant vector field on GL_n, and $(\delta_A)_e = A$. A short calculation shows that $\delta_A\delta_B - \delta_B\delta_A = \delta_{[A,B]}$. $\square$

There is a different way to understand this construction. Regard $A \in \operatorname{Lie} G$ as a derivation $A\colon \mathcal{O}(G) \to \mathbb{C}$ in $e \in G$. Then

$$\delta_A\colon \mathcal{O}(G) \xrightarrow{\mu^*} \mathcal{O}(G)\otimes\mathcal{O}(G) \xrightarrow{\operatorname{id}\otimes A} \mathcal{O}(G).$$

In fact, it is easy to see that $\delta := (\operatorname{id}\otimes A)\circ\mu^*$ is a derivation of $\mathcal{O}(G)$, and that

$$\delta_g = \operatorname{ev}_g \circ\delta = (\operatorname{ev}_g \otimes A)\circ\mu^* = A\circ\lambda_g^* = d\lambda_g(A)$$

where $\operatorname{ev}_g(f) = f(g)$ is the evaluation at $g \in G$.

Remark 4.5.2. If we us the action on G by right multiplication, $(h,g) \mapsto \rho(h,g) := gh^{-1}$, then, for any $A \in \operatorname{Lie} G$, we can construct in a similar way a *right-invariant* vector field $\tilde{\delta}_A$ on G:

$$(\tilde{\delta}_A)_g := (d\rho_{g^{-1}})_e A \in T_gG.$$

In this case we get $\tilde{\delta}_{[A,B]} = [\tilde{\delta}_B, \tilde{\delta}_A]$ which shows that $A \mapsto \tilde{\delta}_A$ is a anti-homomorphism of Lie algebras. Replacing in the above description of δ_A the map $\operatorname{id}\otimes A$ by $A\otimes\operatorname{id}$ we get the following description of $\tilde{\delta}_A$:

$$\tilde{\delta}_A\colon \mathcal{O}(G) \xrightarrow{\mu^*} \mathcal{O}(G)\otimes\mathcal{O}(G) \xrightarrow{A\otimes\operatorname{id}} \mathcal{O}(G).$$

Exercise 4.5.3. Let $\mu\colon G \to H$ be a surjective homomorphism of algebraic groups.

(1) For any $A \in \operatorname{Lie} G$ we have $d\mu(\delta_A) = \delta_B$ where $B := d\mu(A) \in \operatorname{Lie} H$, i.e. $d\mu_g(\delta_A)_g = (\delta_B)_{\mu(g)}$ for all $g \in G$.
(2) Use (1) to give another proof that $\operatorname{Lie}\mu$ is a homomorphism of Lie algebras.

Exercises

For the convenience of the reader we collect here all exercises from Chapter II.

EXERCISE. Suppose $[\ ,\ ]\colon L \times L \to L$ is an alternating bilinear map where L is a two-dimensional vector space. Show that

(a) L is a Lie algebra, i.e., the JACOBI-identity is satisfied.

(b) If $[\ ,\ ] \neq 0$, then there is a basis $\{u, v\}$ of L with $[u, v] = v$.

Thus all noncommutative two-dimensional Lie algebras are isomorphic.

EXERCISE. For a general $\mathbb{C}$-algebra A a *derivation* is a linear map $\delta\colon A \to A$ such that $\delta(ab) = a\,\delta(b) + \delta(a)\,b$ for $a, b \in A$. Show that the vector space of derivations $\operatorname{Der}(A) \subseteq \operatorname{End}_{\mathbb{C}}(A)$ is a Lie subalgebra of $\operatorname{End}_{\mathbb{C}}(A)$. (This generalizes Example 4.1.2.)

EXERCISE. Describe the tangent spaces in E_n of the subgroups $B_n, T_n, U_n \subseteq \mathrm{GL}_n$ and show that they are Lie subalgebras of M_n.

EXERCISE. Let $\beta\colon V \times V \to \mathbb{C}$ be a symmetric or alternating bilinear form which might be degenerate. Define

$$G(\beta) := \{g \in \mathrm{GL}_n \mid \beta(gv, gw) = \beta(v, w) \text{ for } v, w \in V\}.$$

Then

$$\operatorname{Lie} G(\beta) = \{A \in \operatorname{End}(V) \mid \beta(Av, w) + \beta(v, Aw) = 0 \text{ for all } v, w \in V\},$$

and this is a Lie subalgebra of $\mathfrak{gl}(V)$.

EXERCISE. Consider the quadratic form $q(x, y, z) := xz$ on $\mathbb{C}^3$ and define

$$G(q) := \{g \in \mathrm{GL}_3 \mid q(ga) = q(a) \text{ for all } a \in \mathbb{C}^3\}$$

as in the previous exercise. Describe $G(q)$ and its Lie algebra $\operatorname{Lie} G(q)$. Is $G(q)$ connected? And what is $\dim G(q)$?

EXERCISE. Let $N \subseteq G$ be a closed normal subgroup. Show that $\operatorname{Lie} N \subseteq \operatorname{Lie} G$ is an ideal, i.e. $[A, B] \in \operatorname{Lie} N$ for $A \in \operatorname{Lie} G$ and $B \in \operatorname{Lie} N$.
(Hint:)

EXERCISE. Let $\mu\colon G \to H$ be a surjective homomorphism of algebraic groups.

(1) For any $A \in \operatorname{Lie} G$ we have $d\mu(\delta_A) = \delta_B$ where $B := d\mu(A) \in \operatorname{Lie} H$, i.e. $d\mu_g(\delta_A)_g = (\delta_B)_{\mu(g)}$ for all $g \in G$.

(2) Use (1) to give another proof that $\operatorname{Lie} \mu$ is a homomorphism of Lie algebras.

CHAPTER III

Group Actions and Representations

Contents

Introduction. In this chapter we introduce the fundamental notion of algebraic group actions on varieties and of linear representations, and we discuss their basic properties. Then we study in detail tori and diagonalizable groups which will play a central role in the following. Finally, we show that there is a very strong relation between (connected) algebraic groups and their Lie algebras.

1. Group Actions on Varieties

1.1. G-Varieties. Let G be an algebraic group, $e \in G$ its identity element, and let X be an affine variety.

DEFINITION 1.1.1. An *action* of G on X is a morphism $\mu\colon G \times X \to X$ with the usual properties:

(i) $\mu(e, x) = x$ for all $x \in X$;
(ii) $\mu(gh, x) = \mu(g, \mu(h, x))$ for $g, h \in G$ and $x \in X$.

We will shortly write gx for $\mu(g, x)$, so that the conditions above are the following:

$$ex = x \text{ for } x \in X, \text{ and } (gh)x = g(hx) \text{ for } g, h \in G \text{ and } x \in X.$$

An affine variety X with an action of G is called a *G-variety*. For any $g \in G$ the map $x \mapsto gx$ is an isomorphism $\mu_g\colon X \xrightarrow{\sim} X$, with inverse $\mu_{g^{-1}}$.

EXAMPLE 1.1.2. The map $\mu\colon \mathrm{GL}(V) \times V \to V$ given by $\mu(g, v) := gv$ is a morphism and thus defines an action of $\mathrm{GL}(V)$ on V. This is clear, because $\mathrm{End}(V) \times V \to V$ is bilinear, hence regular.

This action is *linear* which means that $\mu_g\colon v \mapsto gv$ is a linear map for all $g \in \mathrm{GL}(V)$. It follows that for any homomorphism $\rho\colon G \to \mathrm{GL}(V)$ we obtain a *linear action of G on V*, given by $gv := \rho(g)v$.

EXAMPLE 1.1.3. For an algebraic group G we have the following actions of G on itself:

- By *left multiplication*: $(g, h) \mapsto \lambda_g(h) := gh$;
- By *right multiplication*: $(g, h) \mapsto \rho_g(h) := hg^{-1}$;
- By *conjugation*: $(g, h) \mapsto ghg^{-1}$.

The inverse g^{-1} on the right of the products is necessary in order to satisfy condition (ii) from the definition.

1.2. Fixed Points, Orbits and Stabilizers. Let X be a G-variety. We make the usual definitions.

DEFINITION 1.2.1. (1) An element $x \in X$ is called *fixed point* if $gx = x$ for all $g \in G$. We denote by $X^G := \{x \in X \mid x \text{ fixed point}\}$ the *fixed point set*.
(2) For $x \in X$ we define the *orbit of x* by $Gx := \{gx \mid g \in G\} \subseteq X$ and the *orbit map* $\mu_x\colon G \to X$ by $g \mapsto gx$.
(3) The *stabilizer* of $x \in X$ is defined by $\mathrm{St}_G(x) := G_x := \{g \in G \mid gx = x\}$; it is also called *isotropy group* of x. The *stabilizer* of a subset $Y \subseteq X$ is defined similarly by $\mathrm{St}_G(Y) := \{g \in G \mid gy = y \text{ for all } y \in Y\} = \bigcap_{y \in Y} G_y$.
(4) A subset $Y \subseteq X$ is called *G-stable* if $gY \subseteq Y$ for all $g \in G$.
(5) For a subset $Y \subseteq X$ we define the *normalizer of Y in G* by $\mathrm{N}_G(Y) := \{g \in G \mid gY = Y\}$.
(6) For two G-varieties X, Y a morphism $\varphi\colon X \to Y$ is called *G-equivariant* if $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in X$.

EXAMPLE 1.2.2. On the group G we have the two G-action, by left and by right multiplication (Example 1.1.3). These two G-varieties are G-isomorphic where the G-equivariant isomorphism is given by $g \mapsto g^{-1}$.

PROPOSITION 1.2.3. *Let X be a G-variety.*

(1) *The fixed point set X^G is closed in X.*
(2) *For any $x \in X$ and any subset $Y \subseteq X$ the stabilizers G_x and $\mathrm{St}_G(Y)$ are closed subgroups of G.*

(3) *For any* $x \in X$ *the orbit* Gx *is open in its closure* $\overline{Gx}$.
(4) *If* $Y \subseteq X$ *is closed, then the normalizer* $\mathrm{N}_G(Y)$ *is a closed subgroup of* G.

PROOF. (1) The morphism $\eta_g \colon X \to X \times X$, $x \mapsto (gx, x)$, shows that $X^g = \eta_g^{-1}(\Delta_X)$ is closed, and so $X^G = \bigcap_{g \in G} X^g$ is also closed.

(2) Since $G_x = \mu_x^{-1}(x)$ where μ_x is the orbit map, the stabilizer is a closed subgroup. This implies that $\mathrm{St}_G(Y) = \bigcap_{y \in Y} G_y$ is also a closed subgroup.

(3) The orbit Gx is the image of the morphism μ_x and thus contains a set U which is open and dense in $\overline{Gx}$. It follows that $Gx = \bigcup_{g \in G} gU$ is open in $\overline{Gx}$.

(4) The set $A := \{g \in G \mid gY \subseteq Y\} = \bigcap_{y \in Y} \mu_y^{-1}(Y)$ is a closed subset of G, and so $\mathrm{N}_G(Y) = A \cap A^{-1}$ is closed. □

EXERCISE 1.2.4. Let X be a G-variety, let $H \subseteq G$ an “abstract” subgroup and $\bar{H} \subseteq G$ its closure. Then we have the following:
(1) $X^H = X^{\bar{H}}$;
(2) If $Y \subseteq X$ is closed and H-stable, then Y is also $\bar{H}$-stable.
Is the closeness of Y necessary in (2)?

EXERCISE 1.2.5. Consider the action of GL_n on the matrices M_n by conjugation. Show that the stabilizers $(\mathrm{GL}_n)_A$ are connected for all $A \in \mathrm{M}_n$.
(Hint: For $A \in \mathrm{M}_n$, the subset $R_A := \{X \in \mathrm{M}_n \mid AX = XA\} \subseteq \mathrm{M}_n$ is a subalgebra. Now use Proposition II.1.1.11.)

DEFINITION 1.2.6. For an action of G on a variety X the stabilizer $\mathrm{St}_G(X)$ is called the *kernel* of the action. It is a closed normal subgroup (see Exercise 1.3.4 above). The action is called *faithful* if the kernel of the action is trivial. The action is called *free* if the stabilizer G_x of any point $x \in X$ is trivial.

If the G-variety X contains a point x with trivial stabilizer, then the action is clearly faithful. However, it is not true that every faithful action contains points with trivial stabilizer as we can see from the linear action of $\mathrm{GL}(V)$ or $\mathrm{SL}(V)$ on V, for $\dim V \geq 2$.

EXAMPLE 1.2.7. Let G be a finite group acting on an irreducible variety X. If the action is faithful, then the set of points with trivial stabilizer, $\{x \in X \mid G_x = \{e\}\}$, is open and dense in X.

PROOF. If the action is faithful, then X^H is a strict closed subset of X for every nontrivial subgroup $H \subseteq G$. Since X is irreducible the union $\bigcup_{H \neq \{e\}} X^H$ is a strict closed subset, and its complement has the required property. □

EXERCISE 1.2.8. Give an example of a faithful action of a finite group G which does not admit points with trivial stabilizer. Is this possible if G is commutative?

EXERCISE 1.2.9. Consider the standard action of O_2 on $\mathbb{C}^2$. Then the curve $H := \mathcal{V}(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$ is stable under O_2, and the action of O_2 on H is faithful, transitive, but not free.

PROPOSITION 1.2.10. *Let* X *be a* G*-variety, and suppose that there exists a* G*-equivariant morphism* $\varphi \colon X \to G$ *where* G *acts by left multiplication on* G*. Then there is a* G*-equivariant isomorphism* $G \times \varphi^{-1}(e) \xrightarrow{\sim} X$ *given by* $(g, y) \mapsto gy$*. In particular, the* G*-action on* X *is free.*

PROOF. One easily checks that the inverse morphism of $(g, y) \mapsto gy$ is given by $x \mapsto (\varphi(x), \varphi(x)^{-1}x)$. □

The proposition rises the interesting question whether a variety X with a free action of an algebraic group G looks locally like $G \times S$. The answer is yes for a certain class of groups, the so-called *reductive special groups* G, but no in general, e.g. for finite groups. Examples of reductive special groups are tori, GL_n, SL_n and Sp_{2m}, and product of those. We will discuss the case of tori in Section 3.

EXERCISE 1.2.11. Consider the faithful action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $t(x,y) := (t^p x, t^q y)$ where $p, q \in \mathbb{Z}$ are coprime.

(1) Determine the (nonempty) open set $U \subseteq \mathbb{C}^2$ where the action is free.
(2) Show that U can be covered by at most two $\mathbb{C}^*$-invariant special open sets isomorphic to $\mathbb{C}^* \times S$.

(Hint: The answers depend on whether $|p| = |q| = 1$ or not.)

EXERCISE 1.2.12. Show that a reductive special group G is connected.
(Hint: The action of $G \subseteq \mathrm{GL}(V)$ on $\mathrm{GL}(V)$ by left multiplication is free, and every open set of $\mathrm{GL}(V)$ is irreducible.)

1.3. Orbit map and dimension formula. Let X be a G-variety, $x \in X$, and let $\mu_x : G \to X$ be the orbit map. The image of μ_x is the orbit Gx and the fibers are the left cosets of the stabilizer G_x:

$$\mu_x^{-1}(hx) = \{g \in G \mid gx = hx\} = hG_x.$$

In particular, we obtain a bijection $G/G_x \xrightarrow{\sim} Gx$ between the left cosets G/G_x and the orbit Gx. Therefore, we have the following *dimension formula for orbits* (see Theorem A.3.4.1)

$$\dim Gx = \dim \overline{Gx} = \dim G - \dim G_x.$$

Note that the stabilizer of $y = gx \in Gx$ is a conjugate subgroup of G_x, namely $G_y = G_{gx} = gG_x g^{-1}$.

EXERCISE 1.3.1. Let X be a G-variety and $Y \subseteq X$ a G-stable subset. Then the closure $\bar{Y}$ is also G-stable.

Another consequence is the existence of closed orbits.

COROLLARY 1.3.2. *Let X be a G-variety and $x \in X$. Then the orbit closure $\overline{Gx}$ contains a closed orbit.*

PROOF. If $Gx = \overline{Gx}$, we are done. Otherwise $\dim(\overline{Gx} \setminus Gx) < \dim \overline{Gx}$, because Gx is open and dense in $\overline{Gx}$ (Proposition 1.2.3(3) and Exercise A.3.1.12), and we can proceed by induction on $\dim \overline{Gx}$, because $\overline{Gx}$ is G-stable (Exercise 1.3.1). □

REMARK 1.3.3. If G is connected and X a G-variety, then every irreducible component of X is G-stable. In fact, every $g \in G$ permutes the irreducible components X_i, so that $\mathrm{N}_G(X_i) \subseteq G$ has finite index, hence contains G° (Remark 1.4.3(3)).

EXERCISE 1.3.4. Let X be a G-variety.

(1) If $N \subseteq G$ is a normal subgroup, then the fixed point set X^N is G-stable.
(2) For any $Y \subseteq X$ the stabilizer $\mathrm{St}_G(Y)$ is a normal subgroup of the normalizer $\mathrm{N}_G(Y)$.

EXERCISE 1.3.5. Consider the standard representation of GL_2 on $\mathbb{C}^2$. Describe the orbits for the actions of the following closed subgroups $G \subseteq \mathrm{GL}_2$:

$$\text{(a) } G = \mathrm{SL}_2, \quad \text{(b) } G = \mathbb{C}^+ = U_2, \quad \text{(c) } G = \{\left[\begin{smallmatrix} t & \\ & t^{-1}\end{smallmatrix}\right] \mid t \in \mathbb{C}^*\}.$$

Which orbits are closed, and which one are contained in the closure of another orbit?

EXERCISE 1.3.6. Consider the action of GL_2 (resp. SL_2) by left-multiplication on the matrices M_2, and describe the orbits. Which orbits are closed and which are contained in the closure of other orbits?

1.4. Exercises. For the convenience of the reader we collect here all exercises from the first section.

EXERCISE. Let X be a G-variety, let $H \subseteq G$ an "abstract" subgroup and $\bar{H} \subseteq G$ its closure. Then we have the following:

(1) $X^H = X^{\bar{H}}$;
(2) If $Y \subseteq X$ is closed and H-stable, then Y is also $\bar{H}$-stable.

Is the closeness of Y necessary in (2)?

EXERCISE. Consider the action of GL_n on the matrices M_n by conjugation. Show that the stabilizers $(\mathrm{GL}_n)_A$ are connected for all $A \in \mathrm{M}_n$.
(Hint: For $A \in \mathrm{M}_n$, the subset $R_A := \{X \in \mathrm{M}_n \mid AX = XA\} \subseteq \mathrm{M}_n$ is a subalgebra. Now use Proposition II.1.1.11.)

EXERCISE. Give an example of a faithful action of a finite group G which does not admit points with trivial stabilizer.

EXERCISE. Consider the standard action of O_2 on $\mathbb{C}^2$. Then the curve $H := \mathcal{V}(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$ is stable under O_2, and the action of O_2 on H is faithful, transitive, but not free.

EXERCISE. Consider the faithful action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $t(x, y) := (t^p x, t^q y)$ where $p, q \in \mathbb{Z}$ are coprime.

(1) Determine the (nonempty) open set $U \subseteq \mathbb{C}^2$ where the action is free.
(2) Show that U can be covered by at most two $\mathbb{C}^*$-invariant special open sets isomorphic to $\mathbb{C}^* \times S$.

(Hint: The answers depend on whether $|p| = |q| = 1$ or not.)

EXERCISE. Show that a special group G is connected.
(Hint: The action of $G \subseteq \mathrm{GL}(V)$ on $\mathrm{GL}(V)$ by left multiplication is free, and every open set of $\mathrm{GL}(V)$ is irreducible.)

EXERCISE. Let X be a G-variety and $Y \subseteq X$ a G-stable subset. Then the closure $\bar{Y}$ is also G-stable.

EXERCISE. Let X be a G-variety.

(1) If $N \subseteq G$ is a normal subgroup, then the fixed point set X^N is G-stable.
(2) For any $Y \subseteq X$ the stabilizer $\mathrm{St}_G(Y)$ is a normal subgroup of the normalizer $\mathrm{N}_G(Y)$.

EXERCISE. Consider the standard representation of GL_2 on $\mathbb{C}^2$. Describe the orbits for the actions of the following closed subgroups $G \subseteq \mathrm{GL}_2$:

$$\text{(a) } G = \mathrm{SL}_2, \quad \text{(b) } G = \mathbb{C}^+ = U_2, \quad \text{(c) } G = \{\left[\begin{smallmatrix} t & \\ & t^{-1}\end{smallmatrix}\right] \mid t \in \mathbb{C}^*\}.$$

Which orbits are closed, and which one are contained in the closure of another orbit?

EXERCISE. Consider the action of GL_2 (resp. SL_2) by left-multiplication on the matrices M_2, and describe the orbits. Which orbits are closed and which are contained in the closure of other orbits?

2. Linear Actions and Representations

2.1. Linear representation. Let G be an algebraic group and V a finite dimensional $\mathbb{C}$-vector space.

DEFINITION 2.1.1. A *representation* of G on V is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$ of algebraic groups. A representation $\rho\colon G \to \mathrm{GL}(\mathbb{C}^n) = \mathrm{GL}_n(\mathbb{C})$ is sometimes called a *matrix representation* of G.

Two representations $\rho\colon G \to \mathrm{GL}(V)$ and $\mu\colon G \to \mathrm{GL}(W)$ are called *equivalent* if there is a linear isomorphism $\varphi\colon V \xrightarrow{\sim} W$ such that $\varphi(\rho(g)v) = \mu(g)\varphi(v)$ for all $g \in G$, $v \in V$.

In the literature one also finds the notion of a *rational representation*. We will only use this when we have to talk about "abstract" representations, as in the following easy lemma whose proof is left to the reader.

LEMMA 2.1.2. *Let G be an algebraic group and $\rho\colon G \to \mathrm{GL}_n$ an abstract homomorphism of groups. Then ρ is a rational representation if and only if the matrix coefficients $\rho_{ij}(g)$ are regular functions on G.*

EXAMPLE 2.1.3. Let $\rho\colon \mathbb{C}^* \to \mathrm{GL}(V)$ be an n-dimensional representation of $\mathbb{C}^*$. *Then ρ is diagonalizable*, i.e. there is a basis of V such that $\rho(\mathbb{C}^*) \subseteq T_n$. Thus ρ is equivalent to a matrix representation of the form

$$t \mapsto \begin{bmatrix} t^{m_1} & & & \\ & t^{m_2} & & \\ & & \ddots & \\ & & & t^{m_n} \end{bmatrix} \quad \text{where } m_1, m_2, \ldots, m_n \in \mathbb{Z}.$$

PROOF. The elements of finite order form a dense subgroup $F \subseteq \mathbb{C}^*$ and the image $\rho(F)$ is commutative and consists of diagonalizable elements. This implies that $\rho(F)$ is simultaneously diagonalizable, i.e., we can find a basis of V such that $\rho(F) \subseteq T_n$. Now the claim follows because $\rho(F)$ is dense in $\rho(\mathbb{C}^*)$. □

We can express this in a slightly different way. Define

$$V_k := \{v \in V \mid \rho(t)(v) = t^k \cdot v \text{ for all } t \in \mathbb{C}^*\}.$$

Then we get $V = \bigoplus_k V_k$, because ρ is diagonalizable. The subspace V_k is called the *weight space of weight k*, and the direct sum decomposition is the *weight space decomposition*.

We have seen in Example 1.1.2 that a representation $\rho\colon G \to \mathrm{GL}(V)$ defines a *linear action* of G on V. In general, a finite dimensional vector space V with a linear action of G is called a *G-module*. It is easy to see with the lemma above that every linear action of G on V defines a rational representation $G \to \mathrm{GL}(V)$. Therefore, we will not distinguish between a representation of G on V and the G-module V, and will freely switch between these points of view, depending on the given situation.

A representation $\rho\colon G \to \mathrm{GL}(V)$ is called *faithful* if $\ker\rho$ is trivial. This means that the linear action of G on V is faithful (see Definition 1.2.6).

EXAMPLE 2.1.4. We have seen in II.4.4 that the differential $\mathrm{Ad}\, g$ of the inner automorphism $\mathrm{Int}\, g\colon h \mapsto ghg^{-1}$ defines a linear action of G on its Lie algebra $\mathrm{Lie}\, G$, the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. The orbits are usually called the *conjugacy classes* in $\mathfrak{g}$. In case $G = \mathrm{GL}_n$ where $\mathfrak{g} = \mathrm{M}_n$ this is the standard action by conjugation on matrices.

EXAMPLE 2.1.5. Let X be a G-variety. For every $g \in G$ and every $x \in X$ the differential $(d\mu_g)_x\colon T_xX \to T_{gx}X$ of $\mu_g\colon x \mapsto gx$ is a linear map with the usual composition property $(d\mu_{gh})_x = (d\mu_g)_{hx} \circ (d\mu_h)_x$. Therefore, for every fixed point $x \in X$ we obtain a representation of G on T_xX called the *tangent representation* in the fixed point x. We will see later in Corollary 2.3.10 that this is a rational representation of G.

EXAMPLE 2.1.6. The representations $\rho\colon \mathbb{C}^+ \to \mathrm{GL}_n$ of the additive group $\mathbb{C}^+$ are in one-to-one correspondence with the nilpotent matrices in M_n. This follows immediately from Proposition II.2.6.1 where we showed that ρ is of the form $s \mapsto \exp(sN)$ with $N \in \mathrm{M}_n$ nilpotent. Moreover, two such representations are equivalent if and only if the corresponding matrices are conjugate.

EXAMPLE 2.1.7. Let G be an algebraic group acting on $\mathbb{C}^n$ by *affine transformations* (Exercise II.2.1.7). Then the induced map $\rho\colon G \to \mathrm{Aff}_n$ is a homomorphism of algebraic groups. In particular, every action of G on $\mathbb{A}^1$ is given by a homomorphism $G \to \mathrm{Aff}_1$.

PROOF. By assumption, $\rho(g)x = A(g)x + b(g)$, and we have to show that $A\colon G \to \mathrm{GL}_n$ and $b\colon G \to \mathbb{C}^n$ are morphisms. This is clear for b, because $b(g) = \rho(g)0$. Since $A(g)a = \rho(g)a - b(g)$ this implies that the map $g \mapsto A(g)a$ is a morphism, for every $a \in \mathbb{C}^n$. Now the claim follows, because $A(g)e_i$ is the ith row of $A(g)$. $\square$

If $\varphi\colon V \to W$ is a linear map, then the *transposed map* $\varphi^t\colon W^* \to V^*$ is defined in the usual way: $\varphi^t(\ell) := \ell \circ \varphi$. If φ is an isomorphism, we set $\varphi^* := (\varphi^t)^{-1}$. In this way we get an isomorphism $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}(V^*)$ of algebraic groups. In fact, choosing a basis in V and the dual basis in V^* the corresponding map $\mathrm{GL}_n \xrightarrow{\sim} \mathrm{GL}_n$ is given by $S \mapsto S^{-t}$. Using the canonical identification $(V^*)^* = V$ we see that $(g^*)^* = g$ for all $g \in \mathrm{GL}(V)$.

If $W \subseteq V$ is a subspace, we define $W^\perp := \{\ell \in V^* \mid \ell|_W = 0\} \subseteq V^*$. It follows that $\dim W + \dim W^\perp = \dim V$ and that $(W^\perp)^\perp = W$. Moreover, if $gW = W$ for some $g \in \mathrm{GL}(V)$, then $g^*W^\perp = W^\perp$. All this is well-known, and the reader is advised to check carefully the details.

2.2. Construction of representations and G-homomorphisms. Starting with two representations $\rho\colon G \to \mathrm{GL}(V)$ and $\mu\colon G \to \mathrm{GL}(W)$ we can construct new representations in the usual way:

- The *direct sum* $\rho \oplus \mu\colon G \to \mathrm{GL}(V \oplus W)$, $g \mapsto \rho(g) \oplus \mu(g)$,
- The *tensor product* $\rho \otimes \mu\colon G \to \mathrm{GL}(V \otimes W)$, $g \mapsto \rho(g) \otimes \mu(g)$,
- The *dual representation* $\rho^*\colon G \to \mathrm{GL}(V^*)$, $g \mapsto \rho(g)^* := (\rho(g)^t)^{-1}$, also called the *contragredient representation*,
- The k^{th} *symmetric power* $S^k\rho\colon G \to \mathrm{GL}(S^k(V))$ for all $k \geq 0$,
- The k^{th} *exterior power* $\wedge^k\rho\colon G \to \mathrm{GL}(\bigwedge^k V)$ for $0 \leq k \leq \dim V$.

If $H \subseteq G$ is a closed subgroup and $U \subseteq V$ an H-stable subspace, then we obtain representations of H on the subspace U, a *subrepresentation*, and on the quotient space V/U, a *quotient representation*:

$$\rho'\colon H \to \mathrm{GL}(U), h \mapsto \rho(h)|_U \quad \text{and} \quad \bar{\rho}\colon H \to \mathrm{GL}(V/U), h \mapsto \overline{\rho(h)}.$$

We leave it to the reader to check that all these representations are again rational. (This follows immediately from Lemma 2.1.2 above by choosing a suitable basis.) In the language of G-modules these constructions correspond to the *direct sum* and the *tensor product* of two G-modules, the *dual module*, the *symmetric* and the *exterior power* of a G-module, and *submodules* and *quotient modules* of a G-module.

The language of G-modules has the advantage that we can not only talk about submodules and quotient modules, but more generally about homomorphisms between two G-modules V and W.

DEFINITION 2.2.1. A G-equivariant linear map $\varphi\colon V \to W$ between two G-modules V and W is called a *G-homomorphism* or a *G-linear map*. A bijective G-homomorphism is called a *G-isomorphism*. Clearly, two G-modules V, W are G-isomorphic if and only if the corresponding representations are equivalent.

The set of G-homomorphism has the structure of a $\mathbb{C}$-vector space and will be denoted by $\mathrm{Hom}_G(V, W)$. In case of $W = V$ we talk about *G-endomorphisms* and use the notation $\mathrm{End}_G(V)$.

EXAMPLE 2.2.2. Consider the standard representation of SL_2 on $V = \mathbb{C}^2$ with basis $e_1 := (1,0)$, $e_2 := (0,1)$. Then

$$S^k(V) = \bigoplus_{\substack{\alpha\in\mathbb{N}^2\\ |\alpha|=k}} \mathbb{C}e^\alpha \text{ where } e^\alpha = e_1^{\alpha_1}e_2^{\alpha_2} \text{ and } |\alpha| := \alpha_1+\alpha_2,$$

and the linear action of $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $ge^\alpha = (ae_1+ce_2)^{\alpha_1}(be_1+de_2)^{\alpha_2}$. For the dual representation $V^* = \mathbb{C}x \oplus \mathbb{C}y$, we get $S^k(V^*) = \mathbb{C}[x,y]_m$, the *binary forms of degree* k, and the action of g is given by $gx = dx - by$, $gy = ay - cx$, because

$$g^{-t} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} : \begin{cases} x \mapsto dx - by, \\ y \mapsto ax - cy. \end{cases}$$

EXERCISE 2.2.3. Show that there is a matrix $S \in \mathrm{GL}_2$ such that $Sg^tS^{-1} = g^{-1}$ for all $g \in \mathrm{SL}_2$. Deduce from this that the SL_2-representations $S^m(V)$ and $S^m(V^*)$ are equivalent for all $m \in \mathbb{N}$.

EXERCISE 2.2.4. The same statement as in the previous exercise does not hold for GL_2 and neither for SL_n if $n > 2$.
(Hint: Apply the conjugation to the diagonal matrices!)

EXERCISE 2.2.5. Let V, W be two G-modules. Then there is a natural linear action of G on the space of linear maps $\mathrm{Hom}(V,W)$ given by $(g\varphi)(v) := g(\varphi(g^{-1}v))$. Show that

(1) $\mathrm{Hom}(V,W)$ is a G-module.
(2) $\mathrm{Hom}_G(V,W) = \mathrm{Hom}(V,W)^G$.
(3) There is a G-linear isomorphism $V^* \otimes W \xrightarrow{\sim} \mathrm{Hom}(V,W)$.
(Hint: The isomorphism is induced by $\lambda \otimes w \mapsto \varphi_{\lambda,w}$ where $\varphi_{\lambda,w}(v) = \lambda(v)w$.)

2.3. The regular representation. If an abstract group G acts on a space X then we get a representation of G on the $\mathbb{C}$-valued functions on X in the usual way: $(g,f) \mapsto gf$ where $gf(x) := f(g^{-1}x)$. This representation is called the *regular representation of* G on the functions on X.

If G is an algebraic group acting on an affine variety X and if f is a regular function on X, then gf is also regular. In fact, the action is given by a morphism $\mu\colon G \times X \to X$ whose comorphism $\mu^*\colon \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$ has the following property: If $\mu^*(f) = \sum_i h_i \otimes f_i$, then

$$(*) \qquad gf(x) = f(g^{-1}x) = \sum_i h_i(g^{-1})f_i(x), \text{ and so } gf = \sum_i h_i(g^{-1})f_i \in \mathcal{O}(X).$$

DEFINITION 2.3.1. An (abstract) representation of an algebraic group G on a complex vector space F is said to be *locally finite* if for every $f \in F$ the linear span $\langle gf \mid g \in G\rangle$ is finite dimensional. It is called *rational* if for every finite dimensional G-stable subspace $V \subseteq F$ the map $G \to \mathrm{GL}(V)$ is a homomorphism of algebraic groups. Such a vector space F is called a *locally finite rational G-module.*

A linear map $\mu\colon F \to F'$ between two locally finite rational G-modules is a *G-homomorphism* if it is G-equivariant, i.e. $\mu(gv) = g\mu(v)$ for all $v \in F$ and all $g \in G$.

PROPOSITION 2.3.2. *Let X be a G-variety. Then the representation of G on the regular functions $\mathcal{O}(X)$ is locally finite and rational.*

PROOF. The formula $(*)$ above shows that $\langle gf \mid g \in G\rangle \subseteq \sum_i \mathbb{C}f_i$, and so the representation is locally finite. Now let $V \subseteq \mathcal{O}(X)$ be a finite dimensional G-stable subspace, and let $f_1, \dots f_n$ be a basis of V. Writing $\mathcal{O}(V) = V \oplus W$ and using again the formula $(*)$ one sees that $\mu^*(V) \subseteq \mathcal{O}(G) \otimes V$, i.e. $gf_j = \sum_i h_{ij}(g^{-1})f_i$

for $j = 1, \dots, n$. Thus the representation of G on V is given by the map $g \mapsto H(g)$ where $H(g) = (h_{ij}(g^{-1}))_{ij}$ is an $n \times n$ matrix whose entries are regular functions on G. $\square$

EXAMPLE 2.3.3. From the natural representation of $\mathrm{GL}(V)$ on V we obtain the regular representation on the graded algebra $\mathcal{O}(V) = \bigoplus_{d \geq 0} \mathcal{O}(V)_d$ where $\mathcal{O}(V)_d$ are the homogeneous polynomial functions of degree d on V. These subspaces are stable under $\mathrm{GL}(V)$, and there is a canonical isomorphism $\mathcal{O}(V) \simeq S^d(V^*)$ of $\mathrm{GL}(V)$-modules.
In fact, the multilinear map $V^* \times \cdots \times V^* \to \mathcal{O}(V)$ given by $(\ell_1, \dots, \ell_d)(v) := \ell_1(v) \cdots \ell_d(v)$ is $\mathrm{GL}(V)$-equivariant and symmetric, and has its image in $\mathcal{O}(V)_d$. Thus it defines a $\mathrm{GL}(V)$-homomorphism $S^d(V^*) \to \mathcal{O}(V)_d$ which is easily seen to be an isomorphism by choosing a basis of V.

EXERCISE 2.3.4. Work out the proof indicated above in Example 2.3.3.

The proposition above has a number of interesting consequences. The first one will be used quite often to reduce questions about general G-actions on varieties to the case of linear representations on vector spaces.

COROLLARY 2.3.5. *Let X be a G-variety. Then X is G-isomorphic to a G-stable closed subvariety of a G-module V.*

PROOF. Choose a finite dimensional G-stable subspace $W \subseteq \mathcal{O}(X)$ which generates $\mathcal{O}(X)$. Then the canonical homomorphism $p\colon S(W) \to \mathcal{O}(X)$ is surjective and G-equivariant. Since $S(W) = \mathcal{O}(W^*)$, the coordinate ring of the dual representation W^* of W, it follows from the previous corollary that p is the comorphism of a closed G-equivariant embedding $\mu\colon X \hookrightarrow W^*$. $\square$

EXERCISE 2.3.6. Consider $X := \mathrm{GL}_2$ as a GL_2-variety where GL_2 acts by left-multiplication. Find a GL_2-module V which contains X as a GL_2-stable closed subset. (Hint: Look at pairs $(g, h) \in \mathrm{M}_2 \oplus \mathrm{M}_2$ such that $gh = E_2$.)

COROLLARY 2.3.7. *Let X, Y be G-varieties and $\varphi\colon X \to Y$ a morphism. Then φ is G-equivariant if and only if $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a G-homomorphism.*

PROOF. This follows immediately from the two formulas

$$\varphi^*(gf)(x) = gf(\varphi(x)) = f(g^{-1}\varphi(x)) \quad \text{and} \quad g\varphi^*(f)(x) = f(\varphi(g^{-1}x))$$

which show that φ is G-equivariant if and only if $\varphi^*(gf) = g\varphi^*(f)$ for all $f \in \mathcal{O}(Y)$ and $g \in G$. $\square$

EXERCISE 2.3.8. Let X, Y be G-varieties and $\varphi\colon X \to Y$ a morphism. Let $W \subseteq \mathcal{O}(Y)$ be a finite dimensional G-stable subset which generates $\mathcal{O}(Y)$. If $\varphi^*\colon W \to \mathcal{O}(X)$ is a G-homomorphism, then φ is G-equivariant.

COROLLARY 2.3.9. *Let $\varphi\colon G \twoheadrightarrow H$ be a surjective homomorphism, and let X be a G-variety. Assume that $\ker\varphi$ acts trivially on X. Then X is an H-variety.*

PROOF. We can assume that X is a G-stable closed subset of a G-module V. By assumption, $X \subseteq W := V^{\ker\varphi}$, and W is G-stable as well. Thus $\ker\varphi \subseteq \ker\rho$ where $\rho\colon G \to \mathrm{GL}(W)$ is the corresponding representation of G on W. Now the claim follows from the mapping property (Proposition II.2.1.10) which implies that ρ factors through φ. $\square$

COROLLARY 2.3.10. *Let X be a G-variety and let $x \in X^G$ be a fixed point. Then the maximal ideal $\mathfrak{m}_x$ and all its powers $\mathfrak{m}_x^d$ are G-stable subspaces of $\mathcal{O}(X)$, and the representation of G on $\mathfrak{m}_x/\mathfrak{m}_x^d$ is rational. In particular, the tangent representation of G on T_xX is rational, dual to the representation on $\mathfrak{m}_x/\mathfrak{m}_x^2$.*

Proof. Most of the statements are clear. We only show that the canonical map $T_xX = \mathrm{Der}_x(\mathcal{O}(X)) \to (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, $\delta \mapsto \delta' := \delta|_{\mathfrak{m}_x/\mathfrak{m}_x^2}$, is G-equivariant. Recall that the action on T_xX is given by $\delta \mapsto d\mu_g(\delta)$ where $\mu_g(y) = gy$ for $y \in X$. For $f \in \mathfrak{m}_x$ and $\bar{f} := f + \mathfrak{m}_x^2$ we have

$$g\delta'(\bar{f}) = \delta'(g^{-1}\bar{f}) = \delta'(\overline{g^{-1}f}) = \delta(g^{-1}f) = \delta(\mu_g^*(f)) = (d\mu_g\,\delta)(f),$$

hence $(d\mu_g\,\delta)' = g\,\delta'$. $\square$

Definition 2.3.11. Let X be a G-variety. A regular function $f \in \mathcal{O}(X)$ is called *G-invariant* (shortly *invariant*) if $f(gx) = f(x)$ for all $g \in G$ and $x \in X$, i.e. if f is constant on the G-orbits. The G-invariant functions form the fixed point set $\mathcal{O}(X)^G$ which is a subalgebra of $\mathcal{O}(X)$.

A *semi-invariant* is a regular function $f \in \mathcal{O}(X)$ with the property that the subspace $\mathbb{C}f \subseteq \mathcal{O}(X)$ is stable under G. If $f \neq 0$, then there is a well-defined character $\chi\colon G \to \mathbb{C}^*$ such that $f(gx) = \chi(g)\cdot f(x)$ for all $g \in G$ and $x \in X$. We express this by saying that f is a *semi-invariant with character* χ.

Let V be a finite dimensional vector space and $\alpha\colon V \times V \to \mathbb{C}$ a nondegenerate bilinear form. Then α defines an isomorphism $\tilde{\alpha}\colon V \xrightarrow{\sim} V^*$ by $\tilde{\alpha}(v)(w) := \alpha(w, v)$. If we choose a basis $(v_1, \ldots, v_n)$ of V and the dual basis $(v_1^*, \ldots, v_n^*)$ of V^*, then $\tilde{\alpha}\colon \mathbb{C}^n \to \mathbb{C}^n$ is given by the matrix $A := (\alpha(v_i, v_j))_{i,j}$.

Exercise 2.3.12. The isomorphism $\tilde{\alpha}\colon V \xrightarrow{\sim} V^*$ allows to define a nondegenerate bilinear form α^* on V^* in the obvious way. Show that $\widetilde{\alpha^*} \circ \tilde{\alpha} = \mathrm{id}_V$.
(Hint: With respect to the bases of V and V^* as above and identifying $(V^*)^*$ with V the linear map $\widetilde{\alpha^*}$ is given by the matrix A^{-1}.)

Lemma 2.3.13. *If V is a G-module and if α is G-invariant, then $\tilde{\alpha}\colon V \xrightarrow{\sim} V^*$ is G-isomorphism.*

Proof. This follows from the equalities

$$\tilde{\alpha}(gv)(w) = \alpha(w, gv) = \alpha(g^{-1}w, v) = \tilde{\alpha}(v)(g^{-1}w) = (g\tilde{\alpha}(v))(w)$$

for $v, w \in V$ and $g \in G$. $\square$

2.4. Subrepresentations of the regular representation. For a G-variety X we might ask which representations of G occur in the regular representation on $\mathcal{O}(X)$. For $X = G$ (with respect to left or right multiplication) there is the following partial answer.

Proposition 2.4.1. *Let V be a G-module and assume that V^* is cyclic, i.e. there is an $\ell \in V^*$ such that $\langle G\ell\rangle = V^*$. Then V occurs as a subrepresentation of $\mathcal{O}(G)$, with respect to left or right multiplication. In particular, every simple G-module occurs in $\mathcal{O}(G)$.*

(Recall that a G-module $V \neq \{0\}$ is *simple* if it does not have a submodule different from $\{0\}$ and V.)

Proof. Recall that $\mathcal{O}(G)$ as a G-module with respect to the left multiplication is isomorphic to $\mathcal{O}(G)$ as a module with respect to the right multiplication where the isomorphism is given by the coinverse $\iota\colon \mathcal{O}(G) \xrightarrow{\sim} \mathcal{O}(G)$ (Example 1.2.2). Thus it suffices to consider the left multiplication on G.

For $\ell \in V^*$ and $v \in V$ define $f_{\ell,v} \in \mathcal{O}(G)$ by $f_{\ell,v}(g) := \ell(g^{-1}v)$. Now the map $v \mapsto f_{\ell,v}$ is a G-homomorphism $\varphi_\ell\colon V \to \mathcal{O}(G)$. In fact,

$$f_{\ell,hv}(g) = \ell(g^{-1}hv) = f_{\ell,v}(h^{-1}g) = hf_{\ell,v}(g).$$

It is easy to see that φ_ℓ is injective if ℓ generates V^* as a G-module (see the next exercise). $\square$

EXERCISE 2.4.2. Show that the kernel of the map $v \mapsto f_{\ell,v}$ is equal to $\langle G\ell\rangle^{\perp} \subseteq V$.

COROLLARY 2.4.3. *Every G-module V of dimension $\leq n$ occurs as a submodule of $\mathcal{O}(G)^{\oplus n}$ with respect to left or right multiplication.*

PROOF. We choose generators $\ell_1, \ldots, \ell_m$ of the G-module V^* and use the functions $f_{\ell_i,v} \in \mathcal{O}(G)$ from the proof above to define a G-homomorphism $\varphi\colon V \to \mathcal{O}(G)^{\oplus m}$, $v \mapsto (f_{\ell_1,v}, \ldots, f_{\ell_m,v})$. Since the ℓ_i generate V^* as a G-module it follows that φ is injective. □

EXERCISE 2.4.4. If $V \subseteq \mathcal{O}(G)$ is a finite dimensional G-submodule, then V^* is cyclic. (Hint: Look at the linear function $\ell := \mathrm{ev}_e|_V : v \mapsto v(e)$.)

EXERCISE 2.4.5. Let W be a G-module W. If W^* can be generated, as a G-module, by m elements, then there exists a G-equivariant embedding of W into $\mathcal{O}(G)^{\oplus m}$. Conversely, if $W \subseteq \mathcal{O}(G)^m$ a finite dimensional submodule, then W^* can be generated, as a G-module, by m elements.

Finally, we can prove now what we announced after the definition of an algebraic group (Remark II.1.2.1), namely that a "group object" in the category of affine varieties is an algebraic group in the sense of our definition.

PROPOSITION 2.4.6. *Let H be an affine variety with a group structure such that the multiplication $\mu\colon H \times H \to H$ and the inverse $\iota\colon H \to H$ are morphisms. Then H is isomorphic to a closed subgroup of some* $\mathrm{GL}(V)$.

PROOF. We first remark that the notion of a group action on a variety and of a locally finite and rational representation does not use that the group G is a closed subgroup of some GL_n. It makes perfectly sense for H, and the proof of Proposition 2.3.2 carries over to H without any changes.

Now choose a finite dimensional linear subspace $V \subseteq \mathcal{O}(H)$ which is stable under the action of H by right multiplication and which generates the coordinate ring $\mathcal{O}(H)$. This defines a rational representation $\rho\colon H \to \mathrm{GL}(V)$. For $v \in V$ define the function $f_v \in \mathcal{O}(\mathrm{GL}(V))$ by $f_v(g) := gv(e)$ where $e \in H$ is the identity element. Then

$$\rho^*(f_v)(h) = f_v(\rho(h)) = (\rho(h)v)(e) = v(h), \quad \text{hence } \rho^*(f_v) = v.$$

Thus the image of ρ^* contains V. It follows that the comorphism $\rho^*\colon \mathcal{O}(\mathrm{GL}(V)) \to \mathcal{O}(H)$ is surjective, and so ρ is a closed immersion. □

REMARK 2.4.7. There is the following nice generalization of the above result which is due to PALAIS, see [**Pal78**]. Let H be an affine variety with a group structure. Assume that the left-multiplications and the right-multiplications with elements from H are morphisms. *Then H is an algebraic group.*
The question goes back to MONTGOMERY who proved this in the setting of topological groups with underlying complete metric spaces, see [**Mon36**].

If X is an irreducible G-variety, then G also acts on the field $\mathbb{C}(X)$ of rational functions on X, by field automorphisms. In particular, if G is finite and the action faithful, then $\mathbb{C}(X)/\mathbb{C}(X)^G$ is a Galois extension with Galois group G. In general, $\mathbb{C}(X)^G$ is a finitely generated field over $\mathbb{C}$, and the transcendence degree of $\mathbb{C}(X)$ over $\mathbb{C}(X)^G$ is bounded by $\dim G$. In fact, one can show that

$$\mathrm{tdeg}_{\mathbb{C}(X)^G} \mathbb{C}(X) = \max\{\dim Gx \mid x \in X\}.$$

This is a consequence of a theorem of ROSENLICHT, see [**Spr89**, IV.2.2 Satz von ROSENLICHT]. We will prove this in the next chapter (section IV.**??**) in a special case, namely for the so-called linearly reductive groups G. The case of tori and diagonalizable groups will be handled in the following section where we prove a stronger result, see Proposition 3.6.1.

2.5. Exercises. For the convenience of the reader we collect here all exercises from the second section.

Exercise. Show that there is a matrix $S \in \mathrm{GL}_2$ such that $Sg^tS^{-1} = g^{-1}$ for all $g \in \mathrm{SL}_2$. Deduce from this that the SL_2-representations $S^m(V)$ and $S^m(V^*)$ are equivalent for all $m \in \mathbb{N}$.
Finally show that this does not hold for GL_2 and neither for SL_n, $n > 2$.

Exercise. Let V, W be two G-modules. Then there is a natural linear action of G on the space of linear maps $\mathrm{Hom}(V, W)$ given by $(g\varphi)(v) := g(\varphi(g^{-1}v))$. Show that

(1) $\mathrm{Hom}(V, W)$ is a G-module.
(2) $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$.
(3) There is a G-linear isomorphism $V^* \otimes W \xrightarrow{\sim} \mathrm{Hom}(V, W)$.
(Hint: The isomorphism is induced by $\lambda \otimes w \mapsto \varphi_{\lambda,w}$ where $\varphi_{\lambda,w}(v) = \lambda(v)w$.)

Exercise. Consider $X := \mathrm{GL}_2$ as a GL_2-variety where GL_2 acts by left-multiplication. Find a GL_2-module V which contains X as a GL_2-stable closed subset.
(Hint: Look at pairs $(g, h) \in \mathrm{M}_2 \oplus \mathrm{M}_2$ such that $gh = E_2$.)

Exercise. Let X, Y be G-varieties and $\varphi \colon X \to Y$ a morphism. Let $W \subseteq \mathcal{O}(Y)$ be a finite dimensional G-stable subset which generates $\mathcal{O}(Y)$. If $\varphi^* \colon W \to \mathcal{O}(X)$ is a G-homomorphism, then φ is G-equivariant.

Exercise. Let $\alpha \colon V \times V \to \mathbb{C}$ be a nondegenerate bilinear form on a finite dimensional vector space V. Then the corresponding isomorphism $\tilde{\alpha} \colon V \xrightarrow{\sim} V^*$ allows to define a nondegenerate bilinear form α^* on V^* in the obvious way, and thus an isomorphism $\widetilde{\alpha^*} \colon V^* \to (V^*)^* = V$. Show that $\widetilde{\alpha^*} \circ \tilde{\alpha} = \mathrm{id}_V$.
(Hint: With respect to a basis of V and the dual basis of V^* the linear map $\tilde{\alpha}$ is given by the matrix A corresponding to the form α, and $\widetilde{\alpha^*}$ is given by the matrix A^{-1}.)

Exercise. Show that the kernel of the map $v \mapsto f_{\ell,v}$ is equal to $\langle G\ell \rangle^\perp \subseteq V$.

Exercise. If $V \subseteq \mathcal{O}(G)$ is a finite dimensional G-submodule, then V^* is cyclic.
(Hint: Look at the linear function $\ell := \mathrm{ev}_e|_V : v \mapsto v(e)$.)

Exercise. Let W be a G-module W. If W^* can be generated, as a G-module, by m elements, then there exists a G-equivariant embedding into $\mathcal{O}(G)^{\oplus m}$. Conversely, if $V \subseteq \mathcal{O}(G)^m$ a finite dimensional submodule, then V^* can be generated, as a G-module, by m elements.

3. Tori and Diagonalizable Groups

In this section we first study actions of the multiplicative group $\mathbb{C}^*$ on an affine variety X and prove the finite generation of the algebra of invariants $\mathcal{O}(X)^{\mathbb{C}^*}$ using Gordans's Lemma. This allows to define an algebraic quotient $\pi \colon X \to X/\!\!/\mathbb{C}^*$, and we show some of its properties. In the remaining part we discuss tori and diagonalizable groups, and prove the anti-equivalence between diagonalizable groups and finitely generated abelian groups, given by $D \mapsto \mathcal{X}(D)$, the character group of D.

3.1. $\mathbb{C}^*$-actions and quotients. Let X be a variety with an action of the multiplicative group $\mathbb{C}^*$. For $k \in \mathbb{Z}$ define

$$\mathcal{O}(X)_k := \{f \in \mathcal{O}(X) \mid tf = t^k \cdot f \text{ for all } t \in \mathbb{C}^*\}.$$

These are the semi-invariants of weight k, see Definition 2.3.11. Since $\mathcal{O}(X)$ is a locally finite and rational $\mathbb{C}^*$-module the weight space decomposition for a representation of $\mathbb{C}^*$ (see Example 2.1.3) implies that we get a similar decomposition for $\mathcal{O}(X)$,

$$\mathcal{O}(X) = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}(X)_k,$$

and this is a grading: $\mathcal{O}(X)_k \cdot \mathcal{O}(X)_m \subseteq \mathcal{O}(X)_{k+m}$. In particular, $\mathcal{O}(X)_0 = \mathcal{O}(X)^{\mathbb{C}^*}$ is the subalgebra of $\mathbb{C}^*$-invariant functions, and every subspace $\mathcal{O}(X)_k$ is a $\mathcal{O}(X)^{\mathbb{C}^*}$-module. Moreover, for any $\mathbb{C}^*$-equivariant morphism $\varphi\colon X \to Y$ we have $\varphi^*(\mathcal{O}(Y)_k) \subseteq \mathcal{O}(X)_k$ for all k.

EXAMPLE 3.1.1. For the standard action of $\mathbb{C}^*$ on $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$ by left multiplication the weight space decomposition is given by

$$\mathbb{C}[x] = \bigoplus_{k \geq 0} \mathbb{C}x^k \quad \text{and} \quad \mathbb{C}[x, x^{-1}] = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}x^k.$$

Note that the weight of $\mathbb{C}x^k$ is $-k$.

EXERCISE 3.1.2. Let V be a (nontrivial) two dimensional $\mathbb{C}^*$-module with weights p, q, i.e. $t(x, y) := (t^p \cdot x, t^q \cdot y)$ for a suitable basis. Determine the weight space decomposition and show the following.

(1) The invariant ring $\mathcal{O}(V)^{\mathbb{C}^*}$ is either $\mathbb{C}$ or a polynomial ring $\mathbb{C}[f]$ in one variable.
(2) If $\mathcal{O}(V)^{\mathbb{C}^*} = \mathbb{C}$, then the weight spaces are finite dimensional.
(3) If $\mathcal{O}(X)^{\mathbb{C}^*} \neq \mathbb{C}$, then the weight spaces are free $\mathcal{O}(X)^{\mathbb{C}^*}$-modules of rank 1.

EXAMPLE 3.1.3. Let X be a $\mathbb{C}^*$-variety. Assume that $\mathcal{O}(X)_i = 0$ for all $i < 0$ and that $\mathcal{O}(X)_0 = \mathbb{C}$. Then $X^{\mathbb{C}^*} = \{x_0\}$ and $\mathfrak{m}_0 := \bigoplus_{i>0} \mathcal{O}(X)_i$ is the maximal ideal of x_0. Moreover, if X is smooth in x_0, then X is $\mathbb{C}^*$-isomorphic to a $\mathbb{C}^*$-module with strictly negative weights.

PROOF. The first part is clear; we only prove the last statement. Since X is smooth in x_0 we can find $d := \dim X$ homogeneous functions $f_1, \ldots, f_d \in \mathfrak{m}_0$ whose images in $\mathfrak{m}_0/\mathfrak{m}_0^2$ form a $\mathbb{C}$-basis. Now we apply Lemma 2.3.3 to see that $\mathcal{O}(X) = \mathbb{C}[f_1, \ldots, f_d]$. Since $\dim X = d$ we see that the f_i are algebraically independet, and the claim follows. □

In general, we have the following statement about the invariants $\mathcal{O}(X)^{\mathbb{C}^*}$ of a $\mathbb{C}^*$-variety X.

LEMMA 3.1.4. *The subalgebra $\mathcal{O}(X)^{\mathbb{C}^*}$ is finitely generated.*

PROOF. Choose homogeneous generators $f_k \in \mathcal{O}(X)_{m_k}$, $k = 1, \ldots, n$, for the algebra $\mathcal{O}(X)$. For $\alpha \in \mathbb{N}^n$ set $f^\alpha := f_1^{\alpha_1} \cdots f_n^{\alpha_n} \in \mathcal{O}(X)$ which is homogeneous of degree $\omega(\alpha) := \sum_k \alpha_k m_k$. It follows that

$$\mathcal{O}(X)_0 = \sum_{\alpha \in \mathbb{N}^n,\, \omega(\alpha)=0} \mathbb{C}f^\alpha.$$

By GORDAN's Lemma (see below) the semigroup $S := \{\alpha \in \mathbb{N}^n \mid \sum_k \alpha_k m_k = 0\}$ is finitely generated, and so $\mathcal{O}(X)_0$ is finitely generated as an algebra. □

LEMMA 3.1.5 (GORDAN's Lemma). *For any subgroup $\Gamma \subseteq \mathbb{Z}^n$, the semigroup $M := \Gamma \cap \mathbb{N}^n$ is finitely generated.*

PROOF. Let $\mathcal{M} \subseteq M$ be the set of minimal element where we use the partial order on $\mathbb{Z}^n$ defined by

$$a \leq b \iff a_i \leq b_i \text{ for all } i.$$

Clearly, $\mathcal{M}$ is a set of generators of M. In order to prove that $\mathcal{M}$ is finite we will show now that any infinite sequence $(a^{(i)})$ of elements from $\mathbb{N}^n$ contains a pair $i < j$ such that $a^{(i)} \leq a^{(j)}$. This is clear for $n = 1$. We can assume that the set $\{a_1^{(i)}\} \subseteq \mathbb{N}$ is infinite, and then choose an infinite subsequence such that $a_1^{(i)} < a_1^{(j)}$ for $i < j$. Now consider the sequence $\bar{a}^{(i)} \in \mathbb{N}^{n-1}$ where $\bar{a} := (a_2, \ldots, a_n) \in \mathbb{N}^{n-1}$. By induction, there exists a pair $i < j$ such that $\bar{a}^{(i)} \leq \bar{a}^{(j)}$. Hence, $a^{(i)} \leq a^{(j)}$, because $a_1^{(i)} < a_1^{(j)}$, and we are done. □

EXERCISE 3.1.6. Show that every $\mathcal{O}(X)_n$ is a finitely generated $\mathcal{O}(X)^{\mathbb{C}^*}$-module. (Hint: Modify the proof of Lemma 3.1.4 above.)

Since $\mathcal{O}(X)^{\mathbb{C}^*}$ is a finitely generated subalgebra of $\mathcal{O}(X)$ there is an affine variety Y and a morphism $\pi\colon X \to Y$ such that the comorphism π^* induces an isomorphism $\mathcal{O}(Y) \xrightarrow{\sim} \mathcal{O}(X)^{\mathbb{C}^*}$. Since Y is uniquely determined up to isomorphism, we will use the notation $X/\!\!/\mathbb{C}^*$ for Y and $\pi_X\colon X \to X/\!\!/\mathbb{C}^*$ for the morphism, and will call $X/\!\!/\mathbb{C}^*$ the *quotient of* X *by* $\mathbb{C}^*$, and π_X *quotient morphism.* This notion is justified by the "universal property" formulated in the following proposition.

PROPOSITION 3.1.7. *Let X be a $\mathbb{C}^*$-variety, and let $\pi_X\colon X \to X/\!\!/\mathbb{C}^*$ be the quotient morphism.*

- **Universal property.** *For any $\mathbb{C}^*$-invariant morphism $\varphi\colon X \to Y$ there is a unique morphism $\bar{\varphi}\colon X/\!\!/\mathbb{C}^* \to Y$ such that $\varphi = \bar{\varphi} \circ \pi_X$.*
- **G-closedness.** *If $Z \subseteq X$ is closed and $\mathbb{C}^*$-stable, then $\pi_X(Z) \subseteq X/\!\!/\mathbb{C}^*$ is closed. In particular, π_X is surjective.*
- **G-separation.** *For any family $(Z_i)_I$ of $\mathbb{C}^*$-stable closed subsets of X, we have $\pi_X(\bigcap_i Z_i) = \bigcap_i \pi_X(Z_i)$. In particular, π_X separates disjoint $\mathbb{C}^*$-stable closed subsets of X.*

PROOF. (1) If $\varphi\colon X \to Y$ is $\mathbb{C}^*$-invariant, then $\varphi^*(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)^{\mathbb{C}^*}$. Hence, there is a homomorphism $\rho\colon \mathcal{O}(Y) \to \mathcal{O}(X/\!\!/\mathbb{C}^*)$ such that $\varphi^* = \pi_X^* \circ \rho$. Since $\rho = \bar{\varphi}^*$ for some morphism $\bar{\varphi}\colon X/\!\!/\mathbb{C}^* \to Y$, we get $\varphi = \bar{\varphi} \circ \pi_X$, and the claim follows.

(2) We can identify $\mathcal{O}(X/\!\!/\mathbb{C}^*)$ with $\mathcal{O}(X)_0$. For every ideal $\mathfrak{b} \subseteq \mathcal{O}(X)_0$ we have $\mathcal{O}(X) \cdot \mathfrak{b} = \bigoplus_n \mathcal{O}(X)_n \cdot \mathfrak{b}$, and so $(\mathcal{O}(X) \cdot \mathfrak{b}) \cap \mathcal{O}(X)_0 = \mathfrak{b}$. Since Z is $\mathbb{C}^*$-stable, the ideal $I(Z) \subseteq \mathcal{O}(X)$ is graded. Moreover, $\overline{\pi_X(Z)} \subseteq X/\!\!/\mathbb{C}^*$ is the zero set of the ideal $I(Z) \cap \mathcal{O}(X)^{\mathbb{C}^*} = I(Z)_0$. For any $y \in \overline{\pi_X(Z)}$, the corresponding maximal ideal $\mathfrak{m}_y \subseteq \mathcal{O}(X)_0$ contains $I(Z)_0$. This implies that

$$\pi_X^{-1}(y) \cap Z = \mathcal{V}_X(\mathfrak{m}_y \mathcal{O}(X) + I(Z)) \neq \emptyset,$$

because $(\mathfrak{m}_y \mathcal{O}(X) + I(Z)) \cap \mathcal{O}(X)_0 = \mathfrak{m}_y + I(Z)_0 = \mathfrak{m}_y \subsetneqq \mathcal{O}(X)_0$. Hence, $\overline{\pi_X(Z)} = \pi_X(Z)$.

(3) The image $\pi_X(\bigcap_i Z_i)$ is closed, by (2), and it is equal to the zero set of the ideal $\mathfrak{b} := (\sum_i I(Z_i))_0$. Since the ideals $I(Z_i)$ are graded we get $\mathfrak{b} = \sum_i I(Z_i)_0$, and the zero set of the latter is $\bigcap_i \overline{\pi_X(Z_i)} = \bigcap_i \pi_X(Z_i)$, again by (2). The claim follows. ☐

EXAMPLE 3.1.8. Consider the linear action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $t(x,y) := (t^p \cdot x, t^q \cdot y$. The action is faithful if and only if p and q are coprime, and in this case the action is free on $\mathbb{C}^2 \setminus \mathcal{V}(xy)$. If $p, q > 0$ or $p, q < 0$, then there are no invariants.

Now assume that p, q are coprime and $p > 0 > q$. Then $f := x^{-q}y^p$ is an invariant and $\mathbb{C}[x,y]^{\mathbb{C}^*} = \mathbb{C}[f]$. Thus $\mathbb{C}^2/\!\!/\mathbb{C}^* \simeq \mathbb{C}$ and the quotient morphism is $f\colon \mathbb{C}^2 \to \mathbb{C}$. Moreover, $f^{-1}(0) = \mathcal{V}(xy)$, the union of the x- and the y-axis, and $f\colon \mathbb{C}^2 \setminus \mathcal{V}(xy) \to \mathbb{C} \setminus \{0\}$ is a trivial $\mathbb{C}^*$-bundle. In fact, if $r, s \in \mathbb{Z}$ are such that $sp - rq = 1$, then we have a $\mathbb{C}^*$-equivariant isomorphism

$$\varphi\colon \mathbb{C}^* \times \mathbb{C} \setminus \{0\} \xrightarrow{\sim} \mathbb{C}^2 \setminus \mathcal{V}(xy), \ (t, z) \mapsto (t^p z^r, t^q z^s).$$

The inverse map is given by $(x, y) \mapsto (x^s y^{-r}, f(x,y))$. This will be generalized in Corollary 3.1.10 below.

A first consequence is the following.

COROLLARY 3.1.9. *For any $y \in X/\!\!/\mathbb{C}^*$, the fiber $\pi_X^{-1}(y)$ of the quotient morphism $\pi_X\colon X \to X/\!\!/\mathbb{C}^*$ contains a unique closed orbit O_y. If $O_y = \{\tilde{y}\}$ where $\tilde{y}$ is a fixed point, then $\pi_X^{-1}(y) = \{x \in X \mid \overline{\mathbb{C}^* x} \ni \tilde{y}\}$. Otherwise, $\pi_X^{-1}(y) = O_y$.*

PROOF. The first statement follows from the separation property and the fact that every closed $\mathbb{C}^*$-stable subvariety of X contains a closed orbit (Corollary 1.3.2).

The $\mathbb{C}^*$-orbits $O \subseteq X$ are either rational curves $\simeq \mathbb{C} \setminus \{0\}$ or fixed points. This implies that $\bar{O} \setminus O$ is either empty or a single fixed point.

If $O_y = \{\tilde{y}\}$ where $\tilde{y} \in X^{\mathbb{C}^*}$, and if $x \in \pi_X^{-1}(y)$, $x \neq \tilde{y}$, then $\mathbb{C}^* x$ is not closed, and so $\overline{\mathbb{C}^* x} = \mathbb{C}^* x \cup \{\tilde{y}\}$.

Finally, if $\dim O_y = 1$, then every orbit in $\pi_X^{-1}(y)$ must be closed of dimension 1, hence equal to O_y. $\square$

Another consequence is the following (cf. Proposition 3.6.1).

COROLLARY 3.1.10. *Assume that $X^{\mathbb{C}^*} = \emptyset$. Then all orbits are closed and of dimension 1, and $X /\!\!/ \mathbb{C}^*$ is the orbit space $X/\mathbb{C}^*$. If the action is free, then $\pi_X \colon X \to X /\!\!/ \mathbb{C}^*$ is a principal $\mathbb{C}^*$-bundle, locally trivial in* ZARISKI*-topology, i.e. there is a finite covering $X /\!\!/ \mathbb{C}^* = \bigcup_i U_i$ by special open sets $U_i \subseteq X /\!\!/ \mathbb{C}^*$ such that $\pi_X^{-1}(U_i) \simeq \mathbb{C}^* \times U_i$.*

PROOF. The first part is clear. For the second, choose a closed orbit O and fix an isomorphism $\mu \colon \mathbb{C}^* \xrightarrow{\sim} O \subseteq X$. We obtain a surjective $\mathbb{C}^*$-homomorphism $\mu^* \colon \mathcal{O}(X) \to \mathbb{C}[t, t^{-1}]$ which implies that there exist semi-invariants $f_1, f_2 \in \mathcal{O}(X)$ with $\mu^*(f_1) = t$ and $\mu^*(f_2) = t^{-1}$. In fact, $\mu^*(\mathcal{O}(X)_k) \subseteq \mathcal{O}(\mathbb{C}^*)_k = \mathbb{C}t^{-k}$ for all k, and so $\mu^*(\mathcal{O}(X)_{-1}) = \mathbb{C}t$ and $\mu^*(\mathcal{O}(X)_1) = \mathbb{C}t^{-1}$.

It follows that $f_1 \colon X \to \mathbb{C}$ is $\mathbb{C}^*$-equivariant and maps O isomorphically onto $\mathbb{C}^* \subseteq \mathbb{C}$. Moreover, $f := f_1 f_2$ is an invariant which does not vanish on O. Hence, $O \subseteq X_f$ and $f_1 \colon X_f \to \mathbb{C}^*$ is $\mathbb{C}^*$-equivariant. Now Proposition 1.2.10 shows that $X_f \simeq \mathbb{C}^* \times f^{-1}(1)$. Since f is an invariant, one easily sees that $\pi_X(X_f) = (X /\!\!/ \mathbb{C}^*)_f = X_f /\!\!/ \mathbb{C}^*$ (see the exercise below), and the claim follows. $\square$

EXERCISE 3.1.11. Let X be a $\mathbb{C}^*$-variety and $\pi_X \colon X \to X /\!\!/ \mathbb{C}^*$ the quotient.

(1) If $Z \subseteq X$ is closed and $\mathbb{C}^*$-stable, then the induced morphism $\pi_X|_Z \colon Z \to \pi_X(Z)$ is the quotient of Z by $\mathbb{C}^*$.
(2) If $f \in \mathcal{O}(X)^{\mathbb{C}^*}$ is an invariant, then $\pi_X(X_f) = (X /\!\!/ \mathbb{C}^*)_f$, and the induced morphism $\pi_X|_{X_f} \colon X_f \to (X /\!\!/ \mathbb{C}^*)_f$ is the quotient of X_f by $\mathbb{C}^*$.

3.2. Tori. An algebraic group isomorphic to $\mathbb{C}^{*n}$ is called an *n-dimensional torus*. The character group $\mathcal{X}(T)$ of a torus T is a torsionfree group of rank $n = \dim T$, i.e. $\mathcal{X}(T) \simeq \mathbb{Z}^n$ (see II.2.2). Moreover, $\mathcal{X}(T) \subseteq \mathcal{O}(T)$ is a $\mathbb{C}$-basis of the coordinate ring (Exercise II.2.2.3).

LEMMA 3.2.1. *Let T, T' be two tori. Then the map $\varphi \mapsto \mathcal{X}(\varphi) := \varphi^*|_{\mathcal{X}(T)}$ is a bijective homomorphism* $\operatorname{Hom}(T, T') \xrightarrow{\sim} \operatorname{Hom}(\mathcal{X}(T'), \mathcal{X}(T))$.

PROOF. It is clear that this map is a homomorphism of groups and that it is injective. In order to prove surjectivity we can assume that $T' = T_n$ and so $\mathcal{X}(T') = \mathcal{X}(T_n) = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$. If $\nu \colon \mathcal{X}(T_n) \to \mathcal{X}(T)$ is a homomorphism and $\chi_j := \nu(\varepsilon_j)$, $j = 1, \ldots, n$, then the homomorphism $\chi := (\chi_1, \ldots, \chi_n) \colon T \to T_n$ has the property that $\mathcal{X}(\chi) = \nu$. $\square$

As a consequence we see that the choice of a $\mathbb{Z}$-basis $\chi_1, \ldots, \chi_n$ of $\mathcal{X}(T)$ determines an isomorphism $T \xrightarrow{\sim} T_n$, and vice versa.

REMARK 3.2.2. Let T be an n-dimensional torus. For every positive integer $d \in \mathbb{N}$ the subgroup $\{t \in T \mid t^d = e\}$ is finite and isomorphic to $(\mathbb{Z}/d\mathbb{Z})^n$. Moreover, the elements of finite order form a dense subset (Exercise II.1.1.7(2)).

PROPOSITION 3.2.3. *For a torus T there exist elements $t \in T$ such that $\overline{\langle t \rangle} = T$. In fact, this holds for any t such that $\chi(t) \neq 1$ for all nontrivial characters χ.*

Proof. We can assume that $T = T_n$. Choose $t = (t_1, \dots, t_n) \in T_n$ such that $\alpha(t) = t_1^{\alpha_1} \cdots t_n^{\alpha_n} \neq 1$ for all $\alpha \neq 0$, e.g., take the t_i to be algebraically independent over $\mathbb{Q}$. We claim that for such an element $t \in T_n$ we have $\overline{\langle t \rangle} = T_n$.

In fact, assume that $f \in \mathbb{C}[\varepsilon_1, \varepsilon_1^{-1}, \dots, \varepsilon_n, \varepsilon_n^{-1}]$, $f \neq 0$, vanishes on the group $\langle t \rangle$. Then $f_t := f(t_1\varepsilon_1, \dots, t_n\varepsilon_n)$ also vanishes on $\langle t \rangle$. Now choose such an $f = \sum_{\alpha \in I} b_\alpha \varepsilon^\alpha$ which contains a minimal number of monomials $\varepsilon^\alpha = \varepsilon_1^{\alpha_1} \cdots \varepsilon_n^{\alpha_n}$. Then $f_t = \sum_{\alpha \in I} b_\alpha \alpha(t) \varepsilon^\alpha$ also vanishes on $\langle t \rangle$, and since all the element $\alpha(t)$ are different it follows that the difference $f_t - cf$, for a suitable $c \in \mathbb{C}$, contains less monomials than f. Hence, by assumption, $f_t = cf$, and so f is a monomial which is clearly a contradiction. □

3.3. Diagonalizable groups. There is a slightly larger class of algebraic groups which share most of the properties of the tori, namely the closed subgroups of tori.

Definition 3.3.1. An algebraic group D is called *diagonalizable* if D is isomorphic to a closed subgroup of T_n.

Since the restriction map $\mathcal{O}(T_n) \to \mathcal{O}(D)$ is surjective, the image of $\mathcal{X}(T_n)$ in $\mathcal{O}(D)$ spans $\mathcal{O}(D)$ and therefore is equal to $\mathcal{X}(D)$, because the characters are linearly independent (Lemma II.2.2.6). Thus, $\mathcal{X}(T_n) \to \mathcal{X}(D)$ is surjective and so $\mathcal{X}(D)$ is a finitely generated abelian group. Moreover, $\mathcal{O}(D)$ is the group algebra of $\mathcal{X}(D)$.

Exercise 3.3.2. Let $d \in \mathrm{GL}(V)$ be a diagonalizable element. Then $D := \overline{\langle d \rangle}$ is a diagonalizable group, and D/D° is cyclic.

Proposition 3.3.3. *An algebraic group D is diagonalizable if and only if $\mathcal{O}(D)$ is linearly spanned by $\mathcal{X}(D)$. In this case $\mathcal{X}(D)$ is a $\mathbb{C}$-basis of $\mathcal{O}(D)$, $\mathcal{X}(D)$ is finitely generated, and $\mathcal{O}(D)$ is the group algebra of $\mathcal{X}(D)$.*

Proof. We have just seen that for a diagonalizable group D the subset $\mathcal{X}(D) \subseteq \mathcal{O}(D)$ is a $\mathbb{C}$-basis. Conversely, if $\mathcal{X}(D)$ linearly spans $\mathcal{O}(D)$, we can find finitely many characters $\chi_1, \dots, \chi_n$ which generate $\mathcal{O}(D)$ as an algebra. This implies that the homomorphism $\chi = (\chi_1, \dots, \chi_n) \colon D \to T_n$ is a closed immersion. □

Corollary 3.3.4. *Let G be an algebraic group and D a diagonalizable group. Then the map* $\mathrm{Hom}(G, D) \to \mathrm{Hom}(\mathcal{X}(D), \mathcal{X}(G))$, $\varphi \mapsto \mathcal{X}(\varphi)$, *is bijective.*

Proof. It is clear that for any homomorphism $\varphi \colon G \to D$ the comorphism $\varphi^* \colon \mathcal{O}(D) \to \mathcal{O}(G)$ is determined by $\mathcal{X}(\varphi) = \varphi^*|_{\mathcal{X}(D)}$, because $\mathcal{X}(D) \subseteq \mathcal{O}(D)$ is a basis. Now let $\nu \colon \mathcal{X}(D) \to \mathcal{X}(G)$ be a homomorphism. Then ν induces an algebra homomorphism of the group algebras $\nu \colon \mathcal{O}(D) \to \mathbb{C}[\mathcal{X}(G)] \subseteq \mathcal{O}(G)$. It remains to see that the corresponding morphism $\varphi \colon G \to D$ is a homomorphism, i.e. that ν commutes with the comultiplication (II.1.3). This is clear because the comultiplication sends a character χ to $\chi \otimes \chi$ (see the following Exercise 3.3.5). □

Exercise 3.3.5. Let G be an algebraic group and $\mu^* \colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ the comultiplication (1.3). If χ is a character of G, then $\mu^*(\chi) = \chi \otimes \chi$.

3.4. Characterization of tori and diagonalizable groups. Among the diagonalizable groups the tori are characterized by several properties.

Proposition 3.4.1. *The following statements for a diagonalizable group D are equivalent.*

(i) *D is a torus.*
(ii) *D is connected.*

(iii) *$\mathcal{X}(D)$ is torsion free.*

PROOF. The implication (i) $\Rightarrow$ (ii) is clear. If D is connected, then $\mathcal{O}(D)$ has no zero divisors and so $\mathcal{X}(D)$ is torsion free, hence (ii) $\Rightarrow$ (iii). If $\mathcal{X}(D)$ is torsion free we choose a basis $\chi_1, \ldots, \chi_n$. Then $\mathcal{O}(D)$ is generated by $\chi_1, \ldots, \chi_n, \chi_1^{-1}, \ldots, \chi_n^{-1}$, and so the homomorphism $\chi = (\chi_1, \ldots, \chi_n)\colon D \to T_n$ is a closed immersion. By construction, $\mathcal{X}(\chi)\colon \mathcal{X}(T_n) \xrightarrow{\sim} \mathcal{X}(D)$ is an isomorphism and so $\chi^*\colon \mathcal{O}(T_n) \to \mathcal{O}(D)$ is injective. If follows that χ is an isomorphism, hence (iii) $\Rightarrow$ (i). $\square$

COROLLARY 3.4.2. *Let D be a diagonalizable group. Then D° is a torus, and there is a finite subgroup $F \subseteq D$ such that the multiplication $D^\circ \times F \xrightarrow{\sim} D$ is an isomorphism.*

PROOF. By the previous proposition D° is a torus, and D/D° is a finite abelian group which we write as a product of cyclic groups: $D/D^\circ = \prod_j \langle d_j \rangle$. It suffices to show that each $d_j \in D/D^\circ$ has a representative in D of the same order. If d_j has order m_j and if $g_j \in D$ is any representative of d_j, then $g_j^{m_j} \in D^\circ$. Since the homomorphism $g \mapsto g^{m_j}\colon D^\circ \to D^\circ$ is surjective (see Exercise II.2.1.5) there is an $h_j \in D^\circ$ such that $h_j^{m_j} = g_j^{m_j}$. Thus $h_j^{-1} g_j \in D$ is a representative of d_j of order m_j. $\square$

EXERCISE 3.4.3. For a diagonalizable group D with D/D° cyclic there exist elements $d \in D$ such that $D = \overline{\langle d \rangle}$.

EXERCISE 3.4.4. Let $d = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$. Describe $D := \overline{\langle d \rangle}$ in terms of a and b. What is D°, and what is D/D°?

PROPOSITION 3.4.5. (1) *A commutative algebraic group D is diagonalizable, if and only if the subgroup D_f of elements of finite order is dense.*

(2) *The image of a diagonalizable group under a homomorphism is diagonalizable.*

(3) *If D is diagonalizable and $\rho\colon D \to \mathrm{GL}_n$ a homomorphism, then the image $\rho(D)$ is conjugate to a subgroup of T_n.*

PROOF. We first remark that for a diagonalizable group D the subgroup D_f of elements of finite order is dense. This is clear for tori (see Exercise II.1.1.7(2)), and thus follows for diagonalizable groups from Corollary 3.4.2 above.

Now let $H \subseteq \mathrm{GL}_n$ be a commutative closed subgroup. If the subgroup H_f of elements of finite order is dense in H, then H is conjugate to a subgroup of T_n. In fact, the elements of H_f are simultaneously diagonalizable, i.e. there is a $g \in \mathrm{GL}_n$ such that $gH_fg^{-1} \subseteq T_n$, and so $gHg^{-1} = g\overline{H_f}g^{-1} = \overline{gH_fg^{-1}} \subseteq T_n$. This proves (3). Assertion (1) follows from this and what we said at the beginning of the proof, and (2) follows from (1). $\square$

COROLLARY 3.4.6. *Let G be an algebraic group, let T be a torus and $\varphi\colon G \twoheadrightarrow T$ a surjective homomorphism. If $\ker\varphi$ is a diagonalizable group, then so is G.*

PROOF. Since there are only finitely many elements of a given order in a diagonalizable group it follows that G° commutes with every element of finite order of $D := \ker\varphi$, hence G° commutes with D (Proposition 3.4.5(1)). But $G = G^\circ \cdot D$, because $\varphi(G^\circ) = \varphi(G) = T$, and so G commutes with D, i.e. $D \subseteq \mathrm{Z}(G)$. Choose an element $g \in G$ such that $\langle \varphi(g) \rangle$ is dense in T (Remark 3.2.2). Then $\langle \ker\varphi, g \rangle \subseteq G$ is commutative and dense in G, hence G is commutative.

For any element $t \in T$ of finite order, there is a preimage $g \in G$ of finite order. In fact, if $g \in G$ is an arbitrary preimage, then $g^n \in D^\circ$ for some $n \geq 1$, and there is a $d \in D^\circ$ such that $d^n = g^n$, because D° is a torus. Hence, $g' := gd^{-1}$ is a preimage of t of finite order. This implies that the elements of finite order are dense in G, hence G is diagonalizable, by Proposition 3.4.5(1). □

Remark 3.4.7. Let $\rho\colon D \to \mathrm{GL}(V)$ be a representation of a diagonalizable group D. For $\chi \in \mathcal{X}(D)$ we define the *weight space*

$$V_\chi := \{v \in V \mid \rho(t)v = \chi(t) \cdot v \text{ for all } t \in D\} \subseteq V.$$

This is a D-stable subspace, and $V_\chi \neq (0)$ for only finitely many characters χ. Moreover, we have

$$V = \bigoplus_{\chi \in \mathcal{X}(D)} V_\chi.$$

This is the so-called *weight space decomposition*, and the characters $\chi \in \mathcal{X}(D)$ with $V_\chi \neq (0)$ are called the *weights of V*.

Proof. It is clear that $V_\chi \subseteq V$ is a D-stable subspace and that the sum $\sum_\chi V_\chi$ is direct. Since every representation of a diagonalizable group is diagonalizable it follows that $\sum_\chi V_\chi = V$. □

This weight space decomposition can be carried over to the coordinate ring of a D-variety X, because the representation of D on $\mathcal{O}(X)$ is locally finite and rational (Proposition 2.3.2). For the special case of the multiplicative group $\mathbb{C}^*$ we have discussed this in the first section 3.1.

Proposition 3.4.8. *Let D be a diagonalizable group acting on a variety X. Then we have the following weight space decomposition:*

$$\mathcal{O}(X) = \bigoplus_{\chi \in \mathcal{X}(D)} \mathcal{O}(X)_\chi, \quad \mathcal{O}(X)_\chi := \{f \in \mathcal{O}(X) \mid tf = \chi(t) \cdot f \text{ for all } t \in D\}.$$

This decomposition is a grading over $\mathcal{X}(D)$, i.e. $\mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_{\chi'} \subseteq \mathcal{O}(X)_{\chi+\chi'}$. In particular, $\mathcal{O}(X)_0 = \mathcal{O}(X)^D$ is the subalgebra of D-invariant functions, and each $\mathcal{O}(X)_\chi$ is a $\mathcal{O}(X)^D$-module. Moreover, for any D-equivariant morphism $\varphi\colon X \to Y$ we have $\varphi^*(\mathcal{O}(Y)_\chi) \subseteq \mathcal{O}(X)_\chi$.

Note that, according to our Definition 2.3.11, the elements from $\mathcal{O}(X)_\chi$ are the semi-invariants with character χ (or with weight χ).

Denote by $\Lambda_X \subseteq \mathcal{X}(D)$ the weights occurring in $\mathcal{O}(X)$. Clearly, an element $d \in D$ acts trivially on X if and only if $\chi(d) = 1$ for all $\chi \in \Lambda_X$. This implies, as we will see in the next section, that the action is faithful if and only if the $\mathbb{Z}$-span $\langle \Lambda_X \rangle_{\mathbb{Z}}$ is equal to $\mathcal{X}(D)$ (see Theorem 3.5.2).

3.5. Classification of diagonalizable groups. In this section we will show that there is an equivalence between diagonalizable groups and finitely generated abelian groups which is given by the character group. We start with a description of the vanishing ideal of a closed subgroup of a diagonalizable group D

Lemma 3.5.1. *Let D be a diagonalizable group and $E \subseteq D$ a closed subgroup. Then the ideal of E is given by*

$$I(E) = (\chi - 1 \mid \chi \in \mathcal{X}(D) \text{ and } \chi|_E = 1) \subseteq \mathcal{O}(D).$$

In particular, E is equal to the kernel of a homomorphism $D \to T_m$.

Proof. Consider the subgroup of characters vanishing on E,

$$X_E := \{\chi \in \mathcal{X}(D) \mid \chi|_E = 1\} = \ker(\mathrm{res}\colon \mathcal{X}(D) \to \mathcal{X}(E)) \subseteq \mathcal{X}(D).$$

Clearly, $J := (\chi - 1 \mid \chi \in X_E) \subseteq I(E)$. Assume that $J \neq I(E)$, and choose $f \in I(E) \setminus J$, $f = \sum_{i=1}^m a_i \chi_i$ where m is minimal. We claim that this implies that $\chi_j|_E \neq \chi_k|_E$ for every pair $j \neq k$. In fact, if $\chi_1|_E = \chi_2|_E$, then $\chi_1 - \chi_2 = \chi_2(\chi_1\chi_2^{-1} - 1) \in J$, and so

$$f - a_1(\chi_1 - \chi_2) = (a_2 - a_1)\chi_2 + a_3\chi_3 + \cdots + a_m\chi_m \in I(E) \setminus J,$$

contradicting the minimality of m.

But if the characters $\chi_j|_E$ are different, then they are linearly independent (Lemma II.2.2.6), and so $\sum_{i=1}^m a_i\chi_i|_E = f|_E = 0$ implies that $a_i = 0$ for all i. Thus $f = 0$, contradicting the assumption that $f \notin J$. □

Now we can show that there is an anti-equivalence between diagonalizable groups and finitely generated abelian groups which is given by the character group. We have already seen in Proposition II.2.2.7 that $G \mapsto \mathcal{X}(G)$ is a left-exact contravariant functor on all algebraic groups.

THEOREM 3.5.2. *The functor $D \mapsto \mathcal{X}(D)$ defines an anti-equivalence between the diagonalizable groups and the finitely generated abelian groups. This means that every finitely generated abelian group is isomorphic to the character group of a diagonalizable group and that the natural map* $\operatorname{Hom}(D, E) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{X}(E), \mathcal{X}(D))$ *is an isomorphism of groups. Moreover, a sequence of diagonalizable group*

$$1 \to D' \to D \to D'' \to 1$$

is exact if and only if the induced sequence $0 \to \mathcal{X}(D'') \to \mathcal{X}(D) \to \mathcal{X}(D') \to 0$ *of the character groups is exact.*

PROOF. (1) Let $\Delta := \mathbb{Z}^n \times F$ where F is finite and put $D := T_n \times F$. Then $\mathcal{X}(D) \simeq \Delta$ (see Exercises II.2.2.8 and II.2.2.9), and so every finitely generated abelian group is isomorphic to the character group of some diagonalizable group.

(2) We have already seen in Corollary 3.3.4 that the map $\operatorname{Hom}(D, E) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{X}(E), \mathcal{X}(D))$ is a bijective group homomorphism.

(3) If $1 \to D' \xrightarrow{\varphi} D \xrightarrow{\psi} D'' \to 1$ is an exact sequence of diagonalizable groups, then $0 \to \mathcal{X}(D'') \xrightarrow{\mathcal{X}(\psi)} \mathcal{X}(D) \xrightarrow{\mathcal{X}(\varphi)} \mathcal{X}(D')$ is exact (Proposition II.2.2.7). Moreover, $\mathcal{X}(\varphi)$ is surjective, because $\varphi^* \colon \mathcal{O}(D) \to \mathcal{O}(D')$ is surjective and so the image contains all characters.

(4) Conversely, if the sequence $0 \to \mathcal{X}(D'') \xrightarrow{\mathcal{X}(\psi)} \mathcal{X}(D) \xrightarrow{\mathcal{X}(\varphi)} \mathcal{X}(D') \to 0$ is exact, then φ^* is surjective and ψ^* injective, hence φ is a closed immersion and ψ is surjective. Moreover, the kernel of ψ is equal to $\psi^{-1}(e)$, and the maximal ideal of $\mathfrak{m}_e \subseteq \mathcal{O}(D'')$ is given by $\mathfrak{m}_e = \langle \chi - 1 \mid \chi \in \mathcal{X}(D'') \rangle$. Hence, the kernel $\ker\psi$ is the zero set of the ideal $(\psi^*(\chi) - 1 \mid \chi \in \mathcal{X}(D''))$. Since $\operatorname{im}\mathcal{X}(\psi) = \ker\mathcal{X}(\varphi)$, we finally get

$$(\psi^*(\chi) - 1 \mid \chi \in \mathcal{X}(D'')) = (\eta - 1 \mid \eta \in \mathcal{X}(D), \eta|_{\varphi(D')} = 1\} = I(\varphi(D')),$$

by Lemma 3.5.1 above, and so $\ker\psi = \operatorname{im}\varphi$. □

EXERCISE 3.5.3. Let D be a diagonalizable group acting on a variety X, and denote by $\langle\Lambda_X\rangle_{\mathbb{Z}}$ the $\mathbb{Z}$-span of the weights of X. Show that $\mathcal{X}(D)/\langle\Lambda_X\rangle_{\mathbb{Z}}$ is the character group of the kernel of the action of D on X.

3.6. Invariant rational functions. We finish this section by two results about a variety X with an action of a diagonalizable group D relating the field of D-invariant rational functions on X with the "generic" structure of X as a D-variety.

PROPOSITION 3.6.1. *Let D be a diagonalizable group acting faithfully on an irreducible variety X, and define $X' := \{x \in X \mid D_x$ is trivial$\}$.*

(1) *X' is open and dense in X.*

(2) *If $D = T$ is a torus, then X' can be covered by T-stable special open sets U_i which are T-isomorphic to $T \times S_i$.*
(3) $\operatorname{tdeg}_{\mathbb{C}(X)^D} \mathbb{C}(X) = \dim D$, *and* $\mathbb{C}(X)/\mathbb{C}(X)^{D^\circ}$ *is purely transcendental.*

Proof. (a) We start with the case where D is connected, hence $D = T$ is a torus. Let $x \in X'$ and let $O := Tx$ be the orbit of x. We first show that there is a T-invariant special open set which contains O as a closed orbit. The ideal $\mathfrak{a} := I(\bar{O} \setminus O)$ is T-stable and thus a direct sum of weight spaces: $\mathfrak{a} = \bigoplus_\chi \mathfrak{a}_\chi$. It follows that there exists a semi-invariant $f \in \mathfrak{a}_\chi$ which does not vanish on O. This implies that X_f is T-stable and that $O \subseteq X_f$ is a closed orbit.

(b) Let $T = T_n$, let $O := T_n x \subseteq X$ be a closed orbit for same $x \in X'$, and let $\mu\colon T_n \to X$ be the orbit map. The comorphism $\mu^*\colon \mathcal{O}(X) \to \mathbb{C}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ is T_n-equivariant and surjective. Since the representation of T_n on $\mathcal{O}(X)$ is locally finite and rational, we can find semi-invariants $f_i \in \mathcal{O}(X)$ mapping onto t_i under μ^*. This implies that the morphism $\varphi = (f_1, \dots, f_n)\colon X \to \mathbb{C}^n$ is T_n-equivariant and maps O isomorphically onto $T_n \subseteq \mathbb{C}^n$. By construction, $O \subseteq \varphi^{-1}(T_n) = X_f$ where $f = f_1 \cdots f_n$, and Proposition 1.2.10 implies that $X_f \simeq T_n \times S$.

(c) From (a) and (b) we get statement (2) and, as a consequence, statement (1) for a connected D. Moreover,

$$\mathbb{C}(X)^T = \mathbb{C}(X_f)^T = \mathbb{C}(S) \text{ and } \mathbb{C}(X) = \mathbb{C}(X_f) = \mathbb{C}(S)(t_1, \dots, t_n),$$

hence $\mathbb{C}(X)$ is purely transcendental over $\mathbb{C}(X)^T$ of transcendence degree $n = \dim T$. Since $\mathbb{C}(X)^{D^\circ}/\mathbb{C}(X)^D$ is a Galois extension with Galois group D/D° the claims from (3) also follow.

(d) It remains to prove (1) for a diagonalizable group D. We already know that $X'' := \{x \in X \mid (D^\circ)_x \text{ is trivial}\}$ is open in X. Moreover, $X'' \setminus X' = \bigcup_{g \neq e} (X'')^g$. For any $x \in X''$ the stabilizer D_x maps injectively into D/D° which implies that if $(X'')^g \neq \emptyset$, then the order of g divides $|D/D^\circ|$. But D contains only finitely many elements of a given order, hence the union $\bigcup_{g \neq e} (X'')^g$ is a finite union of closed sets, and we are done. □

Exercise 3.6.2. For every diagonalizable group D and every algebraic group G the map

$$\operatorname{Hom}(G, D) \to \operatorname{Hom}(\mathcal{X}(D), \mathcal{X}(G)), \quad \varphi \mapsto \varphi^*|_{\mathcal{X}(D)},$$

is a bijective homomorphism of groups.

The second result concerns the case where there are no non-constant invariant rational functions.

Proposition 3.6.3. *Let T be a torus acting on an irreducible affine variety X. The following assertions are equivalent.*

(1) *X consists of finitely many T-orbits.*
(2) *X contains a dense T-orbit.*
(3) $\mathbb{C}(X)^T = \mathbb{C}$.
(4) *The multiplicities of $\mathcal{O}(X)$ are ≤ 1.*

Proof. (i) ⇒ (ii): This is clear.

(ii) ⇒ (iii): This is again clear, since every T-invariant rational function f is defined on an open dense set which meets the dense orbit. Hence f is constant.

(iii) ⇒ (iv): Assume that the dimension of the weight space $\mathcal{O}(X)_\chi$ is ≥ 2. Then we can find two linearly independent $p, q \in \mathcal{O}(X)_\chi$. It follows that $r := \frac{p}{q}$ is a non-constant rational invariant.

(iv) $\Rightarrow$ (i): $\mathcal{O}(X)$ is generated by finitely many weight vectors $f_1, \dots, f_k$, $f_i \in \mathcal{O}(X)_{\chi_i}$. It follows that every weight vector of $\mathcal{O}(X)$ is a scalar multiple of a monomial in the f_i. Setting $f := f_1 \cdots f_k$ we see that every weight vector of $\mathcal{O}(X)_f$ is invertible. We claim that this implies that X_f is a T-orbit. In fact, choose any point $x \in X_f$ and consider the orbit map $\mu_x : T \to X_f$, $t \mapsto tx$. Then $\mu_x^* : \mathcal{O}(X)_f \to \mathcal{O}(T)$ is T-equivariant. Since every weight vector h is invertible, h cannot belong to the kernel of μ_x^*, and so μ_x^* is injective. This implies that μ_x is dominant for every $x \in X_f$, hence X_f is an orbit. $\square$

The last step of the proof above also follows from the previous Proposition 3.6.1. In fact, we can assume that the action of T is faithful, hence there is an orbit with trivial stabilizer by Proposition 3.6.1(1). The last statement of this proposition says that $\dim X = \dim T$, and so T has a dense orbit in X.

We already mentioned an important theorem of Rosenlicht which generalizes both propositons above to an action of an arbitrary algebraic group G on an irreducible variety X, see [**Spr89**, IV.2.2 Satz von Rosenlicht].

3.7. Exercises. For the convenience of the reader we collect here all exercises from the third section.

Exercise. Let V be a (nontrivial) two dimensional $\mathbb{C}^*$-module with weights p, q, i.e. $t(x, y) := (t^p \cdot x, t^q \cdot y)$ for a suitable basis. Determine the weight space decomposition and show the following.

(1) The invariant ring $\mathcal{O}(V)^{\mathbb{C}^*}$ is either $\mathbb{C}$ or a polynomial ring $\mathbb{C}[f]$ in one variable.
(2) If $\mathcal{O}(V)^{\mathbb{C}^*} = \mathbb{C}$, then the weight spaces are finite dimensional.
(3) If $\mathcal{O}(X)^{\mathbb{C}^*} \neq \mathbb{C}$, then the weight spaces are free $\mathcal{O}(X)^{\mathbb{C}^*}$-modules of rank 1.

Exercise. Show that every $\mathcal{O}(X)_n$ is a finitely generated $\mathcal{O}(X)^{\mathbb{C}^*}$-module. (Hint: Modify the proof of Lemma 3.1.4.)

Exercise. Let X be a $\mathbb{C}^*$-variety and $\pi_X : X \to X/\!\!/\mathbb{C}^*$ the quotient.

(1) If $Z \subseteq X$ is closed and $\mathbb{C}^*$-stable, then the induced morphism $\pi_X|_Z : Z \to \pi_X(Z)$ is the quotient of Z by $\mathbb{C}^*$.
(2) If $f \in \mathcal{O}(X)^{\mathbb{C}^*}$ is an invariant, then $\pi_X(X_f) = (X/\!\!/\mathbb{C}^*)_f$, and the induced morphism $\pi_X|_{X_f} : X_f \to (X/\!\!/\mathbb{C}^*)_f$ is the quotient of X_f by $\mathbb{C}^*$.

Exercise. Let $d \in \mathrm{GL}(V)$ be a diagonalizable element. Then $D := \overline{\langle d \rangle}$ is a diagonalizable group, and D/D° is cyclic.

Exercise. Let G be an algebraic group and $\mu^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ the comultiplication (1.3). If χ is a character of G, then $\mu^*(\chi) = \chi \otimes \chi$.

Exercise. For a diagonalizable group D with D/D° cyclic there exists an element $d \in D$ such that $D = \overline{\langle d \rangle}$.

Exercise. Let $d = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$. Describe $D := \overline{\langle d \rangle}$ in terms of a and b. What is D°, and what is D/D°?

Exercise. Let D be a diagonalizable group acting on a variety X, and denote by $\langle \Lambda_X \rangle_{\mathbb{Z}}$ the $\mathbb{Z}$-span of the weights of X. Show that $\mathcal{X}(D)/\langle \Lambda_X \rangle_{\mathbb{Z}}$ is the character group of the kernel of the action of D on X.

Exercise. For every diagonalizable group D and every algebraic group G the map

$$\mathrm{Hom}(G, D) \to \mathrm{Hom}(\mathcal{X}(D), \mathcal{X}(G)), \quad \varphi \mapsto \varphi^*|_{\mathcal{X}(D)},$$

is a bijective homomorphism of groups.

4. Jordan Decomposition and Commutative Algebraic Groups

4.1. Jordan decomposition. Let us first recall the following well-known result from linear algebra.

PROPOSITION 4.1.1. *Every matrix $A \in \mathrm{M}_n(\mathbb{C})$ admits a unique decomposition $A = A_s + A_n$ where A_s is diagonalizable, A_n is nilpotent, and $A_sA_n = A_nA_s$. Similarly, every invertible matrix $g \in \mathrm{GL}_n$ can be uniquely written in the form $g = g_sg_u$ where g_s is diagonalizable, g_u is unipotent, and $g_sg_u = g_ug_s$.*

These decompositions are called the *additive* resp. *multiplicative* JORDAN *decomposition*. They are clearly invariant under conjugation.

EXAMPLE 4.1.2. Let G be one of the classical groups $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}_n, \mathrm{SO}_n, \mathrm{Sp}_{2m}$. *Then, for any $g \in G \subseteq \mathrm{GL}_n$ we have $g_s, g_u \in G$.* This follows immediately from the definition of these groups and the uniqueness of the JORDAN decomposition (see the following Exercises 4.1.3 and 4.1.4). We will see below in Corollary 4.1.6 that this holds for every closed subgroup $G \subseteq \mathrm{GL}_n$.

EXERCISE 4.1.3. Let $g = g_sg_u$ be the JORDAN decomposition of some $g \in \mathrm{GL}_n$. Then $g^t = g_s^tg_u^t$ and $g^{-1} = g_s^{-1}g_u^{-1}$ are the JORDAN decompositions as well.

EXERCISE 4.1.4. Let $B \in \mathrm{M}_n(\mathbb{C})$ be an invertible matrix. Define $G(B) := \{g \in \mathrm{GL}_n \mid g^tBg = B\}$. This is a closed subgroup of GL_n. If $g \in G(B)$ and $g = g_sg_u$ its JORDAN decomposition in GL_n, then $g_s, g_u \in G(B)$.
(Hint: $g \in G(B)$ if and only if $BgB^{-1} = g^{-t}$.)

PROPOSITION 4.1.5. *For any $g \in \mathrm{GL}_n$ with* JORDAN *decomposition $g = g_sg_u$ we have $\overline{\langle g\rangle} = \overline{\langle g_s\rangle} \cdot \overline{\langle g_u\rangle}$ and $\overline{\langle g_s\rangle} \cap \overline{\langle g_u\rangle} = \{e\}$. In particular, $g_s, g_u \in \overline{\langle g\rangle}$.*

PROOF. Using the JORDAN normal form we can assume that g is in upper triangular form and that $g_s \in T_n$ and $g_u \in U_n$. Then $\overline{\langle g\rangle} \subseteq \overline{\langle g_s\rangle} \cdot \overline{\langle g_u\rangle} \subseteq T_nU_n = B_n$, and the projection $B_n \to T_n$ induces a surjective homomorphism $\varphi\colon \overline{\langle g\rangle} \to \overline{\langle g_s\rangle}$. If φ is an isomorphism, then $\overline{\langle g\rangle}$ is a diagonalizable group, hence g is diagonalizable by Proposition 3.4.5(3). Otherwise the kernel is a nontrivial subgroup of $\overline{\langle g_u\rangle} \simeq \mathbb{C}^+$, hence $\ker\varphi = \overline{\langle g_u\rangle}$, and so $g_u \in \overline{\langle g\rangle}$ and $g_s = gg_u^{-1} \in \overline{\langle g\rangle}$. □

COROLLARY 4.1.6. *Let $G \subseteq \mathrm{GL}_n$ be a closed subgroup. For any $g \in G$, with* JORDAN *decomposition $g = g_sg_u$, we have $g_s, g_u \in G$.*

EXERCISE 4.1.7. Let $g \in \mathrm{GL}_n$ and denote by $\mathbb{C}[g] \subseteq \mathrm{M}_n(\mathbb{C})$ the subalgebra generated by g. If $g = g_sg_u$ is the JORDAN decomposition, then $g_s, g_u \in \mathbb{C}[g]$. Moreover, there are polynomials $p(t), q(t)$ of degree $\leq n$ such that $g_s = p(g)$ and $g_u = q(g)$.

EXERCISE 4.1.8. With the notation of the previous exercise assume, in addition, that $g \in \mathrm{GL}_n(K)$ for a subfield $K \subseteq \mathbb{C}$. Then $g_s, g_u \in K[g] \subseteq \mathrm{M}_n(K)$.
(Hint: $K[g] \simeq K[t]/(m_g)$ where m_g is the minimal polynomial of g which has coefficients in K. Moreover, there is a finite extension K'/K which contains g_s and g_u. Now use the action of the Galois group and the uniqueness of the JORDAN decomposition.)

EXERCISE 4.1.9. Let $g \in \mathrm{GL}_n$ with JORDAN decomposition $g = g_sg_u$. If the subspace $W \subseteq \mathbb{C}^n$ is stable under g, then W is stable under g_s and under g_u.

4.2. Semisimple elements. We finally want to show that the JORDAN decomposition does not depend on the embedding $G \subseteq \mathrm{GL}_n$. For this we make the following definition.

DEFINITION 4.2.1. An element g of an algebraic group G is called *semisimple* if $\overline{\langle g\rangle}$ is a diagonalizable group.

PROPOSITION 4.2.2. *Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups.*

(1) *If $u \in G$ is unipotent, then so is $\varphi(u) \in H$.*
(2) *If $s \in G$ is semisimple, then so is $\varphi(s) \in H$.*
(3) *An element $g \in \mathrm{GL}_n$ is semisimple if and only if g is diagonalizable.*

PROOF. For any $g \in G$ we have $\varphi(\overline{\langle g \rangle}) = \overline{\langle \varphi(g) \rangle}$. Now (1) follows, because $u \in G$ is unipotent if and only if either $u = e$ or $\overline{\langle u \rangle} \simeq \mathbb{C}^+$ (Definition 2.6.3). Statement (2) follows from Proposition 3.4.5(2), and (3) from Proposition 3.4.5(3). $\square$

COROLLARY 4.2.3. *Let G be an algebraic group and $g \in G$. Then the* JORDAN *decomposition $g = g_s g_u$ is independent of the choice of an embedding $G \subseteq \mathrm{GL}_n$.*

PROOF. This is clear, because the decomposition $\overline{\langle g \rangle} = \overline{\langle g_s \rangle} \cdot \overline{\langle g_u \rangle}$ is independent of the choice of the embedding, by the proposition above. $\square$

EXERCISE 4.2.4. If $g \in G$ is semisimple, then $\overline{\langle g \rangle}/\overline{\langle g \rangle}^\circ$ is cyclic.

EXERCISE 4.2.5. Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups. If $u \in \varphi(G)$ is unipotent, then $\varphi^{-1}(u)$ contains unipotent elements. If $s \in \varphi(G)$ is semisimple, then $\varphi^{-1}(s)$ contains semisimple elements.

4.3. Commutative algebraic groups. For any algebraic group G denote by $G_u \subseteq G$ the set of unipotent elements and by $G_s \subseteq G$ the set of semisimple elements. Embedding G into GL_n we see that $G_u \subseteq G$ is a closed subset whereas G_s can be dense like in the case of $G = \mathrm{GL}_n$. However, for commutative groups the situation is much nicer.

Let us first discuss the case of unipotent group. Recall that an algebraic group U is called unipotent if every element of U is unipotent (Definition II.2.6.3).

EXERCISE 4.3.1. A unipotent group U is connected.
(Hint: For $u \in U$, $u \neq e$, the subgroup $\overline{\langle u \rangle} \cap U^\circ$ has finite index in $\overline{\langle u \rangle} \simeq \mathbb{C}^+$.)

If W is a finite dimensional vector space, then the underlying additive group W^+ is a unipotent group. We claim that every commutative unipotent group has this form.

PROPOSITION 4.3.2. *Let U be a commutative unipotent group of dimension m. Then U is isomorphic to $(\mathbb{C}^+)^m = (\mathbb{C}^m)^+$. More precisely, there is a canonical isomorphism* $\exp\colon \operatorname{Lie} U \xrightarrow{\sim} U$ *of algebraic groups, and it induces an isomorphism* $\mathrm{GL}(\operatorname{Lie} U) \xrightarrow{\sim} \mathrm{Aut}(U)$.

PROOF. If $U' \subseteq U$ is a closed subgroup and $u \in U \setminus U'$, then $\overline{\langle u \rangle} \cap U' = \{e\}$ and $\mathbb{C}^+ \times U' \xrightarrow{\sim} \overline{\langle u \rangle} \cdot U'$. Thus, by induction, there is an isomorphism $(\mathbb{C}^+)^m \simeq U$. It follows from Proposition II.2.6.1 that every $A \in \operatorname{Lie} U$ belongs to the Lie algebra of a subgroup isomorphic to $\mathbb{C}^+$. As a consequence, for every representation $\rho\colon U \to \mathrm{GL}(V)$ the image $d\rho(A)$ is nilpotent for all $A \in \operatorname{Lie} U$.

Now choose a closed embedding $\rho\colon U \hookrightarrow \mathrm{GL}(V)$. Then $\operatorname{Lie} U \subseteq \mathrm{End}(V)$ consists of pairwise commuting nilpotent elements, and so $\exp\colon \operatorname{Lie} U \to \mathrm{GL}(V)$ is an injective homomorphism of algebraic groups (Proposition II.2.5.1). We know from Proposition II.2.6.1 that for every nilpotent $A \in \mathrm{End}(V)$, $A \neq 0$, there is a unique one-dimensional unipotent subgroup $U' \subseteq \mathrm{GL}(V)$ such that $A \in \operatorname{Lie} U'$. This implies that $\exp(\operatorname{Lie} U) = U$. It is now easy to see that this isomorphism is independent of the choice of the embedding of U into some $\mathrm{GL}(V)$.

The last statement is clear, because $\mathrm{End}(\mathbb{C}^+) = \mathbb{C}$. $\square$

REMARK 4.3.3. The proof above shows that for every representation $\rho\colon U \to \mathrm{GL}(V)$ of a commutative unipotent group U the image $d\rho(\operatorname{Lie} U) \subseteq \mathrm{End}(V)$ consists of nilpotent endomorphisms. We will see later that this holds for any unipotent group.

Now we can describe the structure of commutative algebraic groups.

PROPOSITION 4.3.4. *Let H be a commutative algebraic group. Then H_u and H_s are closed subgroups, H_s is a diagonalizable group and H_u a unipotent group, $H_s \cap H_u = \{e\}$, and the multiplication $H_s \times H_u \xrightarrow{\sim} H$ is an isomorphism. In particular,* $\operatorname{Lie} H = \operatorname{Lie} H_s \oplus \operatorname{Lie} H_u$.

PROOF. We have already seen earlier that H_u is a closed subgroup (see Exercise 2.6.6). We claim that H_s is also closed, hence a diagonalizable group (Proposition 3.4.5). In fact, let $F \subseteq \overline{H_s}$ be the subgroup of elements of finite order. Since the elements of finite order in $\overline{\langle g \rangle}$ are dense for any semisimple element $g \in H$, we see that $H_s \subseteq \bar{F} \subseteq \overline{H_s}$, hence $\bar{F} = \overline{H_s}$, and so $H_s = \overline{H_s}$.

It follows that $H_s \cap H_u = \{e\}$, and we obtain an injective homomorphism $H_s \times H_u \to H$. Since any $h \in H$ has a JORDAN decomposition $h = h_s h_u$ with $h_s, h_u \in H$ we get $h_s \in H_s$ and $h_u \in H_u$, and so $H_s \times H_u \xrightarrow{\sim} H$ is an isomorphism. □

COROLLARY 4.3.5. *A one-dimensional connected algebraic group is isomorphic to $\mathbb{C}^*$ or to $\mathbb{C}^+$.*

PROOF. We know that a one-dimensional algebraic group is commutative (Example 1.4.8). Thus the claim follows from the proposition above together with Proposition 3.4.1. □

EXERCISE 4.3.6. A connected commutative algebraic group H is divisible, i.e., the map $h \mapsto h^m$ is surjective for every $m \in \mathbb{Z} \setminus \{0\}$.

EXERCISE 4.3.7. Let $g := \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$. What is the structure of $\overline{\langle g \rangle}$? Show that the subgroup $\overline{\langle g \rangle} \subseteq \mathrm{GL}_2$ is defined by two linear equations.

We have seen that a representation of a diagonalizable group is diagonalizable. This generalizes to commutative groups in the way that every representation is "triagonalizable". We will see later that this holds more generally for connected solvable groups.

PROPOSITION 4.3.8. *Let $\rho\colon H \to \mathrm{GL}(V)$ be an n-dimensional representation of a commutative group H. Then there is a basis of V such that $\rho(H) \subseteq B_n$, $\rho(H_s) \subseteq T_n$, and $\rho(H_u) \subseteq U_n$.*

PROOF. (a) First assume that H is unipotent. If $h \in H$ is a nontrivial element and $W \subseteq V$ an eigenspace of h, then W is H-stable and $W \neq V$. By induction, we can assume that the claim holds for the representation of H on W and on V/W, and the proposition follows in this case.

(b) In general, we decompose V into weight spaces with respect to the diagonalizable group H_s: $V = \bigoplus_{\chi \in \mathcal{X}(H_s)} V_\chi$, see Remark 3.4.7. Since every weight space V_χ is stable under H, the claim follows from (a) applied to the representation of H_u on V_χ. □

4.4. Exercises. For the convenience of the reader we collect here all exercises from the forth section.

EXERCISE. Let $g = g_s g_u$ be the JORDAN decomposition of some $g \in \mathrm{GL}_n$. Then $g^t = g_s^t g_u^t$ and $g^{-1} = g_s^{-1} g_u^{-1}$ are the JORDAN decompositions as well.

EXERCISE. Let $B \in \mathrm{M}_n(\mathbb{C})$ be an invertible matrix. Define $G(B) := \{g \in \mathrm{GL}_n \mid g^t B g = B\}$. This is a closed subgroup of GL_n. If $g \in G(B)$ and $g = g_s g_u$ its JORDAN decomposition in GL_n, then $g_s, g_u \in G(B)$.
(Hint: $g \in G(B)$ if and only if $BgB^{-1} = g^{-t}$.)

EXERCISE. Let $g \in \mathrm{GL}_n$ and denote by $\mathbb{C}[g] \subseteq \mathrm{M}_n(\mathbb{C})$ the subalgebra generated by g. If $g = g_s g_u$ is the JORDAN decomposition, then $g_s, g_u \in \mathbb{C}[g]$. Moreover, there are polynomials $p(t), q(t)$ of degree $\leq n$ such that $g_s = p(g)$ and $g_u = q(g)$.

EXERCISE. With the notation of the previous exercise assume, in addition, that $g \in \mathrm{GL}_n(K)$ for a subfield $K \subseteq \mathbb{C}$. Then $g_s, g_u \in K[g] \subseteq \mathrm{M}_n(K)$.
(Hint: $K[g] \simeq K[t]/(m_g)$ where m_g is the minimal polynomial of g which has coefficients in K. Moreover, there is a finite extension K'/K which contains g_s and g_u. Now use the action of the Galois group and the uniqueness of the JORDAN decomposition.)

EXERCISE. Let $g \in \mathrm{GL}_n$ with JORDAN decomposition $g = g_s g_u$. If the subspace $W \subseteq \mathbb{C}^n$ is stable under g, then W is stable under g_s and under g_u.

EXERCISE. If $g \in G$ is semisimple, then $\overline{\langle g \rangle}/\overline{\langle g \rangle}^\circ$ is cyclic.

EXERCISE. Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups. If $u \in \varphi(G)$ is unipotent, then $\varphi^{-1}(u)$ contains unipotent elements. If $s \in \varphi(G)$ is semisimple, then $\varphi^{-1}(s)$ contains semisimple elements.

EXERCISE. A unipotent group U is connected.
(Hint: For $u \in U$, $u \neq e$, the subgroup $\overline{\langle u \rangle} \cap U^\circ$ has finite index in $\overline{\langle u \rangle} \simeq \mathbb{C}^+$.)

EXERCISE. A connected commutative algebraic group H is divisible, i.e., the map $h \mapsto h^m$ is surjective for every $m \in \mathbb{Z} \setminus \{0\}$.

EXERCISE. Let $g := \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$. What is the structure of $\overline{\langle g \rangle}$? Show that the subgroup $\overline{\langle g \rangle} \subseteq \mathrm{GL}_2$ is defined by two linear equations.

5. The Correspondence between Groups and Lie Algebras

5.1. The differential of the orbit map. Consider a linear action of an algebraic group G on a finite dimensional vector space V, given by a representation $\rho\colon G \to \mathrm{GL}(V)$. It is easy to see that the differential of the action $\rho\colon G \times V \to V$ has the following form:

$$d\rho_{(e,w)}\colon \operatorname{Lie} G \oplus V \to V, \ (A, v) \mapsto d\rho(A)\, w + v.$$

In fact, we can reduce to the case $G = \mathrm{GL}(V)$ where the action extends to a bilinear map $\mathrm{End}(V) \times V \to V$. We will shortly right Aw for $d\rho(A)\, w$.

In particular, the differential of the orbit map $\mu\colon G \to V$, $g \mapsto gw$, is given by

$$d\mu_e\colon \operatorname{Lie} G \to V, \ A \mapsto Aw.$$

This implies the first statement of the following lemma. The second is obtained by considering the action $G \times W \to W$.

LEMMA 5.1.1. (1) *For a fixed point $v \in V^G$ we have $Av = 0$ for all $A \in \operatorname{Lie} G$.*
(2) *If $W \subseteq V$ is G-stable subspace, then $AW \subseteq W$ for all $A \in \operatorname{Lie} G$.*

EXAMPLE 5.1.2. Consider the action of GL_n on the matrices M_n by conjugation, $(g, A) \mapsto gAg^{-1}$. In this case, $d\rho(A) = \operatorname{ad} A = [A, -]$ (Proposition 4.4.1), and so the differential $d\rho_{(E,A)}\colon \mathfrak{gl}_n \oplus \mathrm{M}_n \to \mathrm{M}_n$ is given by

$$(X, B) \mapsto [X, A] + B = XA - AX + B.$$

It follows that the differential $d\mu$ of the orbit map in $A \in \mathrm{M}_n$ has image $[\mathfrak{gl}_n, A]$, and its kernel is $L := \{X \in \mathfrak{gl}_n \mid XA = AX\}$ which is a subalgebra of the matrix algebra M_n. This shows that the stabilizer of A in GL_n is equal to L^*, the invertible elements of L. Hence $L = \operatorname{Lie}(\mathrm{GL}_n)_A$, and $[\mathfrak{gl}_n, A] = T_A C_A$ where $C_A \subseteq \mathrm{M}_n$ is the conjugacy class of A. (The latter follows because the differential of the orbit map is surjective on an open set of GL_n, hence everywhere.) We will prove this for any G-action on a variety in Lemma 5.1.5 below.

Example 5.1.3. Let V be a finite dimensional vector space and $W \subseteq V$ a subspace. Define $N := \mathrm{N}_{\mathrm{GL}(V)} W$ to be the normalizer of W. Then N is a closed connected subgroup and

$$\operatorname{Lie} N = \{A \in \operatorname{End}(V) \mid AW \subseteq W\}.$$

Proof. N is a closed subgroup by Proposition 1.2.3, and $\operatorname{Lie} N \subseteq L := \{A \in \operatorname{End}(V) \mid AW \subseteq W\}$ by the previous lemma. By definition, $N = L \cap \mathrm{GL}(V)$ and thus N is a nonempty open subset of the vector space L. Therefore, N is connected and $\dim N = \dim L$, and so $\operatorname{Lie} N = L$. □

Example 5.1.4. Let $v \in V$ and let $H := \mathrm{GL}(V)_v$ be the stabilizer of v. Then H is connected and

$$\operatorname{Lie} H = \{A \in \operatorname{End}(V) \mid Av = 0\}.$$

Proof. The first statement is clear, since we can assume that $v = e_1$. We know from the lemma above that $\operatorname{Lie} H \subseteq L := \{A \in \operatorname{End}(V) \mid Av = 0\}$. Therefore it suffices to show that $\dim H \geq \dim L$. We can assume that $v \neq 0$ and choose a basis of V containing v. Then H is defined by $n = \dim V$ equations whereas L is defined by n linearly independent linear equations. Thus $\dim L = n^2 - n \leq \dim H$, and so $\operatorname{Lie} H = L$. □

Now we extend these results to an arbitrary G-action on a variety X. Let $x \in X$ and consider the orbit map

$$\mu_x \colon G \to X, \ g \mapsto gx.$$

Lemma 5.1.5. *For the differential* $d\mu_e \colon \operatorname{Lie} \to T_x X$ *of the orbit map* $\mu = \mu_x \colon G \to X$ *we get*

$$\operatorname{im} d\mu_e = T_x(Gx) = T_x(\overline{Gx}) \quad \textit{and} \quad \ker d\mu_e = \operatorname{Lie} G_x.$$

Proof. $G^\circ x$ is a connected component of Gx and so $T_x G^\circ x = T_x Gx$. Thus we can assume that G is connected and that $X = \overline{Gx}$ is irreducible. Since $d\mu_g$ is surjective on a dense open set of G (see A.4.9) it is surjective everywhere. Thus $\operatorname{im} d\mu_e = T_x(Gx) = T_x(\overline{Gx})$.

Now $\dim \ker d\mu_e = \dim \operatorname{Lie} G - \dim T_x Gx = \dim G - \dim Gx = \dim G_x$, by the dimension formula for orbits (1.3). Since $\operatorname{Lie} G_x \subseteq \ker d\mu_e$ we finally get $\operatorname{Lie} G_x = \ker d\mu_e$. □

Exercise 5.1.6. Consider the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on M_2 defined by $(g,h)A := gAh^{-1}$. Calculate the differential of the orbit map in A, determine its image and its kernel, and verify the claims of Lemma 5.1.5.

5.2. Subgroups and subalgebras. The following results show that there are very strong relations between (connected) algebraic groups and their Lie algebras.

Proposition 5.2.1. (1) *Let* $\varphi, \psi \colon G \to H$ *be two homomorphisms. If* G *is connected, then* $d\varphi_e = d\psi_e$ *implies that* $\varphi = \psi$.

(2) *If* $H_1, H_2 \subseteq G$ *are closed subgroups, then* $\operatorname{Lie}(H_1 \cap H_2) = \operatorname{Lie} H_1 \cap \operatorname{Lie} H_2$. *In particular, if* $\operatorname{Lie} H_1 = \operatorname{Lie} H_2$, *then* $H_1^\circ = H_2^\circ$.

(3) *Let* $\varphi \colon G \to H$ *be a homomorphism, and let* $H' \subseteq H$ *be a closed subgroup. Then*

$$\operatorname{Lie} \varphi(G) = d\varphi_e(\operatorname{Lie} G) \quad \textit{and} \quad \operatorname{Lie} \varphi^{-1}(H') = (d\varphi_e)^{-1}(\operatorname{Lie} H').$$

In particular, $\operatorname{Lie} \ker \varphi = \ker d\varphi_e$.

PROOF. (1) We define the following G-action on H: $gh := \varphi(g)h\psi(g)^{-1}$. Then $G_e = \{g \in G \mid \varphi(g) = \psi(g)\}$. For the orbit map $\mu\colon G \to H$ we get, by assumption, that $d\mu_e(A) = d\varphi(A) - d\psi(A) = 0$, and so $\operatorname{Lie} G_e = \ker d\mu_e = \operatorname{Lie} G$, by Lemma 5.1.5. Thus $G_e = G$, because G is connected, and the claim follows.

(2) Here we consider the action of $H_1 \times H_2$ on G given by $(h_1, h_2)g := h_1 g h_2^{-1}$. Then $(H_1 \times H_2)_e \simeq H_1 \cap H_2$. For the orbit map $\mu\colon H_1 \times H_2 \to G$, $(h_1, h_2) \mapsto h_1 h_2^{-1}$ we have $d\mu_e(A_1, A_2) = A_1 - A_2$, and so $\ker d\mu_e \simeq \operatorname{Lie} H_1 \cap \operatorname{Lie} H_2$. Again the claim follows from Lemma 5.1.5 above.

(3) Now consider the action of G on H by $gh := \varphi(g)h$. Then φ is the orbit map in $e \in H$, and so, again by Lemma 5.1.5, $d\varphi_e(\operatorname{Lie} G) = \operatorname{im} d\varphi_e = T_e\varphi(G) = \operatorname{Lie} \varphi(G)$ and $\ker d\varphi_e = \operatorname{Lie} \ker \varphi$. This proves the first claim and shows that the map $d\varphi_e$ has rank $\dim \varphi(G)$. Because of (2) we can replace H' by $H' \cap \varphi(G)$. Then $\varphi^{-1}(H') \to H'$ is surjective with kernel $\ker \varphi$, and so both sides from the second equality have the same dimension. Since $d\varphi_e(\operatorname{Lie} \varphi^{-1}(H')) \subseteq \operatorname{Lie} H'$ the second claim follows also. □

COROLLARY 5.2.2. *The correspondence $H \mapsto \operatorname{Lie} H$ between closed connected subgroups of G and Lie subalgebras of $\operatorname{Lie} G$ is injective and compatible with inclusions and intersections.*

5.3. Representations of Lie algebras. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of the algebraic group G. Then the differential

$$d\rho\colon \operatorname{Lie} G \to \operatorname{End}(V)$$

is a *representation of the Lie algebra* $\operatorname{Lie} G$. This means that

$$d\rho[A, B] = [d\rho(A), d\rho(B)] = d\rho(A) \circ d\rho(B) - d\rho(B) \circ d\rho(A) \text{ for } A, B \in \operatorname{Lie} G.$$

In this way we obtain an *action of* $\operatorname{Lie} G$ *on* V *by linear endomorphisms* which will be shortly denoted by $Av := d\rho(A)v$.

EXERCISE 5.3.1. Let G be a connected group and let $\rho\colon G \to \mathrm{GL}(V)$ and $\mu\colon G \to \mathrm{GL}(W)$ be two representations. Then ρ is equivalent to μ if and only if $d\rho\colon \operatorname{Lie} G \to \operatorname{End}(V)$ is equivalent to $d\mu\colon \operatorname{Lie} G \to \operatorname{End}(W)$.

Now we can extend Examples 5.1.3 and 5.1.4 to arbitrary representations.

PROPOSITION 5.3.2. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. For any $v \in V$ and any subspace $W \subseteq V$ we have*

$$\operatorname{Lie} G_v = \{A \in \operatorname{Lie} G \mid Av = 0\} \text{ and } \operatorname{Lie} \mathrm{N}_G(W) = \{A \in \operatorname{Lie} G \mid AW \subseteq W\}.$$

If G is connected, then v is a fixed point (resp. W is G-stable) if and only if $Av = 0$ (resp. $AW \subseteq W$) for all $A \in \operatorname{Lie} G$.

PROOF. Proposition 5.2.1(3) implies that we can replace G by its image in $\mathrm{GL}(V)$, hence can assume that $G \subseteq \mathrm{GL}(V)$. Then $G_v = \mathrm{GL}(V)_v \cap G$ and $\mathrm{N}_G(W) = \mathrm{N}_{\mathrm{GL}(V)}(W) \cap G$, and the claims follow from Proposition 5.2.1(2) and the Examples 5.1.3 and 5.1.4. □

EXAMPLE 5.3.3. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation, and let $\eta \in \operatorname{End}(V)$. Define $G_\eta := \{g \in G \mid \rho(g) \circ \eta = \eta \circ \rho(g)\}$. Then $\operatorname{Lie} G_\eta = \{A \in \operatorname{Lie} G \mid [d\rho(A), \eta] = 0\}$.

For any Lie algebra L we denote by

$$\mathfrak{z}(L) := \{A \in L \mid [A, B] = 0 \text{ for all } B \in L\}$$

the *center of* L.

EXERCISE 5.3.4. Show that $\mathfrak{z}(L)$ is a characteristic ideal of L, i.e. $\mathfrak{z}(L)$ is stable under every automorphism of the Lie algebra L.

Corollary 5.3.5. *The kernel of the adjoint representation* $\operatorname{Ad}\colon G \to \operatorname{GL}(\operatorname{Lie} G)$ *of a connected group* G *is the center* $\operatorname{Z}(G)$ *of* G*, and* $\operatorname{Lie}\operatorname{Z}(G) = \mathfrak{z}(\operatorname{Lie} G)$*.*

Proof. Since $\operatorname{Ad} g = d(\operatorname{Int} g)$ we get from Proposition 5.2.1(1) that $\operatorname{Ad} g = \operatorname{Ad} e = \mathrm{id}$ if and only if $\operatorname{Int} g = \operatorname{Int} e = \mathrm{id}$ which means that $g \in \operatorname{Z}(G)$. Hence $\operatorname{Z}(G) = \ker \operatorname{Ad}$, and so $\operatorname{Lie}\operatorname{Z}(G) = \ker \operatorname{ad} = \mathfrak{z}(\operatorname{Lie} G)$, by Proposition 5.2.1(3). □

Corollary 5.3.6. *A connected algebraic group* G *is commutative if and only if its Lie algebra* $\operatorname{Lie} G$ *is commutative.*

Example 5.3.7. A one-dimensional connected algebraic group is commutative, because a one-dimensional Lie algebra is commutative (cf. Example II.1.4.8).

Corollary 5.3.8. *Let* $H \subseteq G$ *be a connected subgroup. Then*

$$\operatorname{Lie}\operatorname{N}_G(H) = \{A \in \operatorname{Lie} G \mid [A, \operatorname{Lie} H] \subseteq \operatorname{Lie} H\}.$$

In particular, if G *is connected, then* H *is normal if and only if* $\operatorname{Lie} H$ *is an ideal in* $\operatorname{Lie} G$*.*

Proof. Applying Proposition 5.2.1(3) to the homomorphism $\operatorname{Int} g\colon H \to G$ we see that $\operatorname{Lie} gHg^{-1} = \operatorname{Ad} g(\operatorname{Lie} H)$. Since H is connected this implies that $gHg^{-1} = H$ if and only if $\operatorname{Ad} g(\operatorname{Lie} H) = \operatorname{Lie} H$. Hence $\operatorname{N}_G(H) = \operatorname{N}_G(\operatorname{Lie} H)$, and the claim follows from Proposition 5.3.2 applied to the adjoint representation $\operatorname{Ad}\colon G \to \operatorname{GL}(\operatorname{Lie} G)$. □

Exercise 5.3.9. Let G be a connected noncommutative 2-dimensional algebraic group. Then

(1) $\operatorname{Z}(G)$ is finite;
(2) The unipotent elements G_u form a normal closed subgroup isomorphic to $\mathbb{C}^+$;
(3) There is a subgroup $T \subseteq G$ isomorphic to $\mathbb{C}^*$ such that $G = T \cdot G_u$.

(Hint: Study the adjoint representation $\operatorname{Ad}\colon G \to \operatorname{GL}(\operatorname{Lie} G)$, and use Exercise 4.1.3.)

Exercise 5.3.10. Use the previous exercise to show that every 2-dimensional closed subgroup of SL_2 is conjugate to $B_2' := B_2 \cap \mathrm{SL}_2$.

5.4. Vector fields on G-varieties. Let X be a G-variety. To any $A \in \operatorname{Lie} G$ we associate a vector field ξ_A on X in the following way:

$$(\xi_A)_x := d\mu_x(A) \text{ for } x \in X$$

where $\mu_x\colon G \to X$ is the orbit map in $x \in X$. If $X = V$ is a vector space and $G \subseteq \operatorname{GL}(V)$, then, for $A \in \operatorname{Lie} G \subseteq \operatorname{End}(V)$ we get $(\xi_A)_v = Av \in V = T_v V$ (5.1) which corresponds to the derivation $\partial_{Av,v} \in T_v V$ (see A.4.5 and Example A.4.5.2). This shows that ξ_A is an algebraic vector field and that $A \mapsto \xi_A$ is a linear map. In particular, for a linear function $\ell \in V^*$ we find

$$\xi_A \ell(v) = \partial_{Av,v}\,\ell = \frac{\ell(v + \varepsilon Av) - \ell(v)}{\varepsilon}|_{\varepsilon=0} = \ell(Av).$$

Choosing a basis of V and identifying V with $\mathbb{C}^n$ and $\operatorname{End}(V)$ with M_n we get $\xi_{E_{ij}} = x_j \frac{\partial}{\partial x_i}$, and so

$$\xi_A = \sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_i} = \sum_i (Ax)_i \frac{\partial}{\partial x_i}.$$

In particular, for a linear function $\ell \in V^*$ we find $\xi_A \ell(v) = \ell(Av)$. A simple calculation shows that $\xi_{[A,B]} = [\xi_B, \xi_A]$. Thus we have proved the following result.

Proposition 5.4.1. *For every* $A \in \operatorname{Lie} G$ *the vector field* ξ_A *on* X *is algebraic, and the map* $A \mapsto \xi_A$ *is an antihomomorphism* $\operatorname{Lie} G \to \operatorname{Vec}(X)$ *of Lie algebras.*

EXAMPLE 5.4.2. Consider an action of an algebraic group G on affine n-space $\mathbb{C}^n$:

$$\rho(g)(x_1,\dots,x_n) = (f_1(g,x_1,\dots,x_n),\dots,f_n(g,x_1,\dots,x_n)) \in \operatorname{Aut}(\mathbb{C}^n).$$

Then, for any $A \in \operatorname{Lie} G$, the vector field ξ_A on $\mathbb{C}^n$ is given by

$$(\xi_A)_x = Af_1(x)\frac{\partial}{\partial x_1} + \cdots Af_n(x)\frac{\partial}{\partial x_n}$$

where $Af_i(x)$ is the derivative of the function $g \mapsto f_i(g,x)$ on G with respect to the tangent vector $A \in \operatorname{Lie} G$. It is formally defined by

$$Af_i(x) = \frac{f_i(e+\varepsilon A, x) - f_i(e,x)}{\varepsilon}\Big|_{\varepsilon=0}.$$

Recall that the *divergence* of a vector field $\delta = \sum_i p_i \frac{\partial}{\partial x_i}$ is defined by

$$\operatorname{div}\delta := \sum_i \frac{\partial p_i}{\partial x_i}.$$

We want to show that $\operatorname{div}\xi_A$ is a constant for any vector field ξ_A induced by an action of an algebraic group.

If $\varphi = (f_1,\dots,f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism, then the Jacobian determinant

$$\operatorname{jac}(\varphi) := \det\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

does not vanish in any point of $\mathbb{C}^n$ and thus is a nonzero constant. It is easy to see that the map $\varphi \mapsto \operatorname{jac}(\varphi)$ is a homomorphism of groups, and so the map $\chi\colon g \mapsto \operatorname{jac}(\rho(g))$ is a character of G. We claim that the differential of χ is given by

$$d\chi(A) = \operatorname{div}\xi_A = \sum_i \frac{\partial Af_i(x)}{\partial x_i},$$

and thus $\operatorname{div}\xi_A \in \mathbb{C}$ as we wanted to show.

In order to prove the claim, consider the morphism $\operatorname{jac}\colon (\mathbb{C}[x]_{\le m})^n \to \mathbb{C}[x]_{\le nm}$ where $\mathbb{C}[x]_{\le m} := \{f \in \mathbb{C}[x_1,\dots,x_n] \mid \deg f \le m\}$, and set $e := (x_1,\dots,x_n)$. Then we get

$$\operatorname{jac}(x_1+\varepsilon p_1,\dots,x_n+\varepsilon p_n) = \det\begin{bmatrix} 1+\varepsilon\frac{\partial p_1}{\partial x_1} & \cdots & \varepsilon\frac{\partial p_1}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \varepsilon\frac{\partial p_n}{\partial x_1} & \cdots & 1+\varepsilon\frac{\partial p_n}{\partial x_n} \end{bmatrix} = 1+\varepsilon\sum_i \frac{\partial p_i}{\partial x_i},$$

and so the differential $d\operatorname{jac}_e\colon (\mathbb{C}[x]_{\le m})^n \to \mathbb{C}[x]_{\le nm}$ is given by $d\operatorname{jac}_e(p_1,\dots,p_n) = \sum_i \frac{\partial p_i}{\partial x_i} = \operatorname{div}(\sum_i p_i\frac{\partial}{\partial x_i})$, and the claim follows.

We have seen in Proposition 5.4.1 above that the map $A \mapsto \xi_A$ is an *antihomomorphism* of Lie algebras. This has the following simple explanation.

PROPOSITION 5.4.3. *Let X be a G-variety. Then the regular representation of G on $\mathcal{O}(X)$ defines a locally finite representation of* $\operatorname{Lie} G$ *on $\mathcal{O}(X)$ by derivations which is given by $A \mapsto -\xi_A$.*

PROOF. Again we can assume that X is a vector space with a linear representation of G. For any $f \in \mathcal{O}(V)$ and $A \in \operatorname{Lie} G \subseteq \operatorname{End}(V)$ we get

$$(e+\varepsilon A)f(v) = f((e+\varepsilon A)^{-1}v) = f(v-\varepsilon Av) = f(v) - \xi_A f$$

which shows that $Af = -\xi_A f$. ☐

Exercise 5.4.4. Let X be a G-variety and assume that G is connected. A regular function $f \in \mathcal{O}(X)$ is a G-invariant if and only if $\xi_A f = 0$ for all $A \in \operatorname{Lie} G$.
(Hint: Look at the regular representation of G on $\mathcal{O}(X)$ and use Proposition 5.3.2.)

As in II.4.5 there is another description of the vector field ξ_A. Denote by $\mu\colon G \times X \to X$ the action of G on X, and consider $A \in \operatorname{Lie} G$ as a derivation $\mathcal{O}(G) \to \mathbb{C}$ in e. Then ξ_A, as a derivation of $\mathcal{O}(X)$, is given by the composition

$$\xi_A\colon \mathcal{O}(X) \xrightarrow{\ \mu^*\ } \mathcal{O}(G) \otimes \mathcal{O}(X) \xrightarrow{A \otimes \mathrm{id}} \mathcal{O}(X).$$

In fact, if $\mu^*(f) = \sum_i h_i \otimes f_i$, then $\mu_x^*(f) = \sum_i h_i f_i(x) \in \mathcal{O}(G)$, and so

$$((A \otimes \mathrm{id})\mu^*(f))(x) = (\sum_i Ah_i \otimes f)(x) = \sum_i Ah_i \cdot f_i(x) = A(\mu_x^*(f)) = d\mu_x(A)f.$$

We have seen in II.4.5 that the Lie algebra structure on $T_eG = \operatorname{Lie} G$ can be obtained via left or right invariant vector fields on G. What is the relation with the construction above? If we take the *right action* $\rho_g\colon (g,h) \mapsto hg^{-1}$ and construct the vector field ξ_A as above, then $\xi_A = \tilde{\delta}_A$, where $\tilde{\delta}_A$ is the right-invariant vector field on G with $(\tilde{\delta}_A)_e = A$ (see Remark II.4.5.2). This explains again why we have an antihomomorphism in Proposition 5.4.1.

Example 5.4.5. Let X be a G-variety and $Y \subseteq X$ a closed G-stable subset. Then $(\xi_A)_y \in T_yY$ for all $A \in \operatorname{Lie} G$ and all $y \in Y$. In particular, if $x \in X$ is a fixed point, then $(\xi_A)_x = 0$ for all $A \in \operatorname{Lie} G$.

Exercise 5.4.6. Let X be a G-variety where G is connected. Then $x \in X$ is a fixed point if and only if $(\xi_A)_x = 0$ for all $A \in \operatorname{Lie} G$.

Corollary 5.4.7. *Let X be a G-variety where G is connected, and let $Y \subseteq X$ be an irreducible closed subset. Then Y is G-stable if and only if $(\xi_A)_y \in T_yY$ for all $A \in \operatorname{Lie} G$ and $y \in Y$.*

Proof. The differential of the dominant morphism $\varphi\colon G \times Y \to \overline{GY}$ is given by

$$d\varphi_{(e,y)}\colon \operatorname{Lie} G \oplus T_yY \to T_y\overline{GY},\ (A,v) \mapsto (\xi_A)_y + v.$$

By assumption, the image $d\varphi_{(e,y)}$ is contained in T_yY. On the other hand, there is an open dense set $U \subseteq G \times Y$ where the differential is surjective (A.4.9), and by the G-equivariance of φ there is a point of the form (e,y) such that $d\varphi_{(e,y)}$ is surjective. Hence, $\dim_y \overline{GY} = \dim_y Y$ and so $Y = \overline{GY}$, because $\overline{GY}$ is irreducible. □

With almost the same argument we can show the following.

Corollary 5.4.8. *Let X be a G-variety and $Y \subseteq X$ a locally closed irreducible subset. Assume that $T_yY \subseteq T_yGy$ for all $y \in Y$. Then $Y \subseteq \overline{Gy_0}$ for a suitable $y_0 \in Y$.*

Proof. We can assume that G is connected and that Y is a special open set of $\overline{Y}$, hence an affine variety. Consider the morphism $\mu\colon G \times Y \to \overline{GY} \subseteq X$ and its differential

$$d\mu_{(e,y)}\colon \operatorname{Lie} G \oplus T_yY \to T_y(\overline{GY}), \quad (A,v) \mapsto (\xi_A)_y + v.$$

By assumption, we get $\operatorname{im} d\mu_{(e,y)} = T_y(Gy)$ for every $y \in Y$. As above we can find a point $y_0 \in Y$ such that $d\mu_{(e,y_0)}$ is surjective. Hence, $T_{(e,y_0)}(\overline{GY}) = \operatorname{im} d\mu_{(e,y_0)} = T_{(e,y_0)}(Gy_0)$ which implies that $\dim \overline{GY} \le \dim Gy_0$. Since both varieties, $\overline{GY}$ and $\overline{Gy_0}$, are irreducible, we finally get $\overline{GY} = \overline{Gy_0}$, hence the claim. □

5.5. G-action on vector fields. In section II.4.5 we showed that the G-invariant vector fields on the group G can be used to define the Lie algebra structure on the tangent space T_eG. For an arbitrary G-variety X a vector field $\xi \in \mathrm{Vec}(X)$ is called *G-invariant* if it is invariant under left multiplication $\mu_g : x \mapsto gx$ for all $g \in G$, i.e. if $d\mu_g \xi_x = \xi_{gx}$ for all $g \in G$ and $x \in X$.

EXAMPLE 5.5.1. If V is a G-module and $v, w \in V$, then $d\mu_g \partial_{v,w} = \partial_{gv,gw}$. In fact,

$$(d\mu_g \partial_{v,w})f = \partial_{v,w}(\mu_g^* f) = \frac{\mu_g^* f(v+\varepsilon w) - \mu_g^* f(v)}{\varepsilon}\Big|_{\varepsilon=0} = \frac{f(gv+\varepsilon gw) - f(gv)}{\varepsilon}\Big|_{\varepsilon=0} = (\partial_{gv,gw})f.$$

In particular, for $A \in \mathrm{End}(V)$ we get $d\mu_g \partial_{Av,v} = \partial_{gAv,gv} = \partial_{\mathrm{Ad}\, g(A)gv,gv}$, hence the vector field $\partial_{A,v}$ is G-invariant if and only if $\mathrm{Ad}\, g(A) = A$ for all $g \in G$.

More generally, we can define a linear action of G on the vector fields of a G-variety X by setting

$$(g\delta)_{gx} := d\mu_g \delta_x, \text{ i.e. } g\delta = d\mu_g \circ \delta \circ \mu_{g^{-1}}$$

where we consider δ as section of the tangent bundle $p: TX \to X$ (A.4.5). If we regard δ as a derivation, $\delta \in \mathrm{Vec}(X) = \mathrm{Der}(\mathcal{O}(X))$, then $g\delta := g \circ \delta \circ g^{-1}$, i.e. $(g\delta)f = g(\delta(g^{-1}f))$ for $f \in \mathcal{O}(X)$.

In case of a linear representation of G on V we can identify $\mathrm{Vec}(V)$ with $\mathcal{O}(V) \otimes V$ and find $g\partial_v = \partial_{gv}$ and $g(f\partial_v) = (gf)(\partial_{gv})$. This shows that the representation of G on $\mathrm{Vec}(V)$ is locally finite and rational. Moreover, we get $g\,\partial_{Av,v} = \partial_{\mathrm{Ad}\, g(A)v,v}$ as we have already seen in Example 5.5.1 above.

Choosing an embedding of X into a representation space V we see that $g(\xi_A) = \xi_{\mathrm{Ad}\, g(A)}$, and that the action of G on $\mathrm{Vec}(X)$ is also locally finite and rational. In fact, we have a canonical G-linear surjection $\mathrm{Vec}_X(V) \twoheadrightarrow \mathrm{Vec}(X)$ where $\mathrm{Vec}_X(V) \subseteq \mathrm{Vec}(V)$ is the G-stable subspace of those vector fields $\xi \in \mathrm{Vec}(V)$ which satisfy $\xi f|_X = 0$ for all $f \in I(X)$ (Proposition A.4.5.4). This proves the following proposition, except the description of the action of $\mathrm{Lie}\, G$ on the vector fields.

PROPOSITION 5.5.2. *Let X be a G-variety. The action of G on the vector fields $\mathrm{Vec}(X)$ is locally finite and rational, and the corresponding action of $\mathrm{Lie}\, G$ on $\mathrm{Vec}(X)$ is given by $A\delta = -[\xi_A, \delta]$. For $A \in \mathrm{Lie}\, G$ we have $g\xi_A = \xi_{\mathrm{Ad}\, g(A)}$, hence the Lie algebra homomorphism $\xi : \mathrm{Lie}\, G \to \mathrm{Vec}(X)$ is G-equivariant.*

PROOF. As above, we can assume that X is a linear representation V, that $G \subseteq \mathrm{GL}(V)$ and that $\mathrm{Lie}\, G \subseteq \mathrm{End}(V)$. With the identification $\mathrm{Vec}(V) = \mathcal{O}(V) \otimes V$ we find $A(f\partial_v) = (Af)\partial_v + f\partial_{Av}$. On the other hand, $[\xi_A, f\partial_v] = (\xi_A f)\partial_v + f[\xi_A, \partial_v]$. We claim that $[\xi_A, \partial_v] = -\partial_{Av}$. In fact, choosing coordinates, one reduces to the cases $A = E_{ij}$ and $v = e_k$, i.e. $\xi_A = x_j \frac{\partial}{\partial x_i}$ and $\partial_v = \frac{\partial}{\partial x_k}$. But then an easy calculation shows that $[\xi_{E_{ij}}, \partial_{e_k}] = 0$ for $k \neq j$ and $[\xi_{E_{ij}}, \partial_{e_j}] = -\partial_{e_i}$, and the claim follows. Thus, using Proposition 5.4.3, we finally get

$$[\xi_A, f\partial_v] = (\xi_A f)\partial_v + f[\xi_A, \partial_v] = (-Af)\partial_v - f\partial_{Av} = -A(f\partial_v).$$

□

EXERCISE 5.5.3. Let $f \in \mathbb{C}[y]$ be a polynomial.

(1) The map $\mathbb{C}^+ \times \mathbb{C}^2 \to \mathbb{C}^2$, $s(x, y) := (x + sf(y), y)$, is an action of $\mathbb{C}^+$ on $\mathbb{C}^2$.
(2) Describe the orbits and the fixed points of this action.
(3) Determine the differential of the orbit maps and verify the results of (2).

Finally we prove the following generalization of the fact that a representation of a connected group G on a vector space V is uniquely determined by the representation of the Lie algebra $\operatorname{Lie} G$ on V (see Proposition 5.2.1(1), cf. Exercise 5.3.1).

PROPOSITION 5.5.4. *Let G be a connected algebraic group and X a variety. An action of G on X is uniquely determined by the Lie algebra homomorphism $\operatorname{Lie} G \to \operatorname{Vec}(X)$.*

PROOF. Let $\rho^{(1)}, \rho^{(2)}$ be two actions of G on X and assume that the corresponding vector fields $\xi_A^{(1)}$ and $\xi_A^{(2)}$ are equal for all $A \in \operatorname{Lie} G$. Define an action $\tilde{\rho}$ on $X \times X$ by $\tilde{\rho}(g)(x,y) := (\rho^{(1)}(g)x, \rho^{(2)}(g)y)$. For $A \in \operatorname{Lie} G$ we get $\tilde{\xi}_A = (\xi_A^{(1)}, \xi_A^{(2)}) \in \operatorname{Vec}(X \times X) = \operatorname{Vec}(X) \oplus \operatorname{Vec}(X)$. Hence $(\tilde{\xi}_A)_{(x,x)} \in T_{(x,x)}\Delta X$ where $\Delta X \subseteq X \times X$ is the diagonal. Therefore, by Corollary 5.4.7, ΔX is stable under G and so $\rho^{(1)} = \rho^{(2)}$. $\square$

5.6. Jordan decomposition in the Lie algebra. Let G be an algebraic group and $\mathfrak{g} := \operatorname{Lie} G$ its Lie algebra.

DEFINITION 5.6.1. An element $A \in \mathfrak{g}$ is called *semisimple* resp. *nilpotent* if for every representation $\rho\colon G \to \operatorname{GL}(V)$ the image $d\rho(A) \in \operatorname{End}(V)$ is diagonalizable resp. nilpotent.

We have seen in Proposition 5.4.3 that for a G-variety X the Lie algebra $\mathfrak{g}$ acts on $\mathcal{O}(X)$ by locally finite derivations. This implies the following characterization of semisimple and of nilpotent elements.

LEMMA 5.6.2. *The following statements for an element $A \in \operatorname{Lie} G$ are equivalent:*

(i) *A is semisimple (resp. nilpotent).*
(ii) *For every G-variety X the vector field ξ_A on X is a semisimple (resp. nilpotent) derivation of $\mathcal{O}(X)$.*
(iii) *There is a faithful representation $\rho\colon G \to \operatorname{GL}(V)$ such that $d\rho(A) \in \operatorname{End}(V)$ is diagonalizable (resp. nilpotent).*

EXAMPLE 5.6.3. For a diagonalizable group D every $A \in \operatorname{Lie} D$ is semisimple, because every representation of D is diagonalizable (Proposition 3.4.5). On the other hand, every element of $\operatorname{Lie} \mathbb{C}^+$ is nilpotent, because every representation of $\mathbb{C}^+$ is of the form $s \mapsto \exp(sN)$ where N is nilpotent (Proposition II.2.6.1).

We will see later in chapter IV (section **??**) that for every unipotent group U the Lie algebra $\operatorname{Lie} U$ consists of nilpotent elements. We already know this for commutative unipotent groups (Remark 4.3.3).

PROPOSITION 5.6.4. (1) *If $A \in \operatorname{Lie} G$ is semisimple, then there is a torus $T \subseteq G$ such that $A \in \operatorname{Lie} T \subseteq \operatorname{Lie} G$.*
(2) *If $N \in \operatorname{Lie} G$ is nilpotent, then there is a closed subgroup $U \subseteq G$, $U \xrightarrow{\sim} \mathbb{C}^+$, such that $N \in \operatorname{Lie} U$.*

PROOF. We can assume that $G \subseteq \operatorname{GL}_n$, hence $\operatorname{Lie} G \subseteq \operatorname{M}_n$, and that $A \in \operatorname{M}_n$ is a nonzero diagonal matrix and N a nonzero nilpotent matrix.

(1) We have $\operatorname{Lie} T_n \cap \operatorname{Lie} G \supseteq \mathbb{C}A$, and so $T := (G \cap T_n)^\circ$ is a torus with $\operatorname{Lie} T \supseteq \mathbb{C}A$, by Proposition 5.2.1(2).

(2) We know that the homomorphism $\alpha_N\colon \mathbb{C}^+ \to \operatorname{GL}_n$, $s \mapsto \exp(sN)$, induces an isomorphism $\mathbb{C}^+ \xrightarrow{\sim} U := \alpha_N(\mathbb{C}^+)$ and that $N \in \operatorname{Lie} U$ (Proposition II.2.6.1). It follows that $\operatorname{Lie} U = \mathbb{C}N \subseteq \operatorname{Lie} G$ and so $U \subseteq G$, again by Proposition 5.2.1(2). $\square$

The next result is the analogue of the JORDAN decomposition in algebraic groups for Lie algebras.

PROPOSITION 5.6.5. *Let G be an algebraic group and $\mathfrak{g}$ its Lie algebra. Then every $A \in \operatorname{Lie} G$ has a unique decomposition $A = A_s + A_n$ where $A_s, A_n \in \operatorname{Lie} G$, A_s is semisimple, A_n is nilpotent, and $[A_s, A_n] = 0$.*

PROOF. Let $G \subseteq \mathrm{GL}_n$, $A \in \operatorname{Lie} G \subseteq \mathrm{M}_n$, and let $A = A_s + A_n$ be the additive JORDAN decomposition in M_n. We have to show that $A_s, A_n \in \operatorname{Lie} G$.

We can assume that A_s is a diagonal matrix. Define $D := \{t \in T_n \mid t \circ A_n = A_n \circ t\}$, and $U := \alpha_{A_n}(\mathbb{C}^+) \subseteq \mathrm{GL}_n$ where $\alpha_N(s) := \exp(sN)$, see Proposition II.2.6.1. Then $A_s \in \operatorname{Lie} D$ (Example 5.3.3), $A_n \in \operatorname{Lie} U$, and so $A \in \operatorname{Lie}(D \cdot U) = \operatorname{Lie} D + \operatorname{Lie} U \subseteq \mathrm{M}_n$. Define $H := D \cdot U \cap G \subseteq \mathrm{GL}_n$. Since H is commutative we have $H = H_s \cdot H_u$ and $\operatorname{Lie} H = \operatorname{Lie} H_s \oplus \operatorname{Lie} H_u$ (Proposition 4.3.4). This implies that $A = A' + A''$ where $A' \in \operatorname{Lie} H_s \subseteq \operatorname{Lie} G$ and $A'' \in \operatorname{Lie} H_u \subseteq \operatorname{Lie} G$. Therefore A' is diagonal, A'' is nilpotent and $[A', A''] = 0$. Now the claim follows from the uniqueness of the additive JORDAN decomposition in M_n. □

Here is a nice application. We know that a commutative algebraic group consisting of semisimple elements is a diagonalizable group (Proposition 4.3.4). We want to extend this to an arbitrary (connected) group.

PROPOSITION 5.6.6. *Let G be a connected algebraic group. Assume that all elements of G are semisimple. Then G is a torus.*

PROOF. We have to show that G is commutative. We can assume that $G \subseteq \mathrm{GL}_n$. Proposition 5.6.4 above implies that $\mathfrak{g} := \operatorname{Lie} G \subseteq \mathrm{M}_n$ does not contain nilpotent elements, hence consists of diagonalizable elements, by Proposition 5.6.5. We first claim that every subtorus $T \subseteq G$ lies in the center $\mathrm{Z}(G)$ of G. In fact, we can decompose $\mathfrak{g}$ under the action of T by conjugation, $\mathfrak{g} = \bigoplus_{j=1}^{m} \mathfrak{g}_{\chi_j}$ where $\chi_j \in \mathcal{X}(T)$ are characters of T. If $A \in \mathfrak{g}_{\chi_j}$, then $A^k \in (\mathrm{M}_n)_{(k \cdot \chi_j)}$ and so A is nilpotent in case $\chi_j \neq 0$. Thus $\mathfrak{g} = \mathfrak{g}_0$, i.e. T commutes with $\mathfrak{g}$ and hence with G.

If $g \in G$, then $\overline{\langle g \rangle}$ is a diagonalizable group and so, for a suitable $m \geq 1$, g^m belongs to a subtorus of G, hence to the center $\mathrm{Z}(G)$. This implies that the image of G under the adjoint representation $\operatorname{Ad}\colon G \to \mathrm{GL}(\operatorname{Lie} G)$ is a connected algebraic group whose elements all have finite order, because the kernel of Ad is the center of G (Corollary 5.3.5). Therefore, by Proposition II.1.4.5, $\operatorname{Ad}(G)$ is trivial and thus G is commutative. □

EXERCISE 5.6.7. Use the proposition above to give another proof of Corollary 3.4.6 which says that an extension of a torus by a diagonalizable group is diagonalizable.

5.7. Invertible functions and characters. Let T be an n-dimensional torus. Then $\mathcal{O}(T) \simeq \mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$, and this implies that the group $\mathcal{O}(T)^*$ of invertible functions of $\mathcal{O}(T)$ has the form

$$\mathcal{O}(T)^* = \mathbb{C}^* \cdot \mathcal{X}(T).$$

In particular, an invertible element $f \in \mathcal{O}(T)^*$ with $f(e) = 1$ is a character. We will show now that this holds for every connected algebraic group G. We start with the following lemma.

LEMMA 5.7.1. *Let G be an algebraic group, and let $G' := \overline{\langle G_u \rangle}$ be the closure of the subgroup generated by all unipotent elements of G.*

(1) *There are finitely many unipotent elements $u_1, u_2, \ldots, u_N \in G$ such that $G' = \overline{\langle u_1 \rangle} \cdot \overline{\langle u_2 \rangle} \cdots \overline{\langle u_N \rangle}$.*
(2) *G' is a connected normal subgroup of G.*

(3) $\operatorname{Lie} G'$ *contains all nilpotent elements of* $\operatorname{Lie} G$.
(4) *If* $f \in \mathcal{O}(G)^*$ *is invertible, then* f *is constant on* G'.

PROOF. For (1) see Exercises II.1.4.11.

(2) This follows from (1) and the fact that the set G_u of unipotent elements is stable under conjugation by G.

(3) If $A \in \operatorname{Lie} G$ is nilpotent, then there is one-dimensional unipotent subgroup $U \subseteq G$ such that $A \in \operatorname{Lie} U$ (Proposition 5.6.4(2)).

(4) It follows from (1) that there is a surjective morphism $\mathbb{C}^N \to G'$, and so every invertible function on G' is a constant. □

EXAMPLE 5.7.2. If $G = \mathrm{SL}_n$, SO_n $(n \geq 3)$ or Sp_n then $G = \overline{\langle G_u \rangle}$.

PROOF. (a) Let $U := U_2 \subseteq \mathrm{SL}_2$, and put $X := U \cdot U^t \cdot U$. Then $\dim \overline{X} = 3$, because $U^t \cdot U$ is closed of dimension 2 and does not contain X. Thus $X \cdot X = \mathrm{SL}_2$ by Lemma 1.4.9, and so $\overline{\langle (\mathrm{SL}_2)_u \rangle} = \mathrm{SL}_2$.

(b) For every pair $1 \leq i < j \leq n$ there is an embedding $\mathrm{SL}_2 \hookrightarrow \mathrm{SL}_n$ given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a-1)E_{ii} + bE_{ij} + cE_{ji} + (d-1)E_{jj} + E$. This implies that $\overline{\langle G_u \rangle}$ contains the diagonal elements $t_{ij}(t)$ which, together with the unipotent elements $u_{ij}(s)$, generate SL_n (see II.3.1).

(c) Similarly, one shows that SO_n for $n \geq 3$ and Sp_n are generated by homomorphic images of SL_2. □

The following result is due to ROSENLICHT, see [**KKV89**, Prop. 1.2 and 1.3].

PROPOSITION 5.7.3. *Let* G *be an algebraic group.*

(1) *The character group* $\mathcal{X}(G)$ *is finitely generated.*
(2) *If* G *is connected, then every* $f \in \mathcal{O}(G)^*$ *with* $f(e) = 1$ *is a character, i.e.* $\mathcal{O}(G)^* = \mathbb{C}^* \cdot \mathcal{X}(G)$.

PROOF. We can assume that G is connected since the map $\mathcal{X}(G) \to \mathcal{X}(G^\circ)$ is injective. Let $T \subseteq G$ be a torus of maximal dimension, and set $G' := \overline{\langle G_u \rangle}$. The proposition follows if we show that $G = T \cdot G' = G' \cdot T$. In fact, if $f \in \mathcal{O}(G)^*$ with $f(e) = 1$, then $f|_T \colon T \to \mathbb{C}^*$ is a homomorphism, and so $f \colon G \to \mathbb{C}^*$ is a homomorphism, because $f|_{G'} = 1$ by Lemma 5.7.1 above. In addition, $\mathcal{X}(G) \to \mathcal{X}(T)$ is injective, and so $\mathcal{X}(G)$ is finitely generated.

It remains to show that $G = T \cdot G'$, or, equivalently, that $\operatorname{Lie} G = \operatorname{Lie} G' + \operatorname{Lie} T$. We decompose $\mathfrak{g} := \operatorname{Lie} G$ into weight spaces with respect to the adjoint action of T: $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\chi \neq 0} \mathfrak{g}_\chi$. Embedding G into GL_n we see as in the proof of Proposition 5.6.6 above that the elements from $\mathfrak{g}_\chi$ are nilpotent if $\chi \neq 0$. Thus $\mathfrak{g}_\chi \subseteq \operatorname{Lie} G'$ for $\chi \neq 0$. Moreover, $\mathfrak{g}_0 = \operatorname{Lie} \mathrm{C}_G(T)$ and so $G = \mathrm{C}_G(T) \cdot G'$. If $g \in \mathrm{C}_G(T)$, $g = g_s g_u$, then g_s has finite order since otherwise $T \cdot \overline{\langle g_s \rangle} \subseteq G$ contains a torus of dimension $> \dim T$.

It follows that for every element in $g \in G$ there is an $n \in \mathbb{N}$ such that $g^n \in T \cdot G'$. Since G is connected this implies that $G = T \cdot G'$. In fact, $G = \bigcup_{n \in \mathbb{N}} p_n^{-1}(T \cdot G')$ where $p_n \colon G \to G$ is the power map $g \mapsto g^n$. Hence $G = p_n^{-1}(T \cdot G')$ for some n, and the claim follows because p_n is dominant for $n \neq 0$ (Exercise II.1.4.6). □

5.8. $\mathbb{C}^+$-actions and locally nilpotent vector fields. In this section we study actions of the additive group $\mathbb{C}^+$ and relate them to locally nilpotent vector fields. For more details we refer to the book [**Fre06**] of GENE FREUDENBURG.

DEFINITION 5.8.1. Let X be a variety. A vector field $\delta \in \mathrm{Vec}(X)$ is called *locally nilpotent* if for every $f \in \mathcal{O}(X)$ there is an $m > 0$ such that $\delta^m f = 0$. The subspace of locally nilpotent vector fields will be denoted by $\mathrm{Vec}^{ln}(X) \subseteq \mathrm{Vec}(X)$.

Consider an action ρ of $\mathbb{C}^+$ on X. Then, for any $a \in \operatorname{Lie}\mathbb{C}^+ = \mathbb{C}$ the vector field ξ_a is locally nilpotent. In fact, for a representation $\rho\colon \mathbb{C}^+ \to \mathrm{GL}(V)$ the matrix $A := d\rho(a) \in \operatorname{End}(V)$ is nilpotent (Proposition 2.6.1), and for a linear function $\ell \in V^*$ we have $\xi_a\ell(v) = \ell(Av)$ (see 5.4). Thus the action is locally nilpotent on V^* and thus on $\mathcal{O}(V)$. Embedding X as a $\mathbb{C}^+$-stable closed subvariety into a $\mathbb{C}^+$-module V we see that the same holds for $\mathcal{O}(X)$.

For an action ρ of $\mathbb{C}^+$ on a variety X we denote by $\xi_\rho \in \operatorname{Vec}(X)$ the vector field corresponding to $1 \in \operatorname{Lie}\mathbb{C}^+ = \mathbb{C}$. This means that

$$\xi_\rho f = \frac{d}{ds} f(\rho(s)x)|_{s=0}.$$

Lemma 5.8.2. *For $f \in \mathcal{O}(X)$ we have*

$$f(\rho(s)x) = \sum_{k\geq 0} s^k \xi_\rho^k f \in \mathcal{O}(X)[s].$$

Proof. Set $f(\rho(s)x) = \sum_{k\geq 0} s^k f_k(x) \in \mathcal{O}(X)[s]$. Setting $s = 0$ we get $f_0(x) = f(x)$. Using $f(\rho(s'+s)x) = f(\rho(s')(\rho(s)x) = \sum_{k\geq 0} f_k(\rho(s)x)$ and expanding $(s'+s)^k$ we get

$$f_k(\rho(s)x) = \sum_{j\geq 0} \binom{k+j}{j} s^j f_{k+j}(x)$$

which implies that $\xi_\rho f_k = \frac{d}{dt} f(\rho(s)x)|_{s=0} = k! f_{k+1}$ for all $k \geq 0$. □

Proposition 5.8.3. *Let X be a variety. The map $\rho \mapsto \xi_\rho$ is a bijection*

$$\{\mathbb{C}^+\text{-actions on } X\} \xrightarrow{\simeq} \operatorname{Vec}^{ln}(X).$$

Proof. By Proposition 5.5.4 the map is injective. We have seen in 5.4 that for $A \in \operatorname{End}(V) = \operatorname{Lie}\mathrm{GL}(V)$ the corresponding vector field ξ_A is given by $(\xi_A)_v = Av \in V = T_vV$, and thus, for a linear function $\ell \in V^* \subseteq \mathcal{O}(V)$, we get $\xi_A\ell(v) = \ell(Av)$. Equivalently, $\xi_A\ell = A^t\ell$ where $A^t\colon V^* \to V^*$ is the dual map to A.

Now assume that $\delta \in \operatorname{Vec}(X)$ is a locally nilpotent vector field, and choose a finite dimensional subspace $W \subseteq \mathcal{O}(X)$ which is stable under δ and generates $\mathcal{O}(X)$. Set $V := W^*$ and $A := (\delta|_W)^t \in \operatorname{End}(V)$. Identifying V^* with W, we get a closed immersion $X \hookrightarrow V$ such that $\delta = \xi_A|_X$. In fact, $\xi_A\ell = A^t\ell = \delta(\ell)$ for $\ell \in V^* = W$, by construction. Proposition 2.6.1 now shows that there is a linear representation $\alpha\colon \mathbb{C}^+ \to \mathrm{GL}(V)$ such that $d\alpha(1) = A$. This implies that $\xi_\alpha = \xi_A \in \operatorname{Vec}(X)$, hence X is stable under the action of $\mathbb{C}^+$ defined by α, by Corollary 5.4.7, and $\xi_\alpha = \delta$. □

Remark 5.8.4. There is a more abstract way to construct a $\mathbb{C}^+$-action from a locally nilpotent vector field ξ. Define a linear map (see the formula in Lemma 5.8.2 above)

$$\tau\colon \mathcal{O}(X) \to \mathbb{C}[s] \otimes \mathcal{O}(X),\ \tau(f) := \sum_k \frac{s^k}{k!} \otimes \xi^k f.$$

One easily shows that τ is an algebra homomorphism, and thus defines a morphism $\rho\colon \mathbb{C} \times X \to X$ with the property that $f(\rho(s,x)) = \sum_k \frac{s^k}{k!}\xi^k f$ for all $s \in \mathbb{C}$ and $x \in X$. From this it is not difficult to deduce that ρ is a $\mathbb{C}^+$-action, and the corresponding vector field is ξ, by Lemma 5.8.2.

Example 5.8.5. Consider an action of $\mathbb{C}^+$ on affine n-space $\mathbb{C}^n$:

$$\rho(s)(x_1,\ldots,x_n) = (f_1(s,x_1,\ldots,x_n),\ldots,f_n(s,x_1,\ldots,x_n)).$$

Then the corresponding vector field is

$$\xi_\rho = \left(\frac{\partial f_1}{\partial s}\right)_{s=1} \frac{\partial}{\partial x_1} + \cdots + \left(\frac{\partial f_n}{\partial s}\right)_{s=1} \frac{\partial}{\partial x_n}.$$

Since $\operatorname{jac}(\rho(s)) = 1$ for all $s \in \mathbb{C}^+$ we see that $\operatorname{div} \xi_\rho = 0$ (Example 5.4.2).

Let ρ be a $\mathbb{C}^+$-action on a variety X and let $\xi_\rho \in \operatorname{Vec}^{ln}$ be the corresponding locally nilpotent vector field. Then $\mathcal{O}(X)^{\mathbb{C}^+} = \ker \xi_\rho$ (Exercise 5.4.4). It is known that the invariants are not always finitely generated, see [**Nag59**].

If $f \in \mathcal{O}(X)^{\mathbb{C}^+}$, hence $\xi_\rho f = 0$, then the vector field $f\xi_\rho$ is again locally nilpotent, hence corresponds to another action $\tilde{\rho}$ of $\mathbb{C}^+$ on X which is called a *modification* of ρ.

Exercise 5.8.6. Let ρ be a $\mathbb{C}^+$-action on X, and let $f \in \mathcal{O}(X)^{\mathbb{C}^+}$ be an invariant. Define $\tilde{\rho}(s)x := \rho(f(x)s)x$. Then we have the following.

(1) $\tilde{\rho}$ is a $\mathbb{C}^+$-action on X, and $\xi_{\tilde{\rho}} = f\xi_\rho$.
(2) The orbits if $\tilde{\rho}$ are contained in the orbits of ρ.
(3) For the fixed points we have $X^{\tilde{\rho}} = X^\rho \cup \mathcal{V}_X(f)$.
(4) If X is irreducible and $f \neq 0$ then $\mathcal{O}(X)^{\tilde{\rho}} = \mathcal{O}(X)^\rho$.

A closed subvariety $Z \subseteq X$ of a $\mathbb{C}^+$-variety is called a *global section* if the morphism $\mathbb{C}^+ \times Z \to X$, $(s, z) \mapsto \rho(s)z$, is an isomorphism. In particular, *every orbit O meets Z transversally in a unique point s* which means $O \cap S = \{s\}$ and that $T_sX = T_sO \oplus T_sS$. The next lemma shows that the hypersurface $\mathcal{V}_X(f)$ is a section if $\xi_\rho f = 1$.

Lemma 5.8.7. *For $f \in \mathcal{O}(X)$ the map $f\colon X \to \mathbb{C}^+$ is $\mathbb{C}^+$-equivariant if and only if $\xi_\rho f = 1$. In this case, the map $\mathbb{C}^+ \times \mathcal{V}_X(f) \xrightarrow{\sim} X$, $(s, z) \mapsto \rho(s)z$, is an isomorphism, i.e. $\mathcal{V}_X(f)$ is a global section.*

Proof. Lemma 5.8.2 shows that $\xi_\rho f = 1$ is equivalent to $f(\rho(s)x) = f(x) + s$ which means that $f\colon X \to \mathbb{C}^+$ is $\mathbb{C}^+$-equivariant. The second part follows from Proposition 1.2.10. □

Local sections can be constructed in the following way. Start with a function $f \in \mathcal{O}(X)$ which not an invariant, i.e. $\xi_\rho f \neq 0$. Then there is a $k > 0$ such that $q := \xi_\rho^k f \neq 0$ and $\xi_\rho^{k+1} f = 0$. This implies that q is a nonzero invariant, and setting $p := \xi_\rho^k f$ we get $\xi_\rho \frac{p}{q} = 1$. This shows that $f := \frac{p}{q} \in \mathcal{O}(X_q)$ defines a global section of the $\mathbb{C}^+$-invariant open set $X_q \subseteq X$.

The following theorem collects some of the main properties of $\mathbb{C}^+$-varieties.

Theorem 5.8.8. *Let X be a variety with a non-trivial $\mathbb{C}^+$-action on X.*

(1) *All orbits in X are closed.*
(2) *If X is not an orbit, then there exist nonconstant invariants.*
(3) *If $x \in X$ is an isolated fixed point, then $\{x\}$ is a connected component of X.*
(4) *X admits local sections.*

Now assume in addition that X is irreducible.

(5) *We have $\mathbb{C}(X)^{\mathbb{C}^+} = \operatorname{Quot}(\mathcal{O}(X)^{\mathbb{C}^+})$.*
(6) *The field extension $\mathbb{C}(X)/\mathbb{C}(X)^{\mathbb{C}^+}$ is purely transcendental of degree 1.*
(7) *The invariant ring $\mathcal{O}(X)^{\mathbb{C}^+}$ is multiplicatively closed: If a product $f_1 f_2$ is an invariant, the f_1 and f_2 are both invariants.*

For the proof we will use the following easy lemma.

Lemma 5.8.9. *Let $Y \subsetneqq X$ be a $\mathbb{C}^+$-stable closed nonempty subvariety. Then there is a nonzero invariant $f \in \mathcal{O}(X)^{\mathbb{C}^+}$ which vanishes on Y.*

Proof. The ideal $I(Y) \subseteq \mathcal{O}(X)$ is nonzero and stable under $\mathbb{C}^+$, hence stable under ξ_ρ. Since ξ_ρ is locally nilpotent, there exist nonzero elements $f \in I(Y)$ such that $\xi_\rho f = 0$. This means that f is a nonzero invariant vanishing on Y. □

Proof of Theorem 5.8.8. (1) Let $O \subseteq X$ be an orbit and $\bar{O} \subseteq X$ its closure. If $Y := \bar{O} \setminus O$ is nonempty, then, by the lemma above, there is a nonzero invariant on $\bar{O}$ vanishing on Y, contradicting the fact that every invariant on $\bar{O}$ is constant.

(2) By (1) every orbit is a closed and $\mathbb{C}^+$-stable subset $\neq X$, and so the claim follows again from the lemma above.

(3) Assume that there exists an irreducible $\mathbb{C}^+$-variety X of positive dimension containing an isolated fixed point $x \in X$. Choose such an X of minimal dimension > 0. Then $\{x\} \subsetneqq X$ is a $\mathbb{C}^+$-stable closed nonempty subset, hence there is a nonzero invariant f such that $f(x) = 0$. Every irreducible component of $\mathcal{V}_X(f)$ has dimension $\dim X - 1$, and one of them contains x as an isolated fixed point. Hence $\dim X = 1$, by the minimality of $\dim X$. But then X is either an orbit, or $X = X^{\mathbb{C}^+}$, and in both cases we end up with a contradiction.

(4) The existence of local sections was shown just before the theorem.

(5) Let $r \in \mathbb{C}(X)^{\mathbb{C}^+}$ be an invariant rational function. Then the ideal of denominators $\mathfrak{a} := \{q \in \mathcal{O}(X) \mid qr \in \mathcal{O}(X)\}$ is $\mathbb{C}^+$-stable, and thus contains a nonzero invariant q (see the proof of Lemma 5.8.9). It follows that qr is also an invariant, and so $r \in \operatorname{Quot}(\mathcal{O}(X)^{\mathbb{C}^+})$.

(6) The construction of local sections given above shows that there exist an invariant q and a function $f = \frac{p}{q} \in \mathcal{O}(X_q)$ such that $Z := \mathcal{V}_{X_q}(f)$ is a global section of X_q. This implies that $\mathcal{O}(X_q) = \mathcal{O}(X_q)^{\mathbb{C}^+}[f]$, hence

$$\mathbb{C}(X) = \mathbb{C}(X_q) = \operatorname{Quot}(\mathcal{O}(X_q)) = \operatorname{Quot}(\mathcal{O}(X_q)^{\mathbb{C}^+})(f)$$

and the claim follows, because $\mathcal{O}(X_q)^{\mathbb{C}^+} = (\mathcal{O}(X)^{\mathbb{C}^+})_q$, and so $\operatorname{Quot}(\mathcal{O}(X_q)^{\mathbb{C}^+}) = \operatorname{Quot}(\mathcal{O}(X)^{\mathbb{C}^+}) = \mathbb{C}(X)^{\mathbb{C}^+}$, by (5).

(7) Let $f_1 f_2$ be a nonzero invariant and set $Y := \mathcal{V}_X(f_1 f_2)$. Then $f_1 f_2$ is a nonzero constant on every orbit $O \subseteq X \setminus Y$. This implies that $f_1|_O$ is also constant, because otherwise f_1 would take all values on O, in particular the value zero. Thus f_1 is an invariant on the dense open set $X \setminus Y$, hence an invariant on X. □

Exercise 5.8.10. Let $A \subseteq \mathcal{O}(\mathbb{C}^+) = \mathbb{C}[s]$ be a subalgebra stable under $\mathbb{C}^+$. Then either $A = \mathbb{C}$ or $A = \mathbb{C}[s]$.

Exercise 5.8.11. Use the previous exercise to give another proof that $\mathbb{C}^+$-orbits are closed.

Exercises

For the convenience of the reader we collect here all exercises from Chapter III.

Exercise. Consider the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on M_2 defined by $(g,h)A := gAh^{-1}$. Calculate the differential of the orbit map in A, determine its image and its kernel, and verify the claims of Lemma 5.1.5.

Exercise. Let G be a connected group and let $\rho\colon G \to \mathrm{GL}(V)$ and $\mu\colon G \to \mathrm{GL}(W)$ be two representations. Then ρ is equivalent to μ if and only if $d\rho\colon \operatorname{Lie} G \to \operatorname{End}(V)$ is equivalent to $d\mu\colon \operatorname{Lie} G \to \operatorname{End}(W)$.

Exercise. Show that $\mathfrak{z}(L)$ is a characteristic ideal of L, i.e. $\mathfrak{z}(L)$ is an ideal of L, and it is stable under every automorphism of the Lie algebra L.

Exercise. Let G be a connected, noncommutative 2-dimensional algebraic group. Then

(1) $\mathrm{Z}(G)$ is finite;
(2) The unipotent elements G_u form a normal closed subgroup isomorphic to $\mathbb{C}^+$;
(3) There is a subgroup $T \subseteq G$ isomorphic to $\mathbb{C}^*$ such that $G = T \cdot G_u = G_u \cdot T$.

(Hint: Study the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}\, G)$, and use Exercise 4.1.3.)

EXERCISE. Use the previous exercise to show that every 2-dimensional closed subgroup of SL_2 is conjugate to $B_2 \cap \mathrm{SL}_2$.

EXERCISE. Let X be a G-variety and assume that G is connected. A regular function $f \in \mathcal{O}(X)$ is a G-invariant if and only if $\xi_A f = 0$ for all $A \in \mathrm{Lie}\, G$.
(Hint: Look at the regular representation of G on $\mathcal{O}(X)$ and use Proposition 5.3.2.)

EXERCISE. Let X be a G-variety where G is connected. Then $x \in X$ is a fixed point if and only if $(\xi_A)_x = 0$ for all $A \in \mathrm{Lie}\, G$.

EXERCISE. Let $f \in \mathbb{C}[y]$ be a polynomial.

(1) The map $\mathbb{C}^+ \times \mathbb{C}^2 \to \mathbb{C}^2$, $s(x, y) := (x + sf(y), y)$, is an action of $\mathbb{C}^+$ on $\mathbb{C}^2$.
(2) Describe the orbits and the fixed points of this action.
(3) Determine the differential of the orbit maps and verify the results of (2).

EXERCISE. Use Proposition 5.6.6 to give another proof of Corollary 3.4.6 which says that an extension of a torus by a diagonalizable group is diagonalizable.

EXERCISE. Let ρ be a $\mathbb{C}^+$-action on X, and let $f \in \mathcal{O}(X)^{\mathbb{C}^+}$ be an invariant. Define $\tilde{\rho}(s)x := \rho(f(x)s)x$. Then we have the following.

(1) $\tilde{\rho}$ is a $\mathbb{C}^+$-action on X, and $\xi_{\tilde{\rho}} = f\xi_\rho$.
(2) The orbits if $\tilde{\rho}$ are contained in the orbits of ρ.
(3) For the fixed points we have $X^{\tilde{\rho}} = X^\rho \cup \mathcal{V}_X(f)$.
(4) If X is irreducible and $f \neq 0$ then $\mathcal{O}(X)^{\tilde{\rho}} = \mathcal{O}(X)^\rho$.

EXERCISE. Let $A \subseteq \mathcal{O}(\mathbb{C}^+) = \mathbb{C}[s]$ be a subalgebra stable under $\mathbb{C}^+$. Then either $A = \mathbb{C}$ or $A = \mathbb{C}[s]$.

EXERCISE. Use the previous exercise to give another proof that $\mathbb{C}^+$-orbits are closed.

CHAPTER IV

Invariants and Algebraic Quotients

Contents

Introduction. This fourth chapter is the main part of the book and is where we prove the so-called finiteness theorem. This asserts that for a rational representation of a linearly reductive group G on a vector space V the ring of G-invariant regular functions on V is a finitely generated $\mathbb{C}$-algebra. Here an algebraic group G is called linearly reductive if every rational representation of G is completely reducible. This allows us to define the algebraic quotient $X/\!\!/G$ of a G-variety X which is, in some sense, the best approximation to the orbit space X/G which in general does not have a reasonable structure due to the existence of non-closed orbits.

We give the main properties of the quotient map $\pi_X : X \to X/\!\!/G$ and develop some tools to construct and determine quotients. Using these results we give a couple of additional properties and some characterizations of linear reductive and semi-simple groups. The third section ends with a section on finite groups G, where it is possible to sharpen the results of this chapter. This follows an account of Emmy Noether [**Noe15**].

In the last sections we turn to some examples and applications. First we prove a geometric version of the so-called first fundamental theorem for GL_n. This form of the fundamental theorem for classical groups is due to Thierry Vust [**Vus76**]. As well we describe the "method of the associated cone". Roughly speaking, this allows one to carry the "good" properties of the zero fiber over to the other fibers. The last part of the chapter contains an outline of structure statements and properties which "push down" for quotient maps, together with some results on invariant rational functions.

The finiteness result has a long and interesting history. In the preface we already discussed the period up to 1900. For a complete account we refer to the encyclopedia report of F. Meyer [**Mey99**] in 1899. This first period ended with the two pioneering articles of D. Hilbert *"Über die Theorie der algebraischen Formen"* [**Hil90**] and *"Über die vollen Invariantensysteme"* [**Hil93**] in 1890 and 1893 which brought the theory to a certain conclusion. Some people even speak of the death of the theory, for example Ch.S. Fisher in his exposition: *"A Study in the Sociology of Knowledge"* [**Fis66**] (see also [**DC71, DC70**]). The *"Vorlesungen über Invariantentheorie"* [**Sch68**] of Issai Schur in 1928 give a small glimpse into the types of questions current then, and in this book the theory of binary forms underlies the whole approach. A modern account of this can be found in T.A. Springer's Lecture Notes *"Invariant Theory"* [**Spr77**].

The fundamental work of Issai Schur, Hermann Weyl, and Élie Cartan on the theory of semisimple Lie groups and their representations brought a new impetus to the subject. Weyl gave a proof for the finiteness theorem and the so-called first and second fundamental theorems for all classical groups. In the orthogonal case the finiteness theorem had already been proved by Hurwitz [**Hur97**]. (In this regard see Appendix B). A complete account of the state of the theory around 1940 can be found in Hermann Weyl's famous book *"Classical Groups"* [**Wey39**].

Even quite early the question of a general finiteness theorem had been asked, i.e. whether the ring of G-invariant functions for an arbitrary group G is finitely generated. In his address to the I.M.C. in Paris in 1900 D. Hilbert devoted the fourteenth of his famous twenty three problems to a generalization of this question. He was basing this on Maurer's proof of the finiteness theorem for groups. This work later turned out to be false, and the finiteness question remained open for some time, until in 1959 Masayoshi Nagata found a counterexample ([**Nag59**]; see also [**DC71, DC70**, Chap. 3.2]). In our proof of the finiteness theorem for linear reductive groups G we follow Nagata's account [**Nag64**].

The use of invariants for classification problems via geometry was established by D. Mumford in his book *"Geometric Invariant Theory"* [**MFK94**] whose first edition appeared in 1965. It turns out to be a useful tool in the study of the classification question and the associated "moduli spaces", e.g. in the case of curves, abelian varieties and vector bundles. This fundamental work from 1965 marks the beginning of the "third blossoming" of invariant theory and also caused a reawakening of interest in the classical literature. It had a great influence on the further development of algebraic geometry and even today lies at the foundation of a great deal of research.

1. Isotypic Decomposition

1.1. Completely reducible representations. The concepts of irreducible and completely reducible representations are certainly known from classical group theory. They carry over to representations of algebraic groups without any changes.

Definition 1.1.1. A representation $\rho\colon G \to \mathrm{GL}(V)$, $V \neq \{0\}$, is called *irreducible* if $\{0\}$ and V are the only G-stable subspaces of V. Otherwise it is called *reducible*. The representation ρ is called *completely reducible* if V admits a decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ into G-stable irreducible subrepresentations $V_i \subseteq V$. A G-module V is called *simple* if the corresponding representation ρ is irreducible, and *semisimple* if ρ is completely reducible.

Example 1.1.2. The natural representations of the classical groups $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, $\mathrm{O}(V, q)$, $\mathrm{Sp}(V, \beta)$ on V are irreducible, and the same holds for $\mathrm{SO}(V, q)$ in case $\dim V \geq 3$. In fact, these groups are irreducible subgroups of $\mathrm{GL}(V)$ as defined and proved in section II.3.

Example 1.1.3. Every representation of a finite group G is completely reducible. This is the famous Theorem of Maschke, see [**Art91**, Chap. 9, Corollary 4.9].

A one-dimensional representation of an algebraic group G is clearly irreducible and it is the same as a character, because $\mathrm{GL}(V) = \mathbb{C}^*$ if $\dim V = 1$. Moreover, if $\rho\colon G \to \mathrm{GL}(V)$ is a representation and $\chi \in \mathcal{X}(G)$ a character, then the product $\chi\rho\colon G \to \mathrm{GL}(V)$ given by $g \mapsto \chi(g)\rho(g)$ is again a representation. In the language of G-modules this is the tensor product $\mathbb{C}_\chi \otimes V$ where $\mathbb{C}_\chi$ denotes the one-dimensional G-module corresponding to the character χ. Clearly, $\chi\rho$ is irreducible (resp. completely reducible) if and only if ρ is irreducible (resp. completely reducible).

Example 1.1.4. Every representation of $\rho\colon \mathbb{C}^* \to \mathrm{GL}(V)$ is completely reducible, and the irreducible representations of $\mathbb{C}^*$ are one-dimensional and are of the form $t \mapsto t^j$ for some $j \in \mathbb{Z}$ (Example III.2.1.3). This implies that we get the following canonical decomposition of the $\mathbb{C}^*$-module V which is called the *weight space decomposition* (cf. Remark III.3.4.7):

$$V = \bigoplus_{j \in \mathbb{Z}} V_j, \quad V_j := \{v \in V \mid tv = t^j \cdot v\}$$

($s \cdot v$ denotes the scalar multiplication).

A homomorphism $\lambda\colon \mathbb{C}^* \to G$ is called a *one-parameter subgroup* of G, shortly a 1-PSG. These subgroups will play an important role in connection with the Hilbert Criterion in section **??**. If $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G, then every 1-PSG $\lambda\colon \mathbb{C}^* \to G$ gives rise to a *weight decomposition* of V:

$$V = \bigoplus_{j \in \mathbb{Z}} V_{\lambda,j}, \quad V_{\lambda,j} := \{v \in V \mid \rho(\lambda(t))v = t^j \cdot v\}.$$

Proposition 1.1.5. *For any G-module V the following statements are equivalent.*

(i) *V is semisimple.*
(ii) *V is generated by simple submodules.*
(iii) *Every submodule $W \subseteq V$ has a G-stable complement, i.e. there is a submodule $W' \subseteq V$ such that $V = W \oplus W'$.*

Proof. We will constantly use the fact that every non-zero G-module contains a simple submodule.

(i) $\to$ (ii): This is obvious from the definition.

(ii) $\to$ (iii): If $W \subseteq V$ is a strict submodule, then there is a simple submodule $U \subseteq V$ not contained in W. It follows that $U \cap W = \{0\}$, and so $W + U = W \oplus U$. By induction we can assume that $W \oplus U$ has a G-stable complement, and the claim follows.

(iii) $\to$ (i): Let $W \subseteq V$ be a semisimple submodule of maximal dimension, and let W' be a G-stable complement. If $W' \neq \{0\}$, then W' contains a simple submodule U, and so $W + U = W \oplus U$ is semisimple, contradicting the maximality of W. □

Corollary 1.1.6. *Every submodule and every quotient module of a semisimple G-module is semisimple.*

Proof. The image of a simple module under a G-homomorphism is either trivial or a simple. Hence a quotient module of a semisimple module V is semisimple, by Proposition 1.1.5(ii). If $W \subseteq V$ is a submodule, then W has a G-stable complement, by Proposition 1.1.5(iii), and so W is isomorphic to a quotient module of V. □

Exercise 1.1.7. Let $G \subseteq \mathrm{GL}(V)$ be an arbitrary subgroup and $\bar{G} \subseteq \mathrm{GL}(V)$ its closure. Then V is a simple (resp. semisimple) G-module if and only if V is a simple (resp. semisimple) $\bar{G}$-module.

1.2. Endomorphisms of semisimple modules. The Lemma of Schur already occurred earlier when we studied the classical groups (see section II.3.1).

Proposition 1.2.1 (Lemma of Schur). *Let V, W be two simple G-modules.*

(1) *Every G-homomorphism $\varphi\colon V \to W$ is either an isomorphism or trivial.*
(2) *We have $\mathrm{End}_G(V) = \mathbb{C}$ where we identify $c \in \mathbb{C}$ with $c \cdot \mathrm{id}_V$.*

Proof. If $\varphi\colon V \to W$ is a G-homomorphism between two G-modules, then $\ker\varphi$ is a submodule of V and $\varphi(V)$ a submodule of W. This implies the first claim. If $\varphi\colon V \to V$ is an G-endomorphism and $c \in \mathbb{C}$ an eigenvalue, then $\varphi - c\cdot\mathrm{id}_V$ is a G-endomorphism with a non-trivial kernel. Thus the second claim follows from the first. □

An immediate consequence of Schur's Lemma is the following description of the G-homomorphisms between direct sums $V^{\oplus n} := \underbrace{V \oplus V \oplus \cdots \oplus V}_{n \text{ copies}}$ of a simple module V.

Corollary 1.2.2. *Let V be a simple G-module. For $n, m \geq 1$ we have a canonical isomorphism $\mathrm{M}_{m\times n}(\mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}_G(V^{\oplus n}, V^{\oplus m})$ where the G-endomorphism defined by an $m \times n$-matrix $A = (a_{ij})$ is given by*

$$(v_1, \ldots, v_n) \mapsto (\ldots, \sum_{j=1}^{n} a_{ij} v_j, \ldots).$$

Corollary 1.2.3. *Let V be a simple G-module and W a simple H-module. Then $V \otimes W$ is a simple $G \times H$-module where the linear action of $G \times H$ on $V \otimes W$ is defined by $(g,h)(v \otimes w) := gv \otimes hw$. Similarly, $\mathrm{Hom}(V, W)$ is a simple $G \times H$-module where the action is given by $(g,h)\varphi := h \circ \varphi \circ g^{-1}$.*

Proof. As a G-module, $V \otimes W$ is semisimple and isomorphic to $V^{\oplus m}$ where $m = \dim W$. It follows from the corollary above that the simple G-submodules of $V \otimes W$ are isomorphic to V and of the form $V \otimes w$ with a suitable non-zero $w \in W$. Since $\langle Hw \rangle = W$ we see that the $G \times H$-module generated by any simple G-submodule of $V \otimes W$ is $V \otimes W$, hence the first claim. The second follows, because $\mathrm{Hom}(V, W) \simeq V^* \otimes W$ as $G \times H$-modules (cf. Exercise III.2.2.5(3)). □

Remark 1.2.4. We will see later that every simple $G \times H$-module is of the form $V \otimes W$ (Corollary 1.3.2).

Corollary 1.2.5. *Let $G \subseteq \mathrm{GL}_n$ be an subgroup and denote by $\langle G \rangle \subseteq \mathrm{M}_n(\mathbb{C})$ the linear span of the elements of G. Then $\langle G \rangle$ is a subalgebra, and $\langle G \rangle = \mathrm{M}_n(\mathbb{C})$ if and only if G is an irreducible subgroup, i.e., the matrix representation of G on $\mathbb{C}^n$ is irreducible.*

Proof. It is clear from the definition that $\mathbb{C}[G]$ is a subalgebra. The group $G \times G$ acts linearly on M_n by $(g,h)A := gAh^{-1}$, and $\mathbb{C}[G]$ is a $G \times G$-stable subspace. As a $G \times G$-module, we have an isomorphism $\mathrm{M}_n(\mathbb{C}) \simeq \mathbb{C}^n \otimes (\mathbb{C}^n)^*$. Hence, by the corollary above, $\mathrm{M}_n(\mathbb{C})$ is a simple $G \times G$-module in case $\mathbb{C}^n$ is an irreducible representation of G. On the other hand, if G is reducible, then, by choosing a suitable basis, the elements of G have block form with a zero block in the lower left corner, and so $\mathbb{C}[G] \subsetneqq \mathrm{M}_n(\mathbb{C})$. □

1.3. Isotypic decomposition. The decomposition of a semisimple module into simple modules is in general not unique, as we have seen in the previous section. However, one obtains a canonical decomposition by collecting the isomorphic simple submodules, as we are going to define now.

Let G be an algebraic group and let Λ_G denote the set of isomorphism classes of simple G-modules. If $\lambda \in \Lambda_G$, then a module $W \in \lambda$ is called *simple of type* λ. We use $0 \in \Lambda_G$ to denote the isomorphism class of the one-dimensional *trivial* module.

If V is a G-module and $\lambda \in \Lambda_G$ we define

$$V_\lambda := \sum_{W \subseteq V, W \in \lambda} W \subseteq V.$$

This submodule of V is semisimple and is called *isotypic component of type* λ. By definition, $V_0 = V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$.

Proposition 1.3.1. (1) *Every simple submodule $W \subseteq V_\lambda$ is of type λ. In particular, $V_\lambda \simeq W^d$.*

(2) *If W is simple of type λ, then*

$$\mathrm{Hom}_G(W,V) \otimes W \xrightarrow{\sim} V_\lambda, \;\; \mu \otimes w \mapsto \mu(w),$$

is an isomorphism of G-modules.

(3) *The sum of the isotypic components is direct: $\bigoplus_{\lambda \in \Lambda_G} V_\lambda \subseteq V$, and this sum contains every semisimple submodule of V. In particular, V is semisimple if and only if $V = \bigoplus_{\lambda \in \Lambda_G} V_\lambda$.*

(4) *If $\varphi\colon V \to U$ is a G-homomorphism, then $\varphi(V_\lambda) \subseteq U_\lambda$.*

Proof. (1) Since V_λ is semisimple, a simple submodule $W \subseteq V_\lambda$ has a G-stable complement: $V_\lambda = W \oplus W'$. Therefore, there is a G-equivariant linear projection $p\colon V_\lambda \to W$. Since V_λ is spanned by simple modules of type λ, one of them maps non-trivially to W, and the claim follows from Schur's Lemma (Proposition 1.2.1).

(2) It is clear that $\mu \otimes w \mapsto \mu(w)$ defines a G-homomorphism $\mathrm{Hom}_G(W,V) \otimes W \xrightarrow{\sim} V_\lambda$. In order to prove that this is an isomorphism, we first replace V by V_λ and then V_λ by W^d.

(3) It is clear that $M := \sum_{\lambda \in \Lambda_G} V_\lambda \subseteq V$ is the largest semisimple submodule of V. For each λ the submodule $V_\lambda \subseteq M$ has a G-stable complement V'. It follows that every simple submodule $W \subseteq V$ which is not of type λ must be contained in V'. Hence the claim.

(4) This is clear, since the image of a simple module of type λ is either (0) or simple of type λ. □

Corollary 1.3.2. *Every simple $G \times H$ module is of the form $V \otimes W$ with a simple G-module V and a simple H-module W.*

Proof. Let U be a simple $G \times H$ module and $W \subseteq U$ a simple H-submodule. Considering U as a G-module we see that $\mathrm{Hom}_H(W, U)$ is a G-module and that the map $\mathrm{Hom}_H(W, U) \otimes W \to U$ from Proposition 1.3.1(2) is an injective $G \times H$-homomorphism. $\square$

The decomposition $V = \bigoplus_\lambda V_\lambda$ of a semisimple module V is called *decomposition into isotypic components* or shortly *isotypic decomposition*. In general, the submodule $V_{\mathrm{soc}} := \bigoplus_\lambda V_\lambda \subseteq V$ is called the *socle* of V.

The isotypic decomposition carries over to locally finite and rational G-modules, in particular to the coordinate rings of affine G-varieties. As in the finite dimensional case such a module is called *semisimple* if it is a sum of simple submodules.

Proposition 1.3.3. *Let X be a G-variety and denote by $\mathcal{O}(X)_{\mathrm{soc}} \subseteq \mathcal{O}(X)$ the sum of all simple submodules. Then*

$$\mathcal{O}(X)_{\mathrm{soc}} = \bigoplus_{\lambda \in \Lambda_G} \mathcal{O}(X)_\lambda$$

where $\mathcal{O}(X)_\lambda$ is the sum of all simple submodules of $\mathcal{O}(X)$ of type λ. Moreover, $\mathcal{O}(X)_0 = \mathcal{O}(X)^G$ is a subalgebra, and each $\mathcal{O}(X)_\lambda$ is a $\mathcal{O}(X)^G$-module.

Proof. Since the G-action on $\mathcal{O}(X)$ is locally finite the first part follows immediately from the proposition above.

For the second part we remark that G acts on $\mathcal{O}(X)$ by algebra automorphisms, i.e. we have $g(f_1 f_2) = (gf_1)(gf_2)$ for $g \in G$ and $f_1, f_2 \in \mathcal{O}(X)$. Hence, if $f_1 \in \mathcal{O}(X)^G$, then the linear map $f \mapsto f_1 f$ is a G-homomorphism, proving the second part of the proposition. $\square$

Exercise 1.3.4. Let V be locally finite rational G-module. If V is semisimple, then every submodule $W \subseteq V$ has a G-stable complement in V.

Definition 1.3.5. A function $f \in \mathcal{O}(X)$ is called *G-invariant* (shortly *invariant*) if f is constant on orbits, i.e. $f(gx) = f(x)$ for all $g \in G$, $x \in X$. Thus $\mathcal{O}(X)_0 = \mathcal{O}(X)^G$ is the subalgebra of G-invariant functions. The $\mathcal{O}(X)^G$-modules $\mathcal{O}(X)_\lambda$ are classically called *modules of covariants.*

Example 1.3.6. The regular representation of $G = \mathrm{GL}(V)$ or $\mathrm{SL}(V)$ on $\mathcal{O}(V)$ stabilizes the homogeneous components $\mathcal{O}(V)_d$. *We claim that the $\mathcal{O}(V)_d$ are simple G-modules, so that $\mathcal{O}(V) = \bigoplus_{d \geq 0} \mathcal{O}(V)_d$ is the isotypic decomposition.*

Proof. Fix a basis of V so that $V = \mathbb{C}^n$ and $\mathcal{O}(V) = \mathbb{C}[x_1, \ldots, x_n]$, and let $W \subseteq \mathbb{C}[x_1, \ldots, x_n]_d$ be a submodule. Then W is stable under $\mathrm{Lie}\, G$. The matrix $E_{ij} \in \mathrm{Lie}\, G$, $i \neq j$, operates as the differential operator $x_j \frac{\partial}{\partial x_i}$ (see III.5.4). If $f \in W$, $f \neq 0$, then, applying successively the operators $x_1 \frac{\partial}{\partial x_i}$, we first see that $x_1^d \in W$ and then, applying the operators $x_j \frac{\partial}{\partial x_1}$, it follows that all monomials of degree d belong to W. $\square$

Exercise 1.3.7. Let V, W be two semisimple G-modules and let $\varphi \colon V \to W$ be a surjective G-homomorphism. Then $\varphi(V_\lambda) = W_\lambda$ for all $\lambda \in \Lambda_G$. Give an example which shows that this does not hold if V is not semisimple.

2. Invariants and Algebraic Quotients

2.1. Linearly reductive groups. We now introduce a special class of groups which share a number of important properties.

Definition 2.1.1. An algebraic group G is *linearly reductive* if every representation $\rho\colon G \to \mathrm{GL}(V)$ is completely reducible. Equivalently, every G-module is semisimple.

Example 2.1.2. A diagonalizable group D is linearly reductive. In fact, every representation of $\rho\colon D \to \mathrm{GL}(V)$ is diagonalizable which means that there is a basis of V such that $\rho(D) \subseteq T_n$ (Proposition II.3.4.5). The isotypic decomposition is given by

$$V = \bigoplus_{\chi \in \mathcal{X}(D)} V_\chi \quad \text{where} \quad V_\chi = \{v \in V \mid \rho(d)v = \chi(d) \cdot v \text{ for all } d \in D\}.$$

Example 2.1.3. A finite group G is linearly reductive, by the Theorem of Maschke (see [**Art91**, Chap. 9, Corollary 4.9]).

Exercise 2.1.4. If G, H are algebraic groups, then $G \times H$ is linearly reductive if and only if G and H are both linearly reductive.

Example 2.1.5. Every representation $\rho\colon \mathbb{C}^+ \to \mathrm{GL}(V)$, $V \neq \{0\}$, contains the trivial representations, because ρ is of the form $s \mapsto \exp(sN)$ with a nilpotent matrix N (see Proposition II.2.6.1). Thus $\mathbb{C}^+$ has a unique simple module, namely the trivial one-dimensional module. In particular, $V_{\mathrm{soc}} = V^{\mathbb{C}^+}$ and so $\mathbb{C}^+$ is not linearly reductive.

Proposition 2.1.6. *Let G be an algebraic group. The following statements are equivalent:*

(i) *G is linearly reductive.*
(ii) *The representation of G on $\mathcal{O}(G)$ (by left or right multiplication) is completely reducible.*
(iii) *For every surjective G-homomorphism $\varphi\colon V \twoheadrightarrow W$ we have $\varphi(V^G) = W^G$.*

Proof. It is clear that (i) implies (ii) and (iii) (see Exercise 1.3.7). Moreover, (ii) implies (i), because every G-module occurs as a submodule of $\mathcal{O}(G)^{\oplus n}$ (Exercise III.2.4.3).

For the implication (iii) $\to$ (i) we show that every submodule $W \subseteq V$ has a G-stable complement (Proposition 1.1.5). Consider the G-homomorphism $\mathrm{Hom}(V, W) \to \mathrm{End}(W)$, $\varphi \mapsto \varphi|_W$ which is clearly surjective. Thus $\mathrm{Hom}_G(V, W) \to \mathrm{End}_G(W)$ is also surjective, and so there is $\varphi\colon V \to W$ such that $\varphi|_W = \mathrm{id}_W$, hence $V = W \oplus \ker\varphi$. $\square$

Remark 2.1.7. If G is linearly reductive and $\varphi\colon V \twoheadrightarrow W$ a surjective G-homomorphism of locally finite and rational G-modules, then we also have $\varphi(V^G) = W^G$. In fact, if $w \in W^G$ and $v \in V$ a preimage of w, then $V' := \langle Gv \rangle$ is a finite dimensional G-module, and $\varphi(V')$ contains w.

Exercise 2.1.8. Use the proposition above to give another proof that a diagonalizable group D is linearly reductive (Example 2.1.2) by showing that $\mathcal{O}(D)$ is a direct sum of one-dimensional submodules.

Exercise 2.1.9. Give an example of a surjective $\mathbb{C}^+$-homomorphism $\varphi\colon V \to W$ such that $\varphi(V^{\mathbb{C}^+}) \neq W^{\mathbb{C}^+}$.

Exercise 2.1.10. Show that G is linearly reductive if and only if there is a G-equivariant linear operator $I\colon \mathcal{O}(G) \to \mathbb{C}$ such that $I(c) = c$ for $c \in \mathbb{C}$.
(This operator can be thought of as the "integral" $f \mapsto \int_G f\,dg$.)

2.2. The coordinate ring of a linearly reductive group. Let G be an algebraic group, and consider G as a $G \times G$-variety with the usual action given by $(g_1, g_2)h := g_1 h g_2^{-1}$. Recall that the simple $G \times G$-modules are of the form $V \otimes W$ where V, W are simple G-modules (Corollary 1.3.2). In particular, $V^* \otimes V$ is a simple $G \times G$-module which is canonically isomorphic to $\mathrm{End}(V)$:

$$V^* \otimes V \xrightarrow{\sim} \mathrm{End}(V) \text{ is induced by } \ell \otimes v \mapsto \varphi_{\ell,v}$$

where $\varphi_{\ell,v}(w) := \ell(w) \cdot v$. The corresponding representation $\rho\colon G \to \mathrm{GL}(V)$ gives a $G \times G$-equivariant morphism $\rho\colon G \to \mathrm{End}(V)$ and thus a $G \times G$-homomorphism $\rho^*\colon \mathrm{End}(V)^* \to \mathcal{O}(G)$ which is injective because $\mathrm{End}(V)$ is simple. It is also clear that for two equivalent representations $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ we get the same images $\rho_1^*(\mathrm{End}(V_1)^*) = \rho_2^*(\mathrm{End}(V_2)^*)$.

For every isomorphism class $\lambda \in \Lambda_G$ we choose a simple module of type λ and denote it by $V(\lambda)$.

Proposition 2.2.1. *Let G be a linearly reductive group. Then the isotypic decomposition of $\mathcal{O}(G)$ as a $G \times G$-module has the form*

$$\mathcal{O}(G) = \bigoplus_{\lambda \in \Lambda_{G \times \Lambda}} \mathcal{O}(G)_\lambda$$

where $\mathcal{O}(G)_\lambda \simeq \mathrm{End}(V(\lambda))^* \simeq V(\lambda) \otimes V(\lambda)^*$.

Proof. Let $W \subseteq \mathcal{O}(G)$ be a simple G-submodule, with respect to the right action of G, and let $\rho\colon G \to \mathrm{GL}(W)$ denote the corresponding representation. The claim follows if we show that W is contained in the image of $\rho^*\colon \mathrm{End}(W)^* \to \mathcal{O}(G)$. For $f \in W$ define the linear map $\ell_f\colon \mathrm{End}(W) \to \mathbb{C}$ by $\ell_f(\varphi) := \varphi(f)(e)$. Then $\rho^*(\ell_f)(h) = \ell_f(\rho(h)) = (\rho(h)f)(e) = f(eh) = f(h)$ and so $\rho^*(\ell_f) = f$. $\square$

If G is a finite group, then $\dim \mathcal{O}(G) = |G|$, and we rediscover the famous formula

$$|G| = \sum_\lambda d_\lambda^2$$

where d_λ is the dimension of the irreducible representation of type λ.

Corollary 2.2.2. *The isotypic decomposition of $\mathcal{O}(G)$ with respect to the left action of G has the form*

$$\mathcal{O}(G) \simeq \bigoplus_{\lambda \in \Lambda_G} V(\lambda)^{\oplus d_\lambda}$$

where $d_\lambda := \dim V(\lambda)$.

Corollary 2.2.3. *Let X be a G-variety containing a dense orbit $Gx \subseteq X$. Then the isotypic decomposition of $\mathcal{O}(X)$ has the form*

$$\mathcal{O}(X) \simeq \bigoplus_{\lambda \in \Lambda_G} V(\lambda)^{m_\lambda} \quad \textit{where} \quad m_\lambda \leq \dim(V(\lambda)^*)^{G_x}.$$

Proof. The orbit map $\mu_x\colon G \to X$ is G-equivariant and dominant, and so $\mu_x^*\colon \mathcal{O}(X) \to \mathcal{O}(G)$ is an injective G-homomorphism. Since μ_x is invariant with respect to the right action of the stabilizer G_x, $\mu_x(gh^{-1}) = \mu_x(g)$ for all $g \in G$ and $h \in G_x$, we see that the image of μ_x^* is contained in the invariants $\mathcal{O}(G)^{G_x}$ with respect to the right action. Hence, $\mu_x^*\colon \mathcal{O}(X)_\lambda \hookrightarrow V(\lambda) \otimes (V(\lambda)^*)^{G_x}$. $\square$

Let X be a G-variety. If the isotypic component $\mathcal{O}(X)_\lambda$ is finite dimensional, then $\mathcal{O}(X)_\lambda \simeq V(\lambda)^{m_\lambda}$. The exponent m_λ is called the *multiplicity of $\lambda \in \Lambda_G$ in X* and will be denoted by $m_\lambda(X)$. If the isotypic component $\mathcal{O}(X)_\lambda$ is not finite dimensional, then we set $m_\lambda(X) := \infty$.

Definition 2.2.4. We say that a G-variety X has *finite multiplicities* if $m_\lambda(X) < \infty$ for all λ, and that X is *multiplicity-free* if $m_\lambda(X) \leq 1$ for all λ.

We will see later in chapter V that multiplicity freeness has an interesting geometric interpretation.

2.3. Hilbert's Finiteness Theorem. The next result is the famous Finiteness Theorem of Hilbert [**Hil90, Hil93**].

Theorem 2.3.1. *Let G be a linearly reductive group and X a G-variety. Then the subalgebra $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$ of invariant functions is finitely generated. Moreover, the isotypic components $\mathcal{O}(X)_\lambda$ are finitely generated $\mathcal{O}(X)^G$-modules.*

Before giving the proof we make some general remarks. Consider the isotypic decomposition $\mathcal{O}(X) = \bigoplus_{\lambda\in\Lambda_G} \mathcal{O}(X)_\lambda$ and the linear projection

$$R\colon \mathcal{O}(X) \to \mathcal{O}(X)^G$$

called the Reynolds-*operator*. The linear map R is a G-equivariant $\mathcal{O}(X)^G$-homomorphism, i.e. $R(pf) = pR(f)$ for $p \in \mathcal{O}(X)^G$ and $f \in \mathcal{O}(X)$. In particular, $R(V) = V^G$ for every G-stable subspace $V \subseteq \mathcal{O}(X)$. As a consequence, we get for every ideal $\mathfrak{a} \subseteq \mathcal{O}(X)^G$:

$$\mathcal{O}(X)\mathfrak{a} \cap \mathcal{O}(X)^G = R(\mathcal{O}(X)\mathfrak{a}) = \mathfrak{a}.$$

This already implies that $\mathcal{O}(X)^G$ is a Noetherian algebra (see A.1.6).

If $\mathfrak{b} \subseteq \mathcal{O}(X)$ is a G-stable ideal, then $p\colon \mathcal{O}(X) \to \mathcal{O}(X)/\mathfrak{b}$ induces a surjection $\mathcal{O}(X)^G \to (\mathcal{O}(X)/\mathfrak{b})^G$ with kernel $\mathfrak{b} \cap \mathcal{O}(X)^G = \mathfrak{b}^G$. Finally, if $(V_i)_{i\in I}$ is a family of G-stable subspaces of $\mathcal{O}(X)$, then $(\sum_i V_i)_\lambda = \sum_i (V_i)_\lambda$ for all $\lambda \in \Lambda_G$. Thus we have proved statements (3)–(5) of the following proposition. The Finiteness Theorem above is contained in the first two statements which we will prove below.

Proposition 2.3.2. *Put $B := \mathcal{O}(X)$ and $A := \mathcal{O}(X)^G \subseteq B$.*

1. *A is a finitely generated $\mathbb{C}$-algebra.*
2. *For every $\lambda \in \Lambda_G$ the isotypic component B_λ is a finitely generated A-module.*
3. *For every G-stable ideal $\mathfrak{b} \subseteq B$ we have $A/(\mathfrak{b}^G) \xrightarrow{\sim} (B/\mathfrak{b})^G$.*
4. *For every ideal $\mathfrak{a} \subseteq A$ we have $B\mathfrak{a} \cap A = \mathfrak{a}$. In particular, A is Noetherian.*
5. *If $(\mathfrak{b}_i)_{i\in I}$ is a family of G-stable ideals of B, then $\sum_i \mathfrak{b}_i^G = (\sum_i \mathfrak{b}_i)^G \subseteq A$.*

For the proof we will need the following useful lemma.

Lemma 2.3.3. *Let $A = \bigoplus_{i\geq 0} A_i$ be a graded $\mathbb{C}$-algebra, $\mathfrak{n} := \bigoplus_{i>0} A_i$, and let $a_1, \ldots, a_n$ be homogeneous elements of $\mathfrak{n}$. Then the following statements are equivalent:*

(i) *$A = A_0[a_1, \ldots, a_n]$.*
(ii) *$\mathfrak{n} = \sum_i Aa_i$.*
(iii) *$\mathfrak{n}/\mathfrak{n}^2 = \sum_i A_0\bar{a}_i$ where $\bar{a}_i := a_i + \mathfrak{n}^2$.*

In particular, if A is Noetherian, then A is finitely generated over A_0.

Proof. The implications (i) → (ii) and (ii) → (iii) are easy. We leave their proof to the reader.

(iii) → (i): By assumption, we have $\mathfrak{n} = \sum_i A_0 a_i + \mathfrak{n}^2$ and so, by induction,

$$\mathfrak{n}^k = \sum_{\sum k_i = k} A_0 a_1^{k_1} \cdots a_n^{k_n} + \mathfrak{n}^{k+1}$$

for all $k \geq 1$. This shows that $A = A_0[a_1, \ldots, a_n] + \mathfrak{n}^k$ for all $k \geq 1$. Now the claim follows, because both sides are graded and $\mathfrak{n}^k$ has no elements in degree $< k$. $\square$

Proof of Proposition 2.3.2. (1) If $X = V$ is a G-module, then $\mathcal{O}(V) = \bigoplus_{i\geq 0}\mathcal{O}(V)_i$ is a graded $\mathbb{C}$-algebra where each graded component is G-stable. Thus $\mathcal{O}(V)^G = \bigoplus_{i\geq 0}\mathcal{O}(V)_i^G$ is again graded, with $\mathcal{O}(V)_0^G = \mathcal{O}(V)_0 = \mathbb{C}$. Since $\mathcal{O}(V)^G$ is Noetherian (Proposition 2.3.2(3)) the lemma above implies that $\mathcal{O}(V)^G$ is finitely generated.

In general, the G-variety X is isomorphic to a G-stable closed subvariety of a G-module V. Therefore, we obtain a G-equivariant surjective homomorphism $\mathcal{O}(V) \twoheadrightarrow \mathcal{O}(X)$ and thus a surjection $\mathcal{O}(V)^G \twoheadrightarrow \mathcal{O}(X)^G$ (Proposition 2.1.6).

(2) Let W be a simple G-module of type λ. Then $\mathcal{O}(X\times W)^G$ is a finitely generated graded $\mathbb{C}$-algebra:

$$\mathcal{O}(X\times W)^G = \bigoplus_{i\geq 0}(\mathcal{O}(X)\otimes\mathcal{O}(W)_i)^G = \mathcal{O}(X)^G \oplus (\mathcal{O}(X)\otimes W^*)^G \oplus \cdots .$$

It follows that $(\mathcal{O}(X)\otimes W^*)^G$ is a finitely generated $\mathcal{O}(X)^G$-module. Now we have seen in Proposition 1.3.1(2) that

$$(\mathcal{O}(X)\otimes W^*)^G \otimes W = \operatorname{Hom}_G(W,\mathcal{O}(X))\otimes W \xrightarrow{\sim} \mathcal{O}(X)_\lambda,$$

and it is easy to see that this linear map is a $\mathcal{O}(X)^G$-module homomorphism. □

A nice application of the lemma above is the following result about smooth cones.

Example 2.3.4. Let $X \subseteq \mathbb{C}^n$ be a closed cone Assume that X is non-singular in 0. Then X is a linear subspace of $\mathbb{C}^n$.

Proof. By assumption, $\mathcal{O}(X)$ is graded, $\mathcal{O}(X) = \bigoplus_{i\geq 0}\mathcal{O}(X)_i$ where $\mathcal{O}(X)_0 = \mathbb{C}$, and $\mathfrak{m}_0 := \bigoplus_{i>0}\mathcal{O}(X)_i$ is the maximal ideal of $0\in X$. Since X is smooth in 0 we know that $\dim \mathfrak{m}_0/\mathfrak{m}_0^2 = \dim X$. Thus we can find $d := \dim X$ homogeneous elements $f_1,\ldots,f_d \in \mathfrak{m}_0$ whose images in $\mathfrak{m}_0/\mathfrak{m}_0^2$ form a $\mathbb{C}$-basis. Then Lemma 2.3.3 implies that $\mathcal{O}(X) = \mathbb{C}[f_1,\ldots,f_d]$. Since $\dim X = d$, the f_i are algebraically independent. It remains to see that $\deg f_i = 1$ for all i. But this is clear, because $\mathfrak{m}_0^2 = \bigoplus_{i>1}\mathcal{O}(X)_i$. □

2.4. Algebraic quotient. We start with the definition of an algebraic quotient of a G-variety X which turns out to be the best algebraic approximation to an orbit space X/G. In the following we assume that G is a linearly reductive.

Definition 2.4.1. Let X be a G-variety. A morphism $\pi\colon X\to Z$ is called *algebraic quotient* (shortly a *quotient*) if the comorphism $\pi^*\colon\mathcal{O}(Z)\to\mathcal{O}(X)$ induces an isomorphism $\mathcal{O}(Z)\xrightarrow{\sim}\mathcal{O}(X)^G$.

It follows from Hilbert's Finiteness Theorem 2.3.1 that algebraic quotients exist. They can be constructed in the following way. Choose a set of generators $f_1,\ldots,f_n$ of the invariants $\mathcal{O}(X)^G$, consider the morphism

$$\pi := (f_1,\ldots,f_n)\colon X\to\mathbb{C}^n,$$

and define $Z := \overline{\pi(X)}$. *Then $\pi\colon X\to Z$ is an algebraic quotient.* It is also clear that a quotient map $\pi\colon X\to Y$ is *G-invariant*, i.e. π is constant on orbits: $\pi(gx) = \pi(x)$ for all $g\in G, x\in X$.
(In fact, if $\pi(gx)\neq\pi(x)$, then there is an $f\in\mathcal{O}(Z)$ such that $f(\pi(gx))\neq f(\pi(x))$ which is a contradiction, because $\pi^*(f)$ is a G-invariant function.)

2.5. Properties of quotients. Let us now collect the main properties of algebraic quotients.

Universal Mapping Property: Let $\pi\colon X \to Z$ be an algebraic quotient. *If $\varphi\colon X \to Y$ is an invariant morphism, i.e. φ is constant on orbits, then there is a unique $\bar\varphi\colon Z \to Y$ such that $\varphi = \bar\varphi \circ \pi$:*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & Z \arrow[d, "\bar\varphi"] \\ & Y \end{tikzcd}$$

Proof. Since φ is invariant we have $\varphi^*(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)^G$, and thus there is a uniquely defined homomorphism $\mu\colon \mathcal{O}(Y) \to \mathcal{O}(Z)$ such that $\pi^* \circ \mu = \varphi^*$. Then the corresponding morphism $\bar\varphi\colon Z \to Y$ has the required property. □

Existence and uniqueness: *An algebraic quotient exists and is unique up to unique isomorphisms.*

Proof. This follows immediately from the universal property. □

Thus we can safely talk about "the" quotient, and we will use the notation

$$\pi_X : X \to X/\!\!/G$$

where we identify $\mathcal{O}(X/\!\!/G) = \mathcal{O}(X)^G$.

G-closedness: *If $X' \subseteq X$ is a closed G-stable subset, then $\pi_X(X') \subseteq X/\!\!/G$ is closed, and the induced morphism $\pi\colon X' \to \pi_X(X')$ is an algebraic quotient.*

Proof. Let $\mathfrak{b} := I(X') \subseteq \mathcal{O}(X)$ be the ideal of X'. Then the ideal of the closure $\overline{\pi_X(X')}$ is given by $\mathfrak{b} \cap \mathcal{O}(X)^G = \mathfrak{b}^G$. Since $\mathcal{O}(X)^G/\mathfrak{b}^G \xrightarrow{\sim} (\mathcal{O}(X)/\mathfrak{b})^G$ by Proposition 2.3.2(3) we see that $\pi\colon X' \to \overline{\pi_X(X')}$ is an algebraic quotient. It remains to show that an algebraic quotient $\pi_X : X \to X/\!\!/G$ is surjective.

Let $y \in X/\!\!/G$ be a point and $\mathfrak{m}_y \subseteq \mathcal{O}(X)^G$ the corresponding maximal ideal. Since $\mathcal{O}(X)\mathfrak{m}_y \cap \mathcal{O}(X)^G = \mathfrak{m}_y$ by Proposition 2.3.2(4), it follows that $\mathcal{O}(X)\mathfrak{m}_y$ is strict ideal of $\mathcal{O}(X)$ and so the fiber $\pi_X^{-1}(y)$ is not empty. □

G-separation: *Let $(C_i)_{i\in I}$ be a family of closed G-stable subsets of X. Then*

$$\pi_X(\bigcap_{i\in I} C_i) = \bigcap_{i\in I} \pi_X(C_i).$$

In particular, the images under the π_X of two disjoint G-stable closed subsets of X are disjoint.

Proof. Let $\mathfrak{b}_i := I(C_i) \subseteq \mathcal{O}(X)$ be the ideal of C_i. Then $\bigcap_{i\in I} C_i$ is defined by $\sum_{i\in I} \mathfrak{b}_i$ and its image in $\pi_X(\bigcap_{i\in I} C_i) \subseteq X/\!\!/G$ by $(\sum_{i\in I} \mathfrak{b}_i)^G$. By Proposition 2.3.2(4), the latter is equal to the ideal $\sum_{i\in I} \mathfrak{b}_i^G$ which defines the closed subset $\bigcap_{i\in I} \pi_X(C_i)$. Thus $\pi_X(\bigcap_{i\in I} C_i) = \bigcap_{i\in I} \pi_X(C_i)$ as claimed. □

2.6. Some consequences. The properties of a quotient map $\pi_X : X \to X/\!\!/G$ formulated above have a number of important consequences.

COROLLARY 2.6.1. *The quotient map $\pi_X : X \to X/\!\!/G$ is submersive, i.e. it is surjective and $X/\!\!/G$ carries the quotient topology.*

PROOF. Let $U \subseteq X/\!\!/G$ be a subset such that $\pi_X^{-1}(U)$ is open in X. Then $A := X \setminus \pi_X^{-1}(U)$ is closed and G-stable and so, by G-closeness, $\pi_X(A) \subseteq X/\!\!/G$ is closed. Thus $U = X/\!\!/G \setminus \pi_X(A)$ is open in $X/\!\!/G$. □

COROLLARY 2.6.2. *Let $\pi_X : X \to X/\!\!/G$ be the quotient and let $f \in \mathcal{O}(X)^G$ be a non-zero divisor. Then $\pi_X(X_f) = (X/\!\!/G)_f$, and $X_f \to (X/\!\!/G)_f$ is a quotient.*

PROOF. We have $\mathcal{O}(X_f) = \mathcal{O}(X)_f$. Since f is G-invariant we get $(\mathcal{O}(X)_f)^G = (\mathcal{O}(X)^G)_f = \mathcal{O}((X/\!\!/G)_f)$ (see Exercise 2.6.3 below), and the claim follows. □

EXERCISE 2.6.3. Let A be an algebra and G a group acting on A by algebra automorphisms. If $f \in A^G$ is a non-zero divisor, then $(A_f)^G = (A^G)_f$.

EXAMPLE 2.6.4. Let X be a G-variety and $\pi_X : X \to X/\!\!/G$ the algebraic quotient. If $\eta : Y \to X/\!\!/G$ is any morphism, then the fiber product $Y \times_{X/\!\!/G} X$ (A.2.6) is a G-variety in a natural way, and the induced morphism $p: Y \times_{X/\!\!/G} X \to Y$ is a quotient:

$$\begin{array}{ccc} Y \times_{X/\!\!/G} X & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Y & \xrightarrow{\eta} & X/\!\!/G \end{array}$$

In fact, $Y \times_{X/\!\!/G} X$ is a closed G-stable subvariety of $Y \times X$ where G acts only on X, and so the quotient is induced by the morphism $\varphi := \mathrm{id}_Y \times \pi_X : Y \times X \to Y \times (X/\!\!/G)$. Since $\varphi(Y \times_{X/\!\!/G} X) \subseteq Y \times (X/\!\!/G)$ coincides with the graph of the morphism η, the projection onto Y induces an isomorphism $\varphi(Y \times_{X/\!\!/G} X) \xrightarrow{\sim} Y$.

COROLLARY 2.6.5. *Every fiber of the quotient map $\pi_X : X \to X/\!\!/G$ contains a unique closed orbit. In particular, the closure of an orbit $Gx \subseteq X$ contains a unique closed orbit.*

PROOF. Since a fiber $\pi_X^{-1}(y)$ is closed an G-stable it contains a closed orbit. Because of G-separation it cannot contain more than one closed orbit. □

The last corollary shows that the quotient $X/\!\!/G$ parametrized the *closed* orbits in X. So if all orbits are closed, e.g. if the group G is finite, then $X/\!\!/G$ can be identified with the orbit space X/G.

DEFINITION 2.6.6. A quotient $\pi_X : X \to X/\!\!/G$ is called a *geometric quotient* if every fiber of π_X is an orbit. This is the case if and only if all orbits in X are closed.

COROLLARY 2.6.7. *Let G be a finite group and X a G-variety. Then $\pi_X : X \to X/\!\!/G$ is a geometric quotient, and π_x is a finite morphism.*

PROOF. Since all G-orbits are closed the quotient is geometric. In order to see that π_X is a finite morphism, we simply remark that every $f \in \mathcal{O}(X)$ satisfies the monic equation $\prod_{g\in G}(x - gf) = 0$ whose coefficients are G-invariants (see Lemma A.3.2.10). □

In general, if $x, x' \in X$ belong to the same fiber $\pi_X^{-1}(y)$ of the quotient map, then the orbit closures $\overline{Gx}$ and $\overline{Gx'}$ both contain a closed orbit which is the unique closed orbit of the fiber, and so $\overline{Gx} \cap \overline{Gx'} \neq \emptyset$. This shows that $X/\!\!/G$ is the quotient space of X with respect to the equivalence relation

$$x \sim x' \iff \overline{Gx} \cap \overline{Gx'} \neq \emptyset.$$

This also explains the notation $X/\!\!/G$ with two slashes.

If V is a G-module with quotient map $\pi_V : V \to V/\!\!/G$, then we get

$$\mathcal{N}_V := \pi_V^{-1}(\pi_V(0)) = \{v \in V \mid \overline{Gv} \ni 0\} \subseteq V.$$

The subset $\mathcal{N}_V$ is a G-stable closed cone in V and is called the *null fiber* or the *null cone* of V.

Example 2.6.8. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ defined by $t(x, y) := (t \cdot x, t^{-1} \cdot y)$. Then

$$\pi : \mathbb{C}^2 \to \mathbb{C}, \quad (x, y) \mapsto xy$$

is the quotient, and all fibers are orbits except the null fiber $\mathcal{N} = \pi^{-1}(0)$ which consists of three orbits, namely $\pi^{-1}(0) = \mathbb{C}^*(1, 0) \cup \mathbb{C}^*(0, 1) \cup \{0\}$. It follows that $\pi : \mathbb{C}^2 \setminus \pi^{-1}(0) \to \mathbb{C} \setminus \{0\}$ is a geometric quotient.

Exercise 2.6.9. Let X be a G-variety and Y an H-variety where G, H are both linearly reductive. Then $X \times Y$ is a $G \times H$-variety, and $\pi_X \times \pi_Y : X \times Y \to X/\!\!/G \times Y/\!\!/H$ is the quotient.

2.7. The case of finite groups. We already noted that a *finite group* is linearly reductive (Theorem of Maschke; cf. 2.1.3) Some of the results proved for linearly reductive groups can be sharpened considerably in the finite case.

The following result was already proved in the previous section (Corollary 2.6.7).

Proposition 2.7.1. *Suppose that G is finite and X is a G-variety. Then the quotient $\pi : X \to X/\!\!/G$ is geometric and π is a finite morphism.*

The Finiteness Theorem (Theorem 2.3.1) can be strengthened to the extent that we can give an explicit system of generators.

To do this let V be a G-module, $(v_1, \dots, v_n)$ a basis of V and $(x_1, \dots, x_n)$ the dual basis where $x_i \in V^* \subseteq \mathcal{O}(V)$. For every $\mu \in \mathbb{N}^n$ we set $x^\mu := x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n} \in \mathcal{O}(V)$, and we consider the *homogeneous invariants*

$$J_\mu := \sum_{g \in G} g x^\mu \in \mathcal{O}(V)^G$$

of degree $|\mu| := \mu_1 + \cdots + \mu_n$. The following result is due to Emmy Noether [**Noe15**].

Theorem 2.7.2. *The invariant ring $\mathcal{O}(V)^G$ is linearly spanned by the invariants J_μ, and is generated by those J_μ with $|\mu| \leq |G|$.*

Thus one sees that the invariants of degree $\leq |G|$ generate the invariant ring. The number of these is less than

$$\binom{\dim V + |G|}{\dim V},$$

but this estimate is far too big. A much better bound was found by Derksen [**Der01**].

Corollary 2.7.3. *The module $\mathcal{O}(V)$ is generated as an $\mathcal{O}(V)^G$-module by the homogeneous elements of degree $< |G|$.*

For the proof we need the following result about symmetric functions, see Exercise I.2.2.2.

LEMMA 2.7.4. *The subalgebra $A \subseteq \mathbb{C}[T_1, ..., T_d]$ of symmetric functions is generated by the power sums*

$$\psi_j := T_1^j + T_2^j + \cdots + T_d^j, \quad j = 1, 2, \ldots, d.$$

PROOF. We have to show that the elementary symmetric functions $s_1, \ldots, s_d$ can be expressed in $\psi_1, \ldots, \psi_d$. This follows by induction from the following formula of NEWTON:

$$(*) \quad \psi_j - s_1\psi_{j-1} + s_2\psi_{j-2} - \cdots + (-1)^{j-1}s_{j-1}\psi_1 + (-1)^j s_j \cdot j = 0, \quad j = 1, 2, \ldots, d.$$

(a) The formula for $j = d$ is clear if we set

$$f(Z) := \prod_{i=1}^{d}(Z - T_i) = Z^d + \sum_{I=1}^{d}(-1)^i s_i Z^{d-i},$$

because this implies

$$0 = \sum_{r=1}^{d} f(T_r) = \psi_d + \sum_{i=1}^{d}(-1)^i s_i \psi_{d-i} \text{ where } s_0 := d.$$

(b) In the case $j < d$ we note that the left hand side of $(*)$ is a symmetric function of degree $\leq j$ and thus is a polynomial $p(s_1, \ldots, s_j)$ in $s_1, \ldots, s_j$. Now we set $T_{j+1} = \cdots = T_d = 0$ and denote this process by means of a bar. Clearly, $\overline{s_i}$ for $i \leq j$ is the ith elementary symmetric function in $T_1, \ldots, T_j$, and $\overline{\psi_i} = T_1^i + \cdots + T_j^i$. From (a) it follows that the left hand side of $(*)$ becomes 0 under this process, and hence $p(\overline{s_1}, \ldots, \overline{s_j}) = 0$. Now $\overline{s_1}, \ldots, \overline{s_j}$ are algebraically independent and thus $p = 0$.

□

PROOF OF THEOREM 2.7.2. If $f = \sum a_\mu X^\mu$ is an invariant, then

$$|G| \cdot f = \sum_{g \in G} gf = \sum a_\mu J_\mu.$$

Thus we get $\mathcal{O}(V)^G = \sum_\mu \mathbb{C}J_\mu$. We have to show that each J_ρ with $|\rho| > |G|$ can be expressed as a polynomial in the J_μ with $|\mu| \leq |G|$. To do this we consider the expressions

$$S_j(X, Z) := \sum_{g \in G}(gX_1 \cdot Z_1 + gX_2 \cdot Z_2 + \cdots + gX_n \cdot Z_n)^j, \quad \text{for } j \in \mathbb{N},$$

with indeterminates $Z_1, \ldots, Z_n$. Clearly

$$S_j(X, Z) = \sum_{\substack{\rho \in \mathbb{N}^n \\ |\rho| = j}} J_\rho \cdot Z^\rho.$$

By Lemma 2.7.4 above the $S_j(X, Z)$ for $j > |G|$ can be expressed as polynomials in the $S_j(X, Z)$ with $j \leq |G|$, and thus the J_ρ with $|\rho| > |G|$ are polynomials in the J_μ with $|\mu| \leq |G|$. □

PROOF OF COROLLARY 2.7.3. It suffices to show that every isotypical component of $\mathcal{O}(V)$ is generated as an $\mathcal{O}(V)^G$-module by elements of degree $< |G|$. Suppose W is a simple G-module of type λ. Then, as we have just seen, $\mathcal{O}(V \oplus W)^G$ is generated by elements of degree $\leq |G|$. Now

$$\begin{aligned} \mathcal{O}(V \oplus W)^G &= (\mathcal{O}(V) \otimes \mathcal{O}(W))^G = \bigoplus_{i \geq 0}(\mathcal{O}(V) \otimes \mathcal{O}(W)_i)^G \\ &= \mathcal{O}(V)^G \oplus (\mathcal{O}(V) \otimes W^*)^G \oplus \cdots \end{aligned}$$

is a grading, and thus the $\mathcal{O}(V)^G$-module $(\mathcal{O}(V)\otimes W^*)^G$ is generated by elements of degree $\le |G|$, i.e. by $\bigoplus_{i<|G|}(\mathcal{O}(V)_i\otimes W^*)^G$. The canonical $\mathcal{O}(V)^G$-module isomorphism (see Proposition 1.3.1(2))

$$(\mathcal{O}(V)\otimes W^*)^G\otimes W\simeq \mathcal{O}(V)_\lambda$$

maps $(\mathcal{O}(V)_i\otimes W^*)^G\otimes W$ onto $(\mathcal{O}(V)_\lambda)_i$, and the claim follows. $\square$

If we consider the usual permutation representation of the *symmetric group* S_n on $\mathbb{C}^n$, then the invariant ring is generated by the invariants of degree $\le n$. One also knows that the coordinate ring $\mathcal{O}(\mathbb{C}^n)$ is generated, as a module over the invariants $\mathbb{C}[s_1,\dots,s_n]$, by homogeneous elements of degree $\binom{n}{2}$. Here the bounds are substantially smaller than those given in the theorem and its corollary above.

The situation is different in the case of the *cyclic group* $G=\langle g\rangle$ of order n with the representation $\rho\colon G\to \mathrm{GL}_1=\mathbb{C}^*$, $g\mapsto \exp(2\pi i/n)$. Here the smallest homogeneous invariant is of degree n, and $1,x,x^2,\dots,x^{n-1}$ form a basis for $\mathcal{O}(\mathbb{C})=\mathbb{C}[x]$ over $\mathcal{O}(\mathbb{C})^G=\mathbb{C}[x^n]$. These are exactly the bounds given by the theorem and its corollary.

In this context there is an interesting result of Barbara Schmid [**Sch89**]. Let us first define the β-invariant of a finite group G. If V is a G-module, then

$$\beta(G,V):=\min\{d\mid \mathcal{O}(V)^G \text{ is generated by invariants of degree } \le d\},$$

and

$$\beta(G):=\max\{\beta(G,V)\mid V \text{ a } G\text{-module}\}.$$

It follows from classical invariant theory that $\beta(G)=\beta(G,V_{\mathrm{reg}})$ where V_{reg} is the regular representation of G. Here is one of the general results of Schmid.

Proposition 2.7.5. *If G is a non-cyclic finite group, then $\beta(G)<|G|$.*

Explicit calculations show that $\beta(S_3)=4$ and $\beta(S_4)\le 12$.

3. The Quotient Criterion and Applications

3.1. Properties of quotients. Let G be a linearly reductive group, X a G-variety and $\pi_X\colon X\to X/\!\!/G$ the quotient.

Proposition 3.1.1. *If X is irreducible, then $X/\!\!/G$ is likewise irreducible. If X is normal, then so is $X/\!\!/G$.*

Proof. The first statement is clear since π_X is surjective. For the second let $f\in\mathbb{C}(X/\!\!/G)$ be integral over $\mathcal{O}(X/\!\!/G)=\mathcal{O}(X)^G$. Then f is integral over $\mathcal{O}(X)$, and so $f\in\mathcal{O}(X)\cap\mathbb{C}(X/\!\!/G)\subseteq\mathcal{O}(X)\cap\mathbb{C}(X)^G=\mathcal{O}(X)^G$. $\square$

Remark 3.1.2. If X is *normal*, then the proof shows that $\mathcal{O}(X/\!\!/G)=\mathcal{O}(X)^G$ is *integrally closed* in $\mathbb{C}(X)^G$. It follows easily from this that $\mathbb{C}(X/\!\!/G)$ is *algebraically closed* in $\mathbb{C}(X)^G$.

But it is possible that $\mathbb{C}(X)^G\supsetneqq\mathbb{C}(X/\!\!/G)$ holds. Suppose, for example, that $X=\mathbb{C}^2$ and $G=\mathbb{C}^*$ with the action given by

$$t(x,y):=(t\cdot x,t\cdot y) \text{ for } t\in\mathbb{C}^* \text{ and } (x,y)\in\mathbb{C}^2.$$

Then $\mathcal{O}(X/\!\!/G)=\mathbb{C}$, while $f=x/y\in\mathbb{C}(X)^G$ is a non-constant invariant rational function.

Remark 3.1.3. If G is also *connected*, then $\mathcal{O}(X/\!\!/G)$ is *integrally closed* in $\mathcal{O}(X)$ and $\mathbb{C}(X/\!\!/G)$ is *algebraically closed* in $\mathbb{C}(X)$.
(In fact, consider the integrality equation resp. the minimal equation. This only has finitely many solutions and the solution set is stable under G. Since G is connected, G leaves every solution fixed.)

Proposition 3.1.4. *Suppose G is connected and has trivial character group. If V is a G-module and $\pi_V : V \to V/\!\!/G$ the quotient, then $\mathcal{O}(V/\!\!/G) = \mathcal{O}(V)^G$ is factorial and $\mathbb{C}(V/\!\!/G) = \mathbb{C}(V)^G$.*

Proof. Suppose $f \in \mathcal{O}(V)^G$ and $f = \prod_{i=1}^{s} f_i^{s_i}$ is the prime factorization of f in $\mathcal{O}(V)$. We are going to show that $f_i \in \mathcal{O}(V)^G$. One has $gf = \prod_{i=1}^{s}(gf_i)^{s_i}$ and thus $gf_i = \mu_i(g)f_j$ for some suitable j and $\mu_i(g) \in \mathbb{C}^*$. Furthermore, $G' := \{g \in G \mid gf_i \in \mathbb{C}^* f_i\}$ is a closed subgroup of G with finite index: $[G : G'] < \#\{\text{irreducible factors of } f\}$. Since G is connected, this implies $G = G'$ and so $gf_i \in \mathbb{C}^* f_i$. It follows that $\mu_i(gh) = \mu_i(g)\mu_i(h)$ for $g, h \in G$, i.e. μ_i is a character of G. Thus, by assumption, one must have $\mu_i(g) = 1$ for every $g \in G$, and so the prime factorization of f in $\mathcal{O}(V)$ yields a prime factorization of f in $\mathcal{O}(V)^G$. This proves the first claim. For the second, let $r \in \mathcal{O}(V)^G$ and write $r = \frac{f}{h}$ where $f, h \in \mathcal{O}(V)$ without common factor. Then, for all $g \in G$, $r = gr = \frac{gf}{gh}$, and so $gf \in \mathbb{C}^* f$ and $gh \in \mathbb{C}^* h$. As above, this implies that $g, h \in \mathcal{O}(V)^G$. ☐

Remark 3.1.5. Instead of requiring that V is a vector space, it is enough to assume that $\mathcal{O}(V)$ is *factorial* with group of units $\mathbb{C}^*$. (The second condition can also be eliminated.) Also it is not necessary that G is linearly reductive.

Remark 3.1.6. If X is an irreducible G-variety and $m := \max\{\dim Gx\}$ is the maximal orbit dimension, then the quotient $\pi \colon X \to X/\!\!/G$ satisfies

$$\dim X/\!\!/G \leq \dim X - m.$$

Under the additional assumption that $\mathbb{C}(X/\!\!/G) = \mathbb{C}(X)^G$ one has equality, and almost every fiber of π contains a dense orbit.
(We know that the set $\{x \in X \mid \dim Gx = m\}$ is open and dense in X (**??**) and so almost every fiber of the quotient map π_X contains an orbit of dimension m. Now the inequality follows from Theorem A.3.4.1. The proof of the second assertion is essentially more difficult (see **??**).

3.2. Some examples revisited. We will now have another look at some examples form the first chapter, using the concept of quotients introduced above.

Example 3.2.1. Suppose Q_n is the $\mathbb{C}$-vector space of *quadratic forms* in n variables with the standard action of SL_n (cf. I.3.1):

$$gq(x) = q(g^{-1}x) \text{ for } g \in \mathrm{SL}_n \text{ and } x \in \mathbb{C}^n.$$

Now Proposition I.3.3.1 shows that the *discriminant* $\Delta \colon Q_n \to \mathbb{C}$ is the quotient of Q_n by SL_n. We would like to look at this in a different way.

First of all Δ is an invariant, hence constant on the orbits. From the universal mapping property of the quotient $\pi \colon Q_n \to Q_n/\!\!/\mathrm{SL}_n$ we therefore get a commutative diagram

$$\begin{tikzcd} Q_n \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & Q_n/\!\!/\mathrm{SL}_n \arrow[d, "\delta"] \\ & \mathbb{C} \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} \mathcal{O}(Q_n) & \mathcal{O}(Q_n)^{\mathrm{SL}_n} \arrow[l, "\pi^*"'] \\ & \mathbb{C}[\Delta] \arrow[ul, "\varphi^*"] \arrow[u, "\delta^*"'] \end{tikzcd}$$

The map δ is surjective, and for $c \in \mathbb{C} \setminus \{0\}$ the set $\Delta^{-1}(c)$ is a closed orbit. Thus $\delta^{-1}(c) = \pi(\Delta^{-1}(c))$ is a point. Hence δ is birational, i.e. $\mathbb{C}(Y) = \mathbb{C}(\Delta)$. This implies that δ is an isomorphism (see the following exercise).

Exercise 3.2.2. Let C be an affine irreducible curve and $\delta \colon C \to \mathbb{C}$ a birational surjective morphism. Then δ is an isomorphism by Igusa's Lemma A.5.6.5.

Exercise 3.2.3. All fibers of the quotient $\Delta\colon Q_n \to \mathbb{C}$ are reduced, the null fiber $\Delta^{-1}(0)$ is normal and the other fibers are smooth.
(Hint: For the normality use the Serre Criterion A.5.7.1.)

The next proposition shows that the structure statement about quotients in the above example, namely the isomorphism $Q_n/\!\!/ \mathrm{SL}_n \xrightarrow{\sim} \mathbb{C}$, follows once one knows the dimensions of the orbits.

Proposition 3.2.4. *If G acts linearly on a vector space V and if G has an orbit of codimension ≤ 1, then the quotient $V/\!\!/G$ is either a point or is isomorphic to $\mathbb{C}$.*

Proof. By Proposition 3.1.1 the quotient $Y := V/\!\!/G$ is irreducible and normal, and Remark 3.1.6 shows that $\dim Y \leq \dim V - \max_{v\in V} \dim Gv \leq 1$. If $\dim Y = 0$, then Y is a point. Now suppose $\dim Y = 1$. Then Y has no singularities (Proposition A.5.6.1). Moreover, $\mathcal{O}(Y) = \mathcal{O}(V)^G = \bigoplus_{i\geq 0} \mathcal{O}(V)_i^G$ is graded with $\mathcal{O}(V)_0^G = \mathbb{C}$, and $\mathfrak{m} := \bigoplus_{i>0} \mathcal{O}(V)_i^G$ is the maximal ideal corresponding to $\pi(0) \in Y$. Now $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional and thus $\mathcal{O}(V)^G = \mathbb{C}[x]$ for a homogeneous $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ (Lemma 2.3.3). $\square$

Remark 3.2.5. The proof above shows that a one-dimensional quotient $V/\!\!/G$ is isomorphic to $\mathbb{C}$. But more is true. Assume that Z is a normal rational G-variety, G is reductive. If $\dim Z/\!\!/G = 1$ and $\mathcal{O}(Z)^* = \mathbb{C}^*$, then $Z/\!\!/G \simeq \mathbb{C}$.

Example 3.2.6. (See section I.4) The group GL_n acts on the $n\times n$-matrices M_n by conjugation. We consider the n symmetric functions in the eigenvalues $S_1, \ldots, S_n$ as functions on M_n. Because

$$(*) \qquad \det(tE - A) = t^n - S_1(A)t^{n-1} + \cdots + (-1)^n S_n(A),$$

the functions $S_1, \ldots, S_n$ are regular on M_n. We have seen in Proposition I.4.1.2 that the invariants $\mathcal{O}(\mathrm{M}_n)^{\mathrm{GL}_n}$ are generated by $S_1, \ldots, S_n$ and that the S_j are algebraically independent. Thus the morphism

$$S\colon M_n \to \mathbb{C}^n, \quad A \mapsto (S_1(A), \cdots, S_n(A))$$

is a quotient of M_n by GL_n. Like in the previous example we want to look at this in a slightly different way.

Let $\pi := \pi_{\mathrm{M}_n}\colon \mathrm{M}_n \to \mathrm{M}_n/\!\!/\mathrm{GL}_n$ be the quotient. Since the S_j are invariant functions the universal mapping property implies that we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{M}_n \arrow[r, "\pi"] \arrow[dr, "S"'] & \mathrm{M}_n/\!\!/\mathrm{GL}_n \arrow[d, "\bar{S}"] \\ & \mathbb{C}^n \end{tikzcd}$$

The matrix $A \in \mathrm{M}_n$ has n distinct eigenvalues if and only if the discriminant of the characteristic polynomial $(*)$ does not vanish, and in this case $S^{-1}(S(A))$ is the conjugacy class of A. This shows that on a dense open set $U \subseteq \mathbb{C}^n$ the fibers of S are orbits, and so $\bar{S}^{-1}(a)$ is a single point of $\mathrm{M}_n/\!\!/\mathrm{GL}_n$ for all $a \in U$. As a consequence, the morphism $\bar{S}$ is birational and surjective. We will see in the next section that this implies that $\bar{S}$ is an isomorphism (Quotient Criterion 3.4), hence $S\colon \mathrm{M}_n \to \mathbb{C}^n$ is the quotient.

3.3. Cosets and quotient groups. Suppose G is an algebraic group and $H \subseteq G$ is a *linearly reductive subgroup*. We have the two actions of H on G, namely by right and by left multiplication:

$$\lambda, \rho\colon H \times G \to G, \quad \lambda(h,g) := hg \quad \text{and} \quad \rho(h,g) := gh^{-1}.$$

Under both of these actions all orbits are closed and isomorphic to H. Thus the corresponding quotient is geometric (Definition 2.6.6). We will denote it in the following by

$$\pi = \pi_\lambda\colon G \to H\backslash G \quad \text{resp.} \quad \pi = \pi_\rho\colon G \to G/H,$$

and we talk as usual about the *right cosets* and the *left cosets*.

Left multiplication of G on itself induces an action of G on G/H, and the quotient map $\pi_\rho\colon G \to G/H$ is G-equivariant. An analogous statement holds for the right multiplication and the right coset space $H\backslash G$.

If H is in addition a normal subgroup, then G/H is an algebraic group with coordinate ring $\mathcal{O}(G/H) = \mathcal{O}(G)^H$. In fact, by the universal mapping property, the multiplication and the inverse define a multiplication and an inverse on G/H, so that G/H is an algebraic group (Proposition 2.4.6). It is the *quotient group* of G by H and has the usual universal property (cf. Proposition II.2.1.10).

Exercise 3.3.1. Let $H \subseteq G$ be a reductive subgroup of an algebraic group G. Show that there is an isomorphism of G-varieties $G/H \xrightarrow{\sim} H\backslash G$.

Exercise 3.3.2. Let G be reductive. Consider the action of G on $G \times G$ by left multiplication: $g(h_1, h_2) := (gh_1, gh_2)$. Then the quotient is given by $\pi\colon G \times G \to G$, $(h_1, h_2) \mapsto h_1^{-1}h_2$. What is the quotient if G acts by left multiplication on the product of n copies of G?

Exercise 3.3.3. Let X be a G-variety, and let $H \subseteq G$ be a closed normal reductive subgroup. If H acts trivially on X, then the induced action of G/H on X is regular.

Exercise 3.3.4. Let X be a G-variety and let $x \in X$ be a point whose stabilizer G_x is reductive. Then the orbit Gx is an affine variety and the orbit map induces an isomorphism $G/G_x \xrightarrow{\sim} Gx$.
(Hint: Use Zariski's Main Theorem A.5.6.7.)

Exercise 3.3.5. Let $T \subseteq \mathrm{GL}_2$ be the torus of diagonal matrices. Describe the quotients $T\backslash M_2/T$ and $T\backslash \mathrm{SL}_2/T$.

3.4. A criterion for quotients. Let X be an irreducible G-variety where G is linearly reductive. If $\varphi\colon X \to Y$ is an invariant morphism, i.e. φ is constant on orbits, we want to find a criterion which guarantees that φ is the quotient.

Proposition 3.4.1. *Assume that Y is normal and that φ is surjective. If there is a dense open set $U \subseteq Y$ such that for every $y \in U$ the fiber $\varphi^{-1}(y)$ contains a unique closed orbit, then φ is the quotient.*

Proof. The universal mapping property gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & X/\!\!/G \arrow[d, "\bar\varphi"] \\ & Y \end{tikzcd}$$

By assumption, $\bar\varphi$ is surjective. Moreover, $\bar\varphi^{-1}(y)$ is one point for every $y \in U$. Thus $\bar\varphi$ is surjective and has degree 1, hence is an isomorphism, because Y is normal (Igusa's Lemma A.5.6.5). $\square$

REMARK 3.4.2. The formulation of IGUSA's Lemma A.5.6.5 used in the proof above shows that the surjectivity of φ can be replaced by the assumption that $\operatorname{codim}_Y \overline{Y \setminus \varphi(X)} \geq 2$.

In order to apply the criterion above to a G-variety X we have to proceed as follows (cf. Examples 3.2.1 and 3.2.6 above).

Quotient Criterion

(1) *Find an invariant morphism $\varphi\colon X \to Y$ which is a candidate for the quotient.* (Quite often $\varphi\colon X \to \mathbb{C}^n$ is given by invariant functions and $Y \subseteq \mathbb{C}^n$ is the image of X.)
(2) *Show that* $\operatorname{codim}_Y \overline{Y \setminus \varphi(X)} \geq 2$.
(3) *Show that Y is normal.* (This might be difficult.)
(4) *Prove that, on an open dense set of Y, the fibers of φ contain a unique closed orbit (e.g. a dense orbit).*

EXAMPLE 3.4.3. Consider the space Q_n of quadratic forms with the linear action of the special orthogonal $\mathrm{SO}_n \subseteq \mathrm{GL}_n$ by substitution (I.3.3). We can identify Q_n with the symmetric matrices $\mathrm{Sym}_n \subseteq \mathrm{M}_n$ where the linear action of SO_n given by $A \mapsto gAg^t = gAg^{-1}$ (I.3.1). This shows that every invariant of M_n under *conjugation by* GL_n defines an O_n-invariant of Q_n. In particular, the quotient map $\pi_{\mathrm{M}_n}\colon \mathrm{M}_n \to \mathbb{C}^n$ of M_n by GL_n restricted to $Q_n = \mathrm{Sym}_n$ is SO_n-invariant.
We claim that the induced morphism

$$\pi_{Q_n}\colon Q_n \to \mathbb{C}^n, \quad q = q_A \mapsto (S_1(A), \ldots, S_n(A))$$

is the quotient of Q_n by SO_n (and by O_n).

PROOF. Since Sym_n contains the diagonal matrices, the map π_{Q_n} is surjective. We claim that a symmetric $n \times n$-matrix A with n distinct eigenvalues is conjugate, under SO_n, to a diagonal matrix. This implies that on an open dense set of $\mathbb{C}^n$ the fibers of π_{Q_n} are orbits and so π_{Q_n} is the quotient map by the Quotient Criterion.

In order to prove the claim, we first remark the two eigenvectors v, w of A with different eigenvalues $\lambda \neq \mu$ are orthogonal. In fact, $v^tAw = (v^tAw)^t = w^tAv$, and $v^tAw = \mu v^tw$ whereas $w^tAv = \lambda w^tv = \lambda v^tw$. As a consequence, at least one of the eigenvectors v_i of A is not isotropic, so we can assume that $v_1^tv_1 = 1$. Now choose a $g \in \mathrm{SO}_n$ such that $gv_1 = e_1$. Then $gAg^{-1} = gAg^t$ has the form

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix}$$

where A' is a symmetric $(n-1) \times (n-1)$-matrix with $n-1$ distinct eigenvalues. Now the claim follows by induction. $\square$

EXAMPLE 3.4.4. Here we look at pairs of quadratic forms, $Q_n \oplus Q_n$, with the diagonal linear action of SO_n given by $(q_A, q_B) \mapsto (gq_A, gq_B)$. Consider the homogeneous polynomial

$$f_{A,B}(s,t) := \det(sA + tB) = \sum_{i=0}^{n} f_i(A,B)s^it^{n-i}$$

which is invariant under the action $(A, B) \mapsto (gAg^t, gBg^t)$.
We claim that the morphism

$$\pi\colon Q_n \oplus Q_n \to \mathbb{C}^{n+1}, \quad (q_A, q_B) \mapsto (f_0(A,B), \ldots, f_n(A,B))$$

is the quotient of $Q_n \oplus Q_n$ by SL_n.

PROOF. Restricting π to pairs of diagonal matrices we see that π is surjective. Now consider the following open dense subset of $Q_n \oplus Q_n$:

$$U := \{(q_A, q_B) \mid q_B \text{ non-degenerate and } f_{A,B} \text{ without multiple factors}\}.$$

Any $(q_A, q_B) \in U$ is equivalent under SL_n to some $(q_{A'}, q_{cE})$, $c \neq 0$, and A' has n distinct eigenvalues, because $\chi_{A'}(t) = f_{A',cE}(-1, \frac{t}{c}) = f_{A,B}(-1, \frac{t}{c})$. The stabilizer of q_{cE} is O_n, and so $(q_{A'}, q_{cE})$ is equivalent to a pair (q_D, q_{cE}) where D is a diagonal matrix (see Example 3.4.3 above). Since $f_{D,cE}(s,t) = \prod_{i=1}^n (\lambda_i s + ct)$ we see that all pairs in the fiber through (q_A, q_B) are equivalent to (q_D, q_{cE}), and the claim follows from the quotient criterion. $\square$

4. The First Fundamental Theorem for GL_n

4.1. A Classical Problem. We consider the vector space $V = \mathbb{C}^n$ with the natural linear GL_n-action. For every pair r, s of natural numbers we get a representation of GL_n on the space

$$L_{r,s} := V^r \oplus (V^*)^s$$

with the contragradient representation on V^*: $(g\ell)(v) = \ell(g^{-1}v)$ for $\ell \in V^*$, $g \in \mathrm{GL}_n$ and $v \in V$.

Classical Problem: *Describe the invariant ring $\mathcal{O}(L_{r,s})^{\mathrm{GL}_n}$ by generators and relations.*

EXAMPLE 4.1.1. For $r = s = 1$ we have the map

$$\pi = \langle\ ,\ \rangle \colon V \oplus V^* \to \mathbb{C}, \quad (v, \ell) \mapsto \langle v, \ell \rangle := \ell(v).$$

Clearly, π is constant on the orbits: $\pi(g(v,\ell)) = \langle gv, g\ell \rangle = (g\ell)(gv) = \ell(g^{-1}gv) = \ell(v) = \pi(v,\ell)$. With the help of the Quotient Criterion (3.4) it is easy to see that π is a quotient. It follows that $\mathcal{O}(V \oplus V^*)^{\mathrm{GL}_n}$ is a polynomial ring in one variable:

$$\mathcal{O}(V \oplus V^*)^{\mathrm{GL}_n} = \mathbb{C}[\langle\ ,\ \rangle].$$

Next we would like to find a candidate for the quotient space $L_{r,s}/\!\!/ \mathrm{GL}_n$. To do this we give a "coordinate free" description of $L_{r,s}$. Suppose U, V, W are three finite dimensional vector spaces. Let

$$L := L(U,V) \times L(V,W)$$

where we have used the following notation:

$$\begin{aligned} L(U,V) &:= \mathrm{Hom}_{\mathbb{C}}(U,V),\\ L_p(U,V) &:= \{\rho \in L(U,V) \mid \operatorname{rk}\rho \leq p\},\\ L'_p(U,V) &:= \{\rho \in L(U,V) \mid \operatorname{rk}\rho = p\}.\end{aligned}$$

The group $G := \mathrm{GL}(V)$ acts linearly on L by

$$g(\alpha,\beta) := (g \circ \alpha, \beta \circ g^{-1}).$$

If one takes $U := \mathbb{C}^r$, $V := \mathbb{C}^n$ and $W := \mathbb{C}^s$, then it is obvious that L and $L_{r,s}$ are canonically $\mathrm{GL}(V)$-isomorphic.

Now we consider the following bilinear map.

$$\pi \colon L \to L(U,W), \quad (\alpha,\beta) \mapsto \beta \circ \alpha.$$

Clearly, π is constant on the orbits and

$$\pi(L) = L_t(U,W),\ t := \min(\dim U, \dim V, \dim W).$$

4.2. First Fundamental Theorem. For a complete proof of the following theorem we need a result from the fifth chapter. However, we can handle some special cases, see Proposition 4.3.1 below.

THEOREM 4.2.1 (First Fundamental Theorem for GL_n). *The mapping*

$$\pi : L(U,V) \times L(V,W) \to L_t(U,W), \quad (\alpha, \beta) \mapsto \beta \circ \alpha,$$

where $t := \min(\dim U, \dim V, \dim W)$*, is the quotient by* $\mathrm{GL}(V)$.

PROOF. By the Quotient Criterion (3.4) it suffices to show the following:

(i) The space $L_t(U,W)$ is normal. This will be proved in **??**, using the method of U-invariants. The irreducibility and a formula for the dimension of $L_t(U,W)$ is given in Lemma 4.2.2 below.
(ii) Every fiber of π contains exactly one closed orbit. This is the assertion of Corollary 5.5.2 below.

□

LEMMA 4.2.2. *The set* $L_p(U,W) := \{\rho \in L(U,W) \mid \operatorname{rk}\rho \le p\} \subseteq L(U,V)$ *is irreducible and closed and has dimension*

$$\dim L_p(U,W) = \begin{cases} \dim U \cdot \dim W & \text{for } p \ge m \\ (\dim U + \dim W - p)p & \text{for } p \le m \end{cases}$$

where $m := \min(\dim U, \dim W)$.

PROOF. Clearly, $L_p(U,W)$ is isomorphic to the set of all $\dim U \times \dim W$-matrices for which $(p+1)\times(p+1)$-minors vanish. This implies that $L_p(U,W)$ is closed in $L(U,W)$.

The group $H := \mathrm{GL}(U) \times \mathrm{GL}(W)$ acts on $L(U,W)$ by $(h,k)\rho := k \circ \rho \circ h^{-1}$. A standard result from linear algebra tells us that ρ and ρ' belong to the same H-orbit exactly when they have the same rank. The sets $L'_p(U,W)$, $p \le m$, are thus the orbits of H. One can easily see from this that

$$(*) \qquad \overline{L'_p(U,W)} = \bigcup_{i \le p} L'_i(U,W) = L_p(U,W).$$

Thus $L_p(U,W)$, as the closure of an orbit of the connected group H, is irreducible. Suppose $p \le m$ and let $U = U' \oplus U''$ be a splitting of U into a direct sum with $\dim U' = p$. We consider the surjective map $\mu\colon L_p(U,W) \to L(U',W)$, $\rho \mapsto \rho|_{U'}$ and determine its fibers over the dense subset $L'_p(U',W)$ of $L(U',W)$:

$$\begin{aligned} \mu^{-1}(\tau) &= \{\rho \in L(U,W) \mid \rho|_{U'} = \tau \text{ and } \rho(U'') \subseteq \tau(U')\} \\ &\simeq L(U'', \tau(U')). \end{aligned}$$

From the dimension formula (A.3.4.7) we now get

$$\begin{aligned} \dim L_p(U,W) &= \dim L(U',W) + \dim L(U'', \tau(U')) \\ &= \dim W \cdot p + (\dim U - p)p = (\dim U + \dim W - p)p. \end{aligned}$$

□

REMARK 4.2.3. The inclusions of the closures of the H-orbits in $L(U,V)$ is given by the following diagram, where $m = \min(\dim U, \dim W)$ as above, see $(*)$.

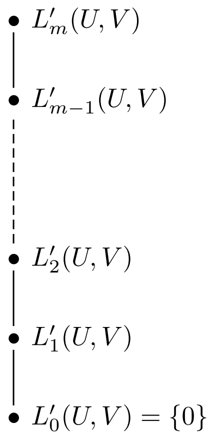


FIGURE 1. Degenerations of orbits in $L(U,V)$

4.3. A special case. Under certain additional assumptions on the dimensions of U, V and W we can now give a complete proof of the First Fundamental Theorem.

PROPOSITION 4.3.1. *If* $\dim V \geq \max(\dim U, \dim W)$*, then*

$$\pi : L(U,V) \times L(V,W) \to L(U,W)$$

is the quotient by $\mathrm{GL}(V)$*.*

PROOF. Clearly, π is surjective and $L(U,W)$ is normal.

(a) First suppose $U = V = W$ and let

$$\pi_0 \colon \mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V)$$

be the multiplication map. For $\rho \in \mathrm{GL}(V)$ one has

$$\pi_0^{-1}(\rho) = \{(\alpha, \beta) \mid \beta \circ \alpha = \rho\} = \{(g, \rho g^{-1}) \mid g \in \mathrm{GL}(V)\}.$$

This shows that over the open dense subset $\mathrm{GL}(V)$ of $\mathrm{End}(V)$ the fiber of π_0 consists of exactly one G-orbit, and the claim follows from the Quotient Criterion (3.4).

(b) If U and W are arbitrary with $\dim U, \dim W \leq \dim V$, then we choose a surjection $\tau \colon V \twoheadrightarrow U$ and an injection $\sigma \colon W \hookrightarrow V$. We thus have a commutative diagram

$$\begin{tikzcd}
L(U,V) \times L(V,W) \arrow[r, "\Phi", "\subseteq"'] \arrow[d, "\pi"] & \mathrm{End}(V) \times \mathrm{End}(V) \arrow[d, "\pi_0"] \\
L(U,W) \arrow[r, "\Psi", "\subseteq"'] & \mathrm{End}(V)
\end{tikzcd}$$

where the two injective linear maps Φ and Ψ are defined by

$$\Phi(\alpha, \beta) := (\alpha \circ \tau, \sigma \circ \beta), \quad \Psi(\rho) := \sigma \circ \rho \circ \tau.$$

Clearly, Φ is G-equivariant and hence identifies $L(U,V) \times L(V,W)$ with a G-stable closed subset of $\mathrm{End}(V) \times \mathrm{End}(V)$ whose image under π_0 is equal to $\Psi(L(U,W))$. The result now follows from the G-closedness of quotient maps (2.5). $\square$

4.4. Orbits in $L(U,V)$. In the rest of this section we study the fibers of π a little closer, in particular, their $\mathrm{GL}(V)$-structure and the question of irreducibility and normality. In the following lemma we present a few simple facts whose proofs are left to the reader.

LEMMA 4.4.1. *For* $\rho, \rho' \in L(U,W)$ *one has:*

(a) $\ker \rho = \ker \rho' \iff \exists k \in \mathrm{GL}(W)$ *such that* $\rho' = k \circ \rho$;
(b) $\operatorname{im} \rho = \operatorname{im} \rho' \iff \exists h \in \mathrm{GL}(U)$ *such that* $\rho' = \rho \circ h1$.

For $\rho \in L(U,W), \alpha \in L(U,V)$ *and* $\beta \in L(V,W)$ *one has:*

(c) $\ker \alpha \subseteq \ker \rho \iff \exists \beta' \in L(V,W)$ *such that* $\beta' \circ \alpha = \rho$;
d) $\operatorname{im} \beta \supseteq \operatorname{im} \alpha \iff \exists \alpha' \in L(U,V)$ *such that* $\beta \circ \alpha' = \rho$

Now we come to the description of the orbits in $L = L(U,V) \times L(V,W)$ and their closures.

Proposition 4.4.2. *Suppose* (α,β) *and* (α',β') *are in* $L := L(U,V)\times L(V,W)$. *Then*

(a) $(\alpha',\beta') \in \mathrm{GL}(V)(\alpha,\beta) \iff \beta'\circ\alpha' = \beta\circ\alpha$, $\ker\alpha' = \ker\alpha$ *and* $\operatorname{im}\beta' = \operatorname{im}\beta$.
(b) $(\alpha',\beta') \in \overline{\mathrm{GL}(V)(\alpha,\beta)} \iff \beta'\circ\alpha' = \beta\circ\alpha$, $\ker\alpha' \supseteq \ker\alpha$ *and* $\operatorname{im}\beta' \subseteq \operatorname{im}\beta$.
(c) $\mathrm{GL}(V)(\alpha,\beta)$ *is closed if and only if* $\ker\alpha = \ker(\beta\circ\alpha)$ *and* $\operatorname{im}\beta = \operatorname{im}(\beta\circ\alpha)$.

Proof. (a) The implication "$\Longrightarrow$" is clear. For the other direction we may assume that $\alpha' = \alpha$ (Lemma 4.4.1(a)). We consider the following decompositions

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3 = V_0 \oplus V_1 \oplus V_2' \oplus V_3'$$

where $\operatorname{im}\alpha = V_0 \oplus V_1$, $\ker\beta = V_1 \oplus V_2$, and $\ker\beta' = V_1 \oplus V_2'$. Then $\beta|_{V_0\oplus V_3}$ and $\beta'|_{V_0\oplus V_3'}$ are both injective with the same image $\beta(V) = \beta'(V)$. By Lemma 4.4.1(b) there is thus an isomorphism

$$\sigma\colon V_0 \oplus V_3 \xrightarrow{\sim} V_0 \oplus V_3'$$

such that $(\beta'|_{V_0\oplus V_3'})\circ\sigma = \beta|_{V_0\oplus V_3}$. Since β and β' agree on V_0, it follows that $\sigma|_{V_0} = \mathrm{id}_{V_0}$. If we now choose any isomorphism $\tau\colon V_2 \xrightarrow{\sim} V_2'$, then we get an automorphism $h\colon V \xrightarrow{\sim} V$, defined by

$$h(v_0,v_1,v_2,v_3) := (v_0,v_1,\tau(v_2),\sigma(v_3)) \quad (v_i \in V_i),$$

which, by construction, gives us what we want, namely

$$h|_{\operatorname{im}\alpha} = \mathrm{id}_{\operatorname{im}\alpha} \ \text{ and } \beta = \beta' \circ h.$$

(b) Again the implication "$\Longrightarrow$" is clear, because $\alpha' \in \overline{\mathrm{GL}(V)\circ\alpha}$ and $\beta' \in \overline{\beta\circ\mathrm{GL}(V)}$ and thus $\ker\alpha' \supseteq \ker\alpha$ and $\operatorname{im}\beta' \subseteq \operatorname{im}\beta$. Therefore, for a fixed ρ, the set $\{(\alpha,\beta) \mid \beta\circ\alpha = \rho\}$ is a fiber of π and so it is closed.

For the other direction let $\rho := \beta\circ\alpha = \beta'\circ\alpha'$. There are decompositions

$$U = U_0 \oplus U_1 \oplus \ker\alpha \quad \text{and} \quad W = W_0 \oplus W_1 \oplus \operatorname{im}\beta'$$

where

$$U_1 \oplus \ker\alpha = \ker\alpha' \quad \text{and} \quad W_1 \oplus \operatorname{im}\beta' = \operatorname{im}\beta.$$

Because $\ker\alpha' \subseteq \ker\rho$ and $\operatorname{im}\rho \subseteq \operatorname{im}\beta'$ the following diagram is commutative for all $\varepsilon \in \mathbb{C}$:

$$\begin{array}{ccc}
U_0 \oplus U_1 \oplus \ker\alpha & \xrightarrow{\ \rho\ } & W_0 \oplus W_1 \oplus \operatorname{im}\beta' \\
{\scriptstyle \tau_\varepsilon := }\big\uparrow{\scriptstyle (\mathrm{id},\varepsilon\cdot\mathrm{id},\mathrm{id})} & & {\scriptstyle \sigma_\varepsilon := }\big\downarrow{\scriptstyle (\mathrm{id},\varepsilon\cdot\mathrm{id},\mathrm{id})} \\
U_0 \oplus U_1 \oplus \ker\alpha & \xrightarrow{\ \rho\ } & W_0 \oplus W_1 \oplus \operatorname{im}\beta'
\end{array}$$

Thus we have

$$\rho = \sigma_\varepsilon \circ \rho \circ \tau_\varepsilon = (\sigma_\varepsilon\circ\beta)\circ(\alpha\circ\tau_\varepsilon) = \beta_\varepsilon \circ \alpha_\varepsilon,$$

where $\alpha_\varepsilon := \alpha\circ\tau_\varepsilon$ and $\beta_\varepsilon := \sigma_\varepsilon\circ\beta$. For $\varepsilon \neq 0$ one clearly has $\ker\alpha_\varepsilon = \ker\alpha$ and $\operatorname{im}\beta_\varepsilon = \operatorname{im}\beta$. By (a) it thus follows that $(\alpha_\varepsilon,\beta_\varepsilon) \in \mathrm{GL}(V)(\alpha,\beta)$ for every $\varepsilon \neq 0$. Hence $(\alpha_0,\beta_0) \in \overline{\mathrm{GL}(V)(\alpha,\beta)}$. Because $\ker\alpha_0 = \ker\alpha'$ and $\operatorname{im}\beta_0 = \operatorname{im}\beta'$ it again follows from (a) that $(\alpha',\beta') \in \mathrm{GL}(V)(\alpha_0,\beta_0)$ and the result is clear.

(c) Suppose $\rho := \beta \circ \alpha, \ker \alpha = \ker \rho$ and $\operatorname{im} \beta = \operatorname{im} \rho$. For some $(\alpha', \beta') \in \overline{\mathrm{GL}(V)(\alpha, \beta)}$ we get by (b) the inclusions $\ker \alpha' \supseteq \ker \rho$ and $\operatorname{im} \beta' \subseteq \operatorname{im} \rho$. On the other hand, since $\beta' \circ \alpha' = \rho$ we have that $\ker \rho \supseteq \ker \alpha'$ and $\operatorname{im} \rho \subseteq \operatorname{im} \beta'$. From (a) it therefore follows that $(\alpha', \beta') \in \mathrm{GL}(V)(\alpha, \beta)$ and thus $\mathrm{GL}(V)(\alpha, \beta)$ is closed.

Now suppose conversely that $\mathrm{GL}(V)(\alpha, \beta)$ is closed. Clearly, one always has a decomposition $\rho = \beta' \circ \alpha'$ with $\ker \alpha' = \ker \rho$ and $\operatorname{im} \beta' = \operatorname{im} \rho$. From (b) it follows that $(\alpha', \beta') \in \overline{\mathrm{GL}(V)(\alpha, \beta)} = \mathrm{GL}(V)(\alpha, \beta)$. Thus $\ker \alpha = \ker \rho$ and $\operatorname{im} \beta = \operatorname{im} \rho$. □

The proofs of the following two corollaries are left as an exercise.

Corollary 4.4.3. *There is exactly one closed orbit in the fiber $\pi^{-1}(\rho)$, namely* $\mathrm{GL}(V)(\alpha_0, \beta_0)$ *where* $\rho = \beta_0 \circ \alpha_0$, $\ker \rho = \ker \alpha_0$, *and* $\operatorname{im} \rho = \operatorname{im} \beta_0$.

Corollary 4.4.4. *The orbit of (α, β) is closed if and only if $V = \operatorname{im} \alpha \oplus \ker \beta$.*

This is fulfilled, for example, *if α is surjective and β is injective.*

4.5. Degenerations of orbits. For a vector space M over $\mathbb{C}$ we now define the *Grassmann manifold* also called *Grassmannian.*

$$\mathrm{Grass}_d(M) := \text{ set of all subspaces of } M \text{ of dimension } d,$$

and

$$\mathrm{Grass}(M) := \text{ set of all subspaces of } M = \bigcup_{d=0}^{\dim M} \mathrm{Grass}_d(M).$$

If $\rho \in \pi(L)$ and $F_\rho := \pi^{-1}(\rho)$ is the fiber of ρ, then we consider the map

$$\Phi\colon F_\rho \to \mathrm{Grass}(\ker \rho) \times \mathrm{Grass}(W/\operatorname{im} \rho), \quad (\alpha, \beta) \mapsto (\ker \alpha, \operatorname{im} \beta / \operatorname{im} \rho).$$

By Proposition 4.4.2(a), the fibers of Φ are exactly the $\mathrm{GL}(V)$-orbits in F_ρ.

Lemma 4.5.1. *The image of Φ consists exactly of those pairs (U_0, W_0) which satiesfy*

$$\text{()} \qquad \operatorname{codim}_\rho U_0 + \dim W_0 \le \dim V - \operatorname{rk} \rho.$$

(Here $\operatorname{codim}_\rho U_0 := \dim \ker \rho - \dim U_0$.)

Proof. Suppose $\rho := \beta \circ \alpha$, $U_0 = \ker \alpha$ and $W_0 = \operatorname{im} \beta / \operatorname{im} \rho$. Since $\alpha(\ker \rho) \subseteq \ker \beta$ it follows that $\operatorname{codim}_\rho U_0 \le \dim \ker \beta$ and thus

$$\operatorname{codim}_\rho U_0 + \dim W_0 \le \dim \ker \beta + \dim \operatorname{im} \beta - \operatorname{rk} \rho = \dim V - \operatorname{rk} \rho.$$

This proves (1).

Conversely suppose $U_0 \subseteq \ker \rho$ and $W_0 \subseteq W/\operatorname{im} \rho$ are given by (1). Let $\tilde{W}_0 \subseteq W$ be the preimage of W_0. Then ρ can be factored as follows:

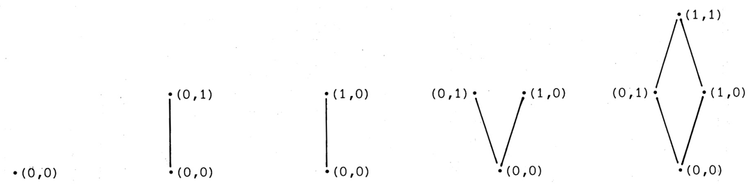
$$U \longrightarrow U/U_0 \xrightarrow{\tilde{\rho}} \tilde{W}_0 \longrightarrow W$$

We have to show that there is a injection $\tilde{\alpha}\colon U/U_0 \to V$ and a surjection $\tilde{\beta}\colon V \to \tilde{W}_0$ with $\tilde{\rho} = \tilde{\beta} \circ \tilde{\alpha}$. Such a pair $(\tilde{\alpha}, \tilde{\beta})$ obviously exists if and only if

$$\dim V \ge \dim(U/U_0 + \dim(\tilde{W}_0 / \operatorname{im} \tilde{\rho}).$$

But the right hand side of this inequality is equal to $\operatorname{rk} \rho + \operatorname{codim}_\rho U_0 + \dim W_0$, and the claim follows. □

Corollary 4.5.2. *The fiber F_ρ is a closed orbit under $\mathrm{GL}(V)$ exactly if ρ is bijective or if* $\operatorname{rk} \rho = \dim V$.

Anzahl der Bahnen in F_ρ	1	2	2	3	4
Inklusions-diagramme					
Bedingungen	rg ρ = dim V oder ρ bijektiv	rg ρ < dim V ρ injektiv codim Im ρ = 1	ρ surjektiv dim Ker ρ = 1	rg ρ = dim V − 1 codim Im ρ = 1 = dim Ker ρ	rg ρ ≤ dim V − 2

Die Fasern F_ρ mit endlich vielen Bahnen

TABLE 1. Fibers of ρ with finitely many H_ρ-orbits

PROOF. If $\dim V = \operatorname{rk}\rho$ and $\rho = \beta \circ \alpha$, then α must be surjective and β injective, and the result follows from Corollary 4.4.4. If $\rho = \beta \circ \alpha$ is bijective, then we get $\ker\alpha = (0) = \ker\rho$ and $\operatorname{im}\beta = W = \operatorname{im}\rho$, and the result follows from Proposition 4.4.2(c).

Conversely if F_ρ consists of exactly one orbit and if $\dim V > \operatorname{rk}\rho$, then it follows from Lemma 4.5.2 above that $\ker\rho = (0)$ and $\operatorname{im}\rho = W$. □

COROLLARY 4.5.3. *The fiber F_ρ consists of finitely many orbits under the action of* $\mathrm{GL}(V)$ *if and only if either* $\operatorname{rk}\rho = \dim V$ *or* $\dim\ker\rho$ *and* $\operatorname{codim}_W \operatorname{im}\rho \le 1$ *hold.*

(This follows easily from Lemma 4.5.1 and the fact that $\operatorname{Grass}(M)$ is finite precisely when $\dim M \le 1$.)

REMARK 4.5.4. On $\operatorname{Grass}(\ker\rho) \times \operatorname{Grass}(W/\operatorname{im}\rho)$ we consider the ordering $\prec$ given by

$$(U_0, W_0) \prec (U_1, W_1) \iff U_0 \supseteq U_1 \text{ and } W_0 \subseteq W_1.$$

Then, by Proposition 4.4.2, one has for $(\alpha_0,\beta_0),(\alpha_1,\beta_1) \in F_\rho$

$$(\alpha_0,\beta_0) \in \overline{\mathrm{GL}(V)(\alpha_1,\beta_1)} \iff \Phi(\alpha_0,\beta_0) \prec \Phi(\alpha_1,\beta_1).$$

If we denote by $F_\rho/\mathrm{GL}(V)$ the set of orbits in F_ρ, along with the ordering given by the closure of the orbits, then the mapping

$$\Phi\colon F_\rho \to \operatorname{Grass}(\ker\rho) \times \operatorname{Grass}(W/\operatorname{im}\rho)$$

induces an *order preserving isomorphism*

$$F_\rho/\mathrm{GL}(V) \xrightarrow{\sim} \Phi(F_\rho) = \{(U,W) \mid \operatorname{codim}_\rho U + \dim W \le \dim V - \operatorname{rk}\rho\}.$$

In Table 1 we present the various cases for which F_ρ only has a finite number of orbits under H_ρ. To do this we have provided the individual orbits in the inclusion diagram with a pair of integers (n,m), which are defined by the following mapping:

$$\theta\colon F_\rho \to \mathbb{N}\times\mathbb{N}, \quad (\alpha,\beta) \mapsto (\operatorname{codim}_\rho \ker\alpha, \operatorname{rk}\beta - \operatorname{rk}\rho).$$

One has

$$\theta(\alpha,\beta) = (\operatorname{codim}_\rho U_0, \dim W_0)$$

where $(U_0, W_0) = \Phi(\alpha, \beta)$, and by Lemma 4.5.1

$$\theta(F_\rho) := \{(n, m) \mid n \le \dim \ker \rho, m \le \dim W - \operatorname{rk} \rho, n + m \le \dim V - \operatorname{rk} \rho\}.$$

Thus the set of orbits in $\theta^{-1}(n, m) \subseteq F_\rho$ for a fixed (n, m) is parametrized by $\mathrm{Grass}_{\dim \ker \rho - n}(\ker \rho) \times \mathrm{Grass}_m(W/\operatorname{im} \rho)$.

4.6. The subgroup H_ρ. Now we want to see that these subsets correspond to the orbits of a particular subgroup H_ρ of $\mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$. To do this we choose decompositions $U = \ker \rho \oplus U_1$, $W = \operatorname{im} \rho \oplus W_1$ and set

$$H_\rho := \mathrm{GL}(\ker \rho) \times \mathrm{GL}(V) \times \mathrm{GL}(W_1) \subseteq \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W).$$

(Every automorphism of $\ker \rho$ (resp. of W_1) is extended to all of U (resp. to all of W) by defining it to be the identity map on U_1 (resp. on $\operatorname{im} \rho$).)

This group H_ρ acts linearly on $L = L(U, V) \times L(V, W)$ by

$$(h, g, k)(\alpha, \beta) := (g \circ \alpha \circ h^{-1}, k \circ \beta \circ g^{-1}).$$

This action coincides on the subgroup $\mathrm{GL}(V) \subseteq H_\rho$ with the given action of $\mathrm{GL}(V)$ on L. Since H_ρ induces the identity on U_1 and on $\operatorname{im} \rho$, the set F_ρ is stable under H_ρ.

Proposition 4.6.1. *The mapping $\theta\colon F_\rho \to \mathbb{N} \times \mathbb{N}$ induces a bijection between F_ρ/H_ρ, the set of H_ρ-orbits in F_ρ, and its image $N_\rho := \theta(F_\rho)$. Moreover, θ is order preserving, i.e. one has $(\alpha', \beta') \in \overline{H_\rho(\alpha, \beta)}$ if and only if $\theta(\alpha', \beta') \prec \theta(\alpha, \beta)$.*

(As before, $(n', m') \prec (n, m)$ if $n' \le n$ and $m' \le m$.)

Proof. We have to show that H_ρ acts transitively on $\theta^{-1}(n, m)$. Now $\theta\colon F_\rho \to \mathbb{N} \times \mathbb{N}$ is the composition

$$\theta = \bar\theta \circ \Phi\colon\ F_\rho \xrightarrow{\ \Phi\ } \mathrm{Grass}(\ker \rho) \times \mathrm{Grass}(W/\operatorname{im} \rho) \xrightarrow{\ \bar\theta\ } \mathbb{N} \times \mathbb{N},$$

where

$$\bar\theta(U_0, W_0) := (\operatorname{codim}_\rho U_0, \dim W_0).$$

Clearly $\mathrm{GL}(\ker \rho) \times \mathrm{GL}(W_1)$ acts transitively on

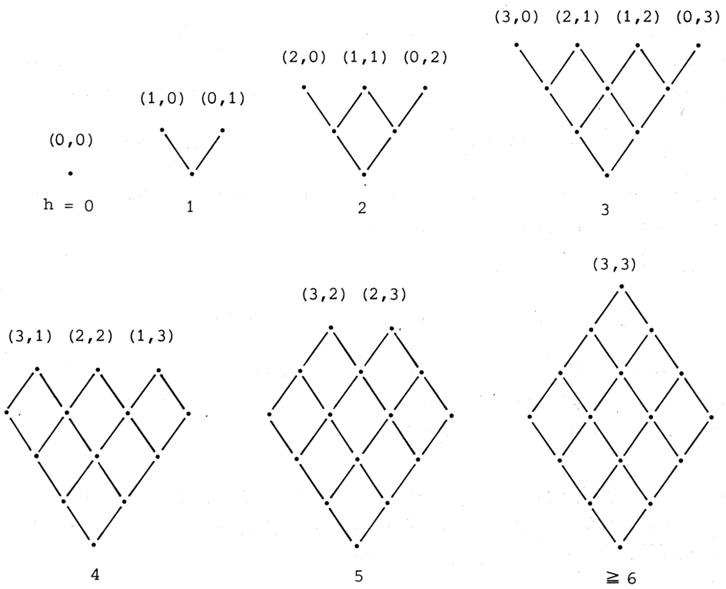
$$\bar\theta^{-1}(m, n) = \{(U_0, W_0) \mid \dim U_0 = \dim \ker \rho - n, \dim W_0 = m\},$$

and the first claim follows.

The second assertion follows easily from Proposition 4.4.2. $\square$

As a consequence we see that the fiber $F = F_\rho$ contains only finitely many H_ρ-orbits. The inclusion diagram for the closures of the H_ρ-orbits is given by the set $N_\rho \subseteq \mathbb{N} \times \mathbb{N}$ with the product ordering on $\mathbb{N} \times \mathbb{N}$ which was just defined.

Example 4.6.2. Suppose $\dim \ker \rho = 3 = \dim W - \operatorname{rk} \rho$. Then we get the following inclusion diagrams of the closure of the H_ρ-orbits in F_ρ (depending on the quantity $h := \dim V - \operatorname{rk} \rho$):

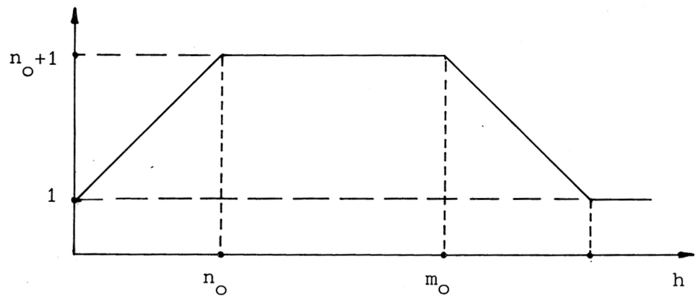


In particular, in this example F_ρ is irreducible if either $\dim V = \operatorname{rk}\rho$ or $\dim V \geq \operatorname{rk}\rho + 6$.

EXERCISE 4.6.3. (1) The number of irreducible components of F_ρ is given by

$$\max(\min(h+1, n_0+1, m_0+1, n_0+m_0-h+1), 1)$$

where $h := \dim V - \operatorname{rk}\rho$, $n_0 := \dim\ker\rho$ and $m_0 = \dim W - \operatorname{rk}\rho$:



(2) For the zero fiber F_0 one has:
   (i) F_0 is irreducible $\iff \dim U + \dim W \leq \dim V$.
   (ii) Suppose $m := \min(\dim U, \dim W) \leq M := \max(\dim U, \dim W)$. Then

$$\#\text{ irreducible components} = \begin{cases} \dim V + 1 & \text{if } m \geq \dim V, \\ m+1 & \text{if } m \leq \dim V \leq M, \\ \max(M + m - \dim V + 1, 1) & \text{if } M \leq \dim V. \end{cases}$$

4.7. Structure of the fiber F_ρ.

PROPOSITION 4.7.1. *The fiber F_ρ is irreducible if and only if one of the following conditions is fulfilled:*

(a) $\operatorname{rk}\rho \geq \dim U + \dim W - \dim V$*,*
(b) $\operatorname{rk}\rho = \dim V$*,*
(c) ρ *is injective or surjective.*

PROOF. Clearly F_ρ is irreducible exactly if N_ρ has a largest element (Proposition 4.6.1). Set

$$h := \dim V - \operatorname{rk}\rho, \quad n_0 := \dim\ker\rho, \quad m_0 := \dim W - \operatorname{rk}\rho.$$

Then $N_\rho = \{(n,m) \le (n_0,m_0) \mid n+m \le h\}$. Thus N_ρ has a largest element if and only if one of the following cases occurs:

(a) $n_0 + m_0 \le h$; then the largest element is (n_0, m_0).
(b) $h = 0$; then the largest element is $(0,0)$.
(c) $n_0 = 0$ resp. $m_0 = 0$; then the largest element is $(d,0)$ resp. $(0,d)$ with $d = \min(n_0,h)$ resp. $d = \min(m_0,h)$.

These three cases correspond exactly to the three cases given in the statement of the proposition. For (a) one should note the relation

$$n_0 + m_0 = \dim\ker\rho + \dim W - \operatorname{rk}\rho = \dim U + \dim W - 2\operatorname{rk}\rho.$$

□

REMARK 4.7.2. Recall that in case (b) the fiber F_ρ is a closed orbit (Corollary 4.5.2).

COROLLARY 4.7.3. *The fibers of π are irreducible on the open, dense subset $L'_t(U,W)$ of $\pi(L) = L_t(U,W)$ where $t := \min(\dim U, \dim V, \dim W)$.*

PROOF. Suppose $\rho \in L'_t(U,W)$, i.e. $\operatorname{rk}\rho = t$. We distinguish three cases.

(1) $\max(\dim U, \dim W) \le \dim V$. This implies $\operatorname{rk}\rho = t \ge \dim U + \dim W - \dim V$ and F_ρ is irreducible by Proposition 4.7.1(a).
(2) $\dim V \le \min(\dim U, \dim W)$. This implies $\operatorname{rk}\rho = \dim V$, and F_ρ is irreducible by Proposition 4.7.1(b).
(3) $\dim U \le \dim V \le \dim W$ resp. $\dim U \ge \dim V \ge \dim W$. This implies $\operatorname{rk}\rho = \dim U$ resp. $\operatorname{rk}\rho = \dim W$, and so ρ is injective (resp. surjective). By Proposition 4.7.1(c) the set F_ρ is irreducible.

□

It remains the question if these fibers are normal or even smooth. A first answer is the following.

PROPOSITION 4.7.4. *If*

$$\operatorname{rk}\rho \ge \dim U + \dim W - \dim V,$$

then the fiber F_ρ is a normal complete intersection (AI.5.7) of dimension

$$\dim F_\rho = (\dim U + \dim W)\cdot\dim V - \dim U \cdot \dim W.$$

For the proof we want to use the normality criterion of SERRE (see AI.5.7.5) and thus we must determine the points $(\alpha,\beta) \in L$ where the tangent map

$$d\pi_{(\alpha,\beta)} \colon L \to L(U,W), \quad (X,Y) \mapsto (\beta\circ X + Y \circ \alpha)$$

is surjective. (As usual we have set $T_{(\alpha,\beta)}(L) = L$ and $T_\rho(L(U,W)) = L(U,W)$, where $\rho = \beta\circ\alpha$. Then

$$\pi(\alpha+\varepsilon X, \beta+\varepsilon Y) = (\beta+\varepsilon Y)\circ(\alpha+\varepsilon X) = \beta\circ\alpha + \varepsilon(\beta\circ X + Y\circ\alpha),$$

and so $d\pi_{(\alpha,\beta)}(X,Y) = \beta\circ X + Y \circ\alpha$.)

LEMMA 4.7.5. *The differential $d\pi_{(\alpha,\beta)} \colon L \to L(U,W)$ is surjective if and only if α is injective or β is surjective.*

PROOF. We set $\delta := d\pi_{(\alpha,\beta)}$ and thus $\delta(X,Y) = \beta\circ X + Y\circ\alpha$. If α is injective (resp. β is surjective), then $\dim U \le \dim V$ (resp. $\dim V \ge \dim W$) and every homomorphism in $L(U,W)$ factors through α (resp. β). On the other hand, for every $\sigma \in \delta(L)$ one clearly has $\sigma(\ker\alpha) \subseteq \operatorname{im}\beta$. Thus, if δ is surjective, then one must either have $\ker\alpha = (0)$ or $\operatorname{im}\beta = W$. □

Proof of Proposition 4.7.4. By the lemma above and Serre's Criterion (Proposition AI.5.7.5) it suffices to show that

$$F'_\rho := \{(\alpha, \beta) \in F_\rho \mid \alpha \text{ is injective or } \beta \text{ surjective}\} \subseteq F_\rho$$

has a complement of codimension ≥ 2. Set $n_0 := \dim \ker \rho$, $m_0 := \dim W - \operatorname{rk} \rho$. By assumption, the subset

$$N_\rho = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid n \leq n_0 \text{ and } m \leq m_0\} \subseteq \mathbb{N} \times \mathbb{N}$$

has the form given in Figure 2 below where $O := \theta^{-1}(n_0, m_0)$, $O_1 = \theta^{-1}(n_0 - 1, m_0)$ and $O_2 = \theta^{-1}(n_0, m_0 - 1)$ are H_ρ-orbits, see Proposition 4.7.1(a).

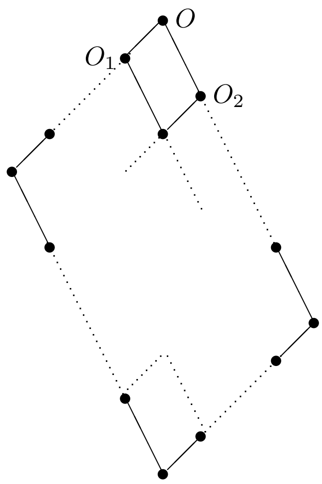


Figure 2. H_ρ-orbits in the fiber F_ρ

By the lemma above we have $(\alpha, \beta) \in F'_\rho$ if and only if $\theta(\alpha, \beta)$ is either of the form (n_0, m) or (n, m_0). In particular,

$$F'_\rho \supseteq (O \cup O_1 \cup O_2) \quad \text{and} \quad F_\rho \setminus O = \overline{O_1} \cup \overline{O_2}.$$

This implies

$$\dim(\overline{F_\rho \setminus F'_\rho}) \leq \dim\left(F_\rho \setminus (O \cup O_1 \cup O_2)\right) < \dim(F_\rho \setminus O) < \dim F_\rho,$$

and thus $\operatorname{codim}_{F_\rho}(\overline{F_\rho \setminus F'_\rho}) \geq 2$. □

Remark 4.7.6. *All irreducible fibers of π are normal.* In fact, if ρ satisfies (b) or (c) of Proposition 4.7.1, then F_ρ is even *smooth*.
(In case (b) the set F_ρ is a $\mathrm{GL}(V)$-orbit and in case (c) one has $F_\rho = F'_\rho$.)

5. Sheets, General Fiber and Null Fiber

We consider a linear representation $\rho\colon G \to \mathrm{GL}(V)$ of a reductive group G and denote by $\pi\colon V \to Y = V/\!\!/G$ the quotient of V by G. In this section we present a few connections between the geometry of the *null fiber* (also called the *null cone*)

$$\mathcal{N} = \mathcal{N}_V := \pi^{-1}(\pi(0))$$

and the geometry of a *general fiber* of π. It will be shown that the null fiber in a certain sense is the "worst" of all the fibers, or otherwise stated, the "good" properties of the null fiber also occur in all the other fibers. In order to study this transformation of a general fiber into the null fiber we first introduce the concept of sheets.

5.1. Sheets. Suppose G is an algebraic group and Z is a G-variety. We consider the union of the G-orbits of a fixed dimension $n \in \mathbb{N}$:

$$Z^{(n)} := \{z \in Z \mid \dim Gz = n\}.$$

These sets are obviously G-stable subsets of Z.

PROPOSITION 5.1.1. *The subsets $Z^{(n)} \subseteq Z$ are locally closed and G-stable. The subset $Z^{\max}$ of orbits of maximal dimension is open in Z.*

PROOF. It is enough to show that for every $n \in \mathbb{N}$ the subset $\{z \in Z \mid \dim Gz > n\}$ is open in Z. This follows from the dimension formula $\dim Gz + \dim G_z = \dim G$ (see III.1.3) with the help of the following lemma. $\square$

LEMMA 5.1.2. *The function $z \mapsto \dim G_z$ is upper semi-continuous, i.e., for every $n \in \mathbb{N}$ the set $\{z \in Z \mid \dim G_z < n\}$ is an open subset of Z.*

PROOF. Since we can embed Z equivariantly into a vector space with a linear action of G (Corollary III.2.3.5) it suffices to prove the lemma for $Z = V$. Now $\operatorname{Lie} G$ also acts on V (see III.5.3), and for $v \in V$ one has

$$\operatorname{Lie} G_v = (\operatorname{Lie} G)_v := \{X \in \operatorname{Lie} G \mid Xv = 0\}$$

(Proposition III.5.3.2). We consider the linear mapping

$$V \to \operatorname{Hom}_{\mathbb{C}}(\operatorname{Lie} G, V), \quad v \mapsto s_v,$$

where $s_v \colon \operatorname{Lie} G \to V$ is given by $X \mapsto Xv$. It follows that $(\operatorname{Lie} G)_v = \ker s_v$ for every $v \in V$, and we have to show that $v \mapsto \dim(\ker s_v)$ is upper semi-continuous. But this is a well-known fact from linear algebra. $\square$

DEFINITION 5.1.3. Suppose G is connected. Then the irreducible components of $Z^{(n)}$ are called the *sheets* of Z. Thus the sheets are *locally closed, irreducible G-stable subsets of Z.*

The notion of sheets arose in the study of conjugacy classes in Lie algebras and goes back to DIXMIER (cf. the original literature [?], [?], [?]). If we consider the classical case of conjugacy classes of matrices (i.e. the operation of GL_n on M_n by conjugation), then one can prove the following.

(a) *The sheets of M_n are pairwise disjoint.*
(b) *Every sheet S contains semi-simple conjugacy classes and exactly one nilpotent conjugacy class.*
(c) *The sheets of M_n are smooth.*

None of these claims is true, in general. For instance, if we consider the adjoint representation of a classical group $G = \mathrm{SO}_n$ or $G = \mathrm{Sp}_n$ on its Lie algebra g, then the sheets in g are not disjoint. Moreover, there are strata which are made up of only one nilpotent conjugacy class, namely the sheets of minimal dimension. And singular sheets occur, for example, in the Lie algebra of the exceptional group G_2. But it was shown by ANDREAS IM HOF that the sheets in the classical Lie algebras are smooth .Compare this with the investigation in [?].

EXAMPLE 5.1.4. **Sheets in the Lie algebra of SL_3.**
Let $G = \mathrm{SL}_3$ act on its Lie algebra

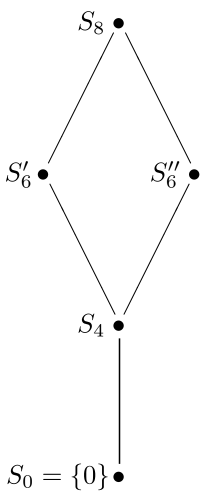
$$\mathfrak{sl}_3 = \{X \in \mathrm{M}_3 \mid \operatorname{tr} X = 0\},$$

by conjugation. (This example was studied in detail in **??**.) Then $\mathfrak{sl}_3$ consists of three *disjoint sheets* having orbit dimensions 0, 4 and 6:

$$\mathfrak{sl}_3 = S_0 \cup S_4 \cup S_6,$$

where S_0 is just the zero matrix, S_4 consists of the semi-simple matrices which have an eigenvalue $\lambda \neq 0$ of multiplicity together with the conjugacy class of $\left[\begin{smallmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{smallmatrix}\right]$ while S_6 contains the rest. Thus it is made up of the semi-simple matrices with three different eigenvalues, the conjugacy classes of the matrices $\left[\begin{smallmatrix} \lambda&1&0\\ 0&\lambda&0\\ 0&0&-2\lambda \end{smallmatrix}\right]$ with $\lambda \neq 0$, and the conjugacy class of $\left[\begin{smallmatrix} 0&1&0\\ 0&0&1\\ 0&0&0 \end{smallmatrix}\right]$. Even in this rather simple example it is not obvious that the sheet S_4 has no singularities!

Example 5.1.5. **Sheets in the Lie algebra of Sp_4.**
The group Sp_4 acts by conjugation on its Lie algebra $\mathfrak{sp}_4$, and $\mathfrak{sp}_4$ has conjugacy classes of dimensions 8, 6, 4 and 0. One gets one sheet $S_8 = \mathfrak{sp}_4^{(8)} = \mathfrak{sp}_4^{\max}$ of dimension 10, the so–called *regular or maximal sheet* (= the sheet of maximal orbit dimension). The closure of S_8 contains two sheets S_6' and S_6'' with orbit dimension 6 which are the so-called *subregular sheets*. These are each 7–dimensional, and $S_6' \cap S_6''$ consists of the nilpotent conjugacy class of dimension 6 with partition $(2, 2)$. The sheet S_4 lies in the closure of S_6' and S_6''; this only contains the nilpotent conjugacy class with partition $(2, 1, 1)$, see Figure 3.



	dimensions of sheet	dimensions of conjucacy classes	nilpotent conjugacy classes in the sheet given by
S_8	10	8	$\left[\begin{smallmatrix} 0&1&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&-1&0 \end{smallmatrix}\right]$
S_6', S_6''	7	6	$\left[\begin{smallmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&-1&0 \end{smallmatrix}\right]$
S_4	4	4	$\left[\begin{smallmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{smallmatrix}\right]$
$S_0 = \{0\}$	0	0	

Figure 3. The sheets in the Lie algebra of Sp_4

Example 5.1.6. **Sheets in pairs of vectors.**
The group SL_2 acts on the space of 2×2-matrices M_2 by multiplication from the left (cf. I.5.1). Then M_2 consists of 3 *disjoint* sheets of orbit dimensions 3, 2 and 0:

$$\mathrm{M}_2 = S_3 \cup S_2 \cup \{0\}$$

where

$$S_3 := \mathrm{GL}_2 \subseteq M_2, \quad S_2 := \{m \in \mathrm{M}_2 \mid m \neq 0 \text{ and } \det m = 0\}.$$

The set S_3 consists exactly of the closed orbits which are not equal to zero. For every orbit $O \subseteq S_2$ one has $\overline{O} = O \cup \{0\}$ and $\overline{O} \simeq \mathbb{C}^2$ where $\mathbb{C}^2$ carries the natural representation of SL_2.

Example 5.1.7. **Sheets in binary forms.** We have a natural action of SL_2 on the binary forms $R_n := \mathcal{O}_n(\mathbb{C}^2)$ (cf. I.6.1). The space $R_1 = (\mathbb{C}^2)^*$ is isomorphic to the natural representation and has two sheets, both consisting in one orbit. The space R_2 is isomorphic to the adjoint representation of SL_2. It also has two sheets, $\{0\}$ and the complement $S_2 := R_2 \setminus \{0\}$ consisting of 2-dimensional orbits.

For $n \geq 3$ the representation R_n consists of three disjoint sheets of orbit dimensions 3, 2, and 0:

$$R_n = S_3 \cup S_2 \cup \{0\}.$$

Let us describe $S_2 = R_n^{(2)}$:

(a) For n odd:

$$S_2 = \{\ell^n \mid \ell \in R_1 \setminus \{0\}\} = O_{x^n} := \text{ orbit of } x^n \subseteq \mathcal{N}_{R_n}$$

(b) For $n = 2m$ even:

$$S_2 = \{(\ell_1 \cdot \ell_2)^m \mid \ell_i \in R_1 \setminus \{0\}\} = O_{x^{2m}} \cup \bigcup_{\lambda \in \mathbb{C}^*} O_{\lambda x^m y^m}.$$

5.2. Finitely many orbits. In this section we prove a first result showing that a "good" property of the null fiber carries over to all fibers of the quotient morphism $\pi \colon V \to V/\!\!/G$.

Proposition 5.2.1. *Assume that the null fiber $\mathcal{N}_V = \pi^{-1}(\pi(0))$ only contains a finite number of orbits. Then*

(1) *Every fiber of π contains only finitely many orbits.*
(2) *$\pi \colon V \to V/\!\!/G$ is equidimensional, i.e. the irreducible components of all fibers of π have the same dimension.*
(3) *Every irreducible component C of a fiber contains a dense orbit of G°.*
(4) *We have $\dim C = \max_{v \in V} \dim Gv = \dim V - \dim V/\!\!/G$.*

Proof. We may assume that G is connected.

(1) If the fiber $F := \pi^{-1}(w)$ contains infinitely many orbits of dimension d for some $w \in V/\!\!/G$, then there is an irreducible component X of $F_d := \{v \in F \mid \dim Gv \leq d\}$ which contains infinitely many orbits of dimension d. In particular, one has $\dim X \geq d+1$. Now consider $\mathbb{C}^* X := \{\lambda X \mid \lambda \in \mathbb{C}^*, x \in X\}$ and its closure $Z := \overline{\mathbb{C}^* X}$. Both sets are irreducible, G-stable and are contained in $V_d := \{v \in V \mid \dim Gv \leq d\}$. By Lemma 5.1.2 the set V_d is closed. Clearly, $0 \in Z$ and $\rho := \pi|_Z \colon Z \to \pi(Z) \subseteq V/\!\!/G$ is a quotient, because of the G-closedness of algebraic quotients (2.5). Since X lies in a fiber of the quotient ρ, one has $\lambda X \subseteq \rho^{-1}(\rho(\lambda x))$ for every $\lambda \in \mathbb{C}^*$ and $x \in X$. This implies $\dim \rho^{-1}(\rho(z)) \geq \dim X \geq d+1$ for every z in the dense subset $\mathbb{C}^* X$ of Z, hence $\dim \rho^{-1}(\rho(0)) \geq d+1$ by the dimension formula AI.3.4.7. Because $\rho^{-1}(\rho(0)) \subseteq V_d$, the fiber $\rho^{-1}(\rho(0))$ must contain an infinite number of orbits, which contradicts the assumption.

(2) Let m be the maximal orbit dimension in V. By Proposition 5.1.1 the set $V^{(m)} = \{v \in V \mid \dim Gv = m\}$ is open (and dense) in V. Thus $\pi(V^{(m)})$ is dense in $V/\!\!/G$, and for every $w \in \pi(V^{(m)})$ the fiber $\pi^{-1}(w)$ contains an orbit of dimension m. Thus $\dim \pi^{-1}(w) \geq m$. By (1) every fiber F of π contains only a finite number of orbits. Thus $\dim F \leq m$. Using again the dimension formula AI.3.4.7 one gets $\dim C = m$ for every irreducible component of F, and the claim follows.

(3) Let C be an irreducible component of a fiber F. By (1), it contains only finitely many orbits, and by (2) it has dimension m. Hence C contains a dense orbit of dimension m.

(4) This follows from the above and the dimension formula AI.3.4.7. □

The same proof yields the following variant of the proposition above.

Proposition 5.2.2. *If every component of the null fiber $\mathcal{N}_V$ contains a dense orbit, then this is true for every fiber of $\pi \colon V \to V/\!\!/G$, and π is equidimensional.*

EXAMPLE 5.2.3. Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ by $t(x,y) := (t \cdot x, t^{-1} \cdot y)$. Then

$$\pi \colon \mathbb{C}^2 \to \mathbb{C}, \quad (x,y) \mapsto xy,$$

is the quotient of $\mathbb{C}^2$ by this $\mathbb{C}^*$-action. The *null fiber* consists of three orbits,

$$\mathcal{N}_V = \mathcal{V}(xy) = (x\text{-axis} \setminus \{0\}) \cup (y\text{-axis} \setminus \{0\}) \cup \{0\}.$$

The other fibers are hyperbolas

$$F_c := \pi^{-1}(c) = \mathcal{V}(xy - c) \text{ where } c \in \mathbb{C}^*,$$

and they are closed orbits.

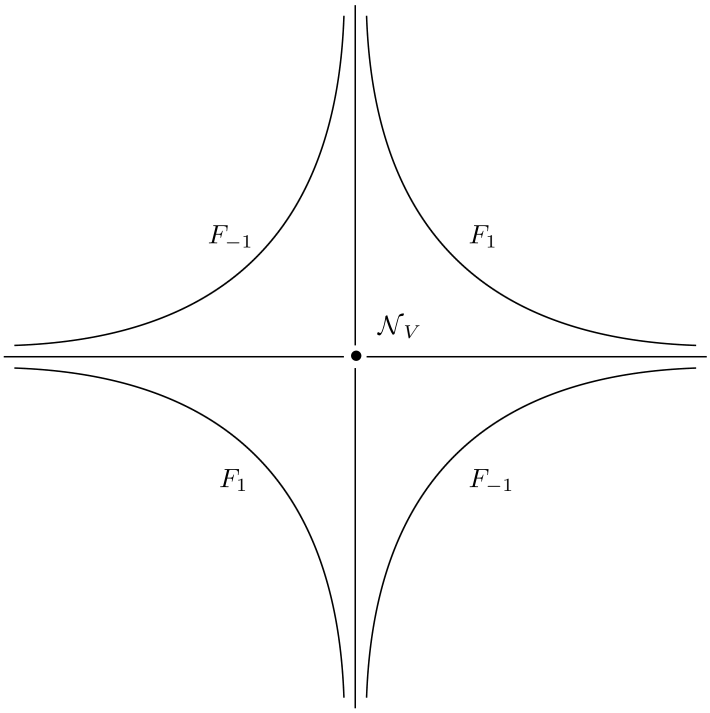


FIGURE 4. General fiber and null fiber

5.3. The associated cone. Now we would like to study the transformation from a general fiber to the null fiber a little closer. Let

$$R := \mathcal{O}(V) = \bigoplus_{i \geq 0} R_i$$

be the coordinate ring of V with its usual grading given by the total degree of the polynomials.

DEFINITION 5.3.1. If $f \in R, f = \sum_{i=0}^{d} f_i$, where $f_i \in R_i$ and $f_d \neq 0$, then we set

$$\operatorname{gr} f := f_d = \textit{homogeneous part of highest degree of } f.$$

If $T \subseteq R$ is a subspace, then

$$\operatorname{gr} T := \langle \operatorname{gr} f \mid f \in T \rangle = \textit{subspace of } R \textit{ spanned by all } \operatorname{gr} f, f \in T.$$

If $\mathfrak{a} \subseteq R$ is an ideal, then $\operatorname{gr} \mathfrak{a}$ is called the *associated graded ideal.*

The following properties are easy to verify. The task of doing this is left to the reader as an exercise. For (2) one uses the fact that R has no zero divisors.

LEMMA 5.3.2 (Properties of the associated graded ideal). *Let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be ideals.*

(1) $\operatorname{gr} \mathfrak{a}$ *is a homogeneous ideal in R. One has $\operatorname{gr} \mathfrak{a} = \mathfrak{a}$ if and only if $\mathfrak{a}$ is homogeneous.*
(2) $\operatorname{gr}(fR) = (\operatorname{gr} f)R$ *for every $f \in R$.*
(3) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\operatorname{gr} \mathfrak{a} \subseteq \operatorname{gr} \mathfrak{b}$.*
(4) *If $\mathfrak{a}$ is G-stable, then $\operatorname{gr} \mathfrak{a}$ is also G-stable.*
(5) $\operatorname{gr} \mathfrak{a} \cdot \operatorname{gr} \mathfrak{b} \subseteq \operatorname{gr}(\mathfrak{a} \cdot \mathfrak{b}) \subseteq \operatorname{gr} \mathfrak{a} \cap \operatorname{gr} \mathfrak{b}$.
(6) $\operatorname{gr} \sqrt{\mathfrak{a}} \subseteq \sqrt{\operatorname{gr} \mathfrak{a}}$.

DEFINITION 5.3.3. Suppose X is an arbitrary subset of V. Then we define the *cone associated to X* to be

$$\mathcal{C}X := \mathcal{V}(\operatorname{gr}(I(X)))$$

where $I(X)$ is the ideal of X.

PROPOSITION 5.3.4. *Let $X \subseteq V$ be a subset.*

(1) *$\mathcal{C}X = \mathcal{C}\bar{X}$, and this is a closed cone in V.*
(2) *The transformation $X \mapsto \mathcal{C}X$ preserves inclusions, takes G-stable subsets to G-stable closed cones and satisfies*

$$\mathcal{C}(X \cup Y) = \mathcal{C}X \cup \mathcal{C}Y.$$

(3) *$\mathcal{C}X \subseteq \overline{\mathbb{C}^* X}$ and $\dim \mathcal{C}X = \dim \bar{X}$.*
(4) *If X is closed and irreducible, then $\mathcal{C}X$ is equidimensional, i.e. all irreducible components have the same dimension.*

PROOF. The first two statements follow directly from the definitions. The first part of (3) is also clear, because $I(\overline{\mathbb{C}^* X})$ is a homogeneous ideal which is contained in $I(X)$ and thus also in $\operatorname{gr}(I(X))$. For the proof of the remaining assertion, we may assume that X is closed.

Consider the vector space $V \oplus \mathbb{C}$ with coordinate ring $\mathcal{O}(V \oplus \mathbb{C}) = R[t]$, along with the grading

$$R[t]_d := \sum_{i=0}^{d} R_i \cdot t^{d-i}.$$

Set

$$X' := \mathbb{C}^*(X \times \{1\}) = \{(\lambda x, \lambda) \in V \oplus \mathbb{C} \mid \lambda \in \mathbb{C}, x \in X\} \subseteq Z := \overline{X'} \subseteq V \oplus \mathbb{C},$$

and let $\eta \colon Z \to \mathbb{C}$ be the map induced by the projection $\operatorname{pr} \colon V \oplus \mathbb{C} \to \mathbb{C}$. We want to show that the following holds:

(i) $\eta^{-1}(\lambda) = \lambda X \times \{\lambda\} \cong X$ for $\lambda \neq 0$.
(ii) $\eta^{-1}(0) = \mathcal{C}X \times \{0\} \cong \mathcal{C}X$.

PROOF OF (i). For a homogeneous element $f = \sum_{i=1}^{d} f_i t^{d-i} \in R[t]_d$ one has $f(\lambda x, \lambda) = \lambda^d \cdot \sum_{i=1}^{d} f_i(x)$ for $\lambda \in \mathbb{C}$. Thus $f \in I(Z)$ implies $\sum_{i=0}^{d} f_i \in I(X)$ and conversely. Since $I(Z)$ is homogeneous, an element $(z, \lambda) \in Z$ for $\lambda \neq 0$ is contained in $\eta^{-1}(\lambda)$ if and only if $z = \lambda x$ for some $x \in X$. This implies (i). $\square$

PROOF OF (ii). For $g = \sum_{i=0}^{d} g_i \in R$, $g_d \neq 0$, set $\tilde{g} := \sum_{j=0}^{d} g_j t^{d-j}$. Then $I(X') = \langle \tilde{g} \mid g \in I(X) \rangle$, and $\tilde{g}(v, 0) = (\operatorname{gr} g)(v)$. For $v \in V$ this implies

$$\begin{aligned} (v,0) \in Z \quad &\iff \quad \tilde{g}(v,0) = 0 \text{ for every } g \in I(X) \\ &\iff \quad (\operatorname{gr} g)(v) = 0 \text{ for every } g \in I(X) \\ &\iff \quad v \in \mathcal{C}X, \end{aligned}$$

i.e. $Z \cap (V \times \{0\}) = \mathcal{C}X \times \{0\}$ which verifies (ii). $\square$

By construction $X' \simeq X \times \mathbb{C}^*$. If X is irreducible, then so is Z, and one has $\dim Z = \dim X + 1$. Because $\mathcal{C}X \times \{0\} \subsetneq Z$, we therefore get $\dim \mathcal{C}X \leq \dim X$. Conversely, it follows from (i) and (ii) that every irreducible component of $\mathcal{C}X$ has dimension $\geq \dim X$ (dimension formula AI.3.4.7). Thus $\mathcal{C}X$ is equidimensional of dimension $\dim X$. Because $\mathcal{C}(X \cup Y) = \mathcal{C}X \cup \mathcal{C}Y$ by (2), it also is true for a reducible $X \subseteq V$ that $\dim \mathcal{C}X = \dim X$. $\square$

Most of all we want to use this "cone construction" $\mathcal{C}X$ in the situation where X is a fiber of a quotient map. In this case $\mathcal{C}X$ has a very simple geometric description. (Cf. **??**; there X is a semi-stable orbit.)

Proposition 5.3.5. *Suppose $X \subseteq V$ is contained in a fiber F of the quotient map $\pi\colon V \to V/\!\!/G$ which is different from the null fiber $\mathcal{N}_V$. Then*

$$\mathcal{C}X = \overline{\mathbb{C}^* X} \cap \mathcal{N}_V = \overline{\mathbb{C}^* X} \setminus \mathbb{C}^* \overline{X}.$$

Proof. We will prove the following statements which imply the claim.

(i) $\mathcal{C}X \subseteq \mathcal{N}_V$;
(ii) $\mathcal{N}_V \cap \mathbb{C}^* \overline{X} = \emptyset$;
iii) $\overline{\mathbb{C}^* X} = \mathbb{C}^* \overline{X} \cup \mathcal{C}X$.

(i) The ideal $\mathfrak{m} := \bigoplus_{i>0} R_i^G$ is the maximal ideal of R^G corresponding to $\pi(0)$, and $\mathcal{N}_V = \mathcal{V}(\mathfrak{m})$. One has to show $\mathfrak{m} \subseteq \operatorname{gr} I(X)$. To do this suppose $f \in \mathfrak{m}$, $f \neq 0$ is homogeneous. Since f is an invariant, $f = c \in \mathbb{C}$ on X. It follows that $f - c \in I(X)$ and thus $\operatorname{gr}(f - c) = f \in \operatorname{gr} I(X)$, i.e. $\mathfrak{m} \subseteq \operatorname{gr} I(X)$.

(ii) Suppose $z \in \mathcal{N}_V \cap \mathbb{C}^* \overline{X}$, i.e. $z = \lambda x$ for some $\lambda \in \mathbb{C}^*$ and $x \in \overline{X}$. Since $\mathcal{N}_V$ is a cone, we get $x = \lambda^{-1} z \in \mathcal{N}_V \cap \overline{X}$. Thus $\pi(x) = \pi(0)$, contradicting the assumption.

(iii) Suppose $z \in \overline{\mathbb{C}^* X} \setminus \mathbb{C}^* \overline{X}$. Since $\mathbb{C}^* \overline{X}$ is dense in $\overline{\mathbb{C}^* X}$ there exist $\lambda_i \in \mathbb{C}^*$ and $x_i \in \overline{X}$ such that $z = \lim_{i\to\infty} \lambda_i x_i$ (cf. Proposition B.1.5.1). By taking a subsequence we may assume that the sequence λ_i converges. (Note that $|\lambda_i| \to \infty$ is not possible, because in this case $x_i \to 0$, and thus $0 \in \overline{X}$, contradicting the hypothesis.) Suppose $\lim_{i\to\infty} \lambda_i = \lambda$. If $\lambda \neq 0$, then

$$\lambda^{-1} z = \lim_{i\to\infty} \lambda_i^{-1} \lambda_i x_i = \lim_{i\to\infty} x_i \in \overline{X}, \text{ i.e. } z \in \mathbb{C}^* \overline{X},$$

contradicting the assumptions. Thus $\lim_{i\to\infty} \lambda_i = 0$.

Now we will show that $z \in \mathcal{C}X$, i.e. $(\operatorname{gr} f)(z) = 0$ for every $f \in I(X)$. Suppose $f = \sum_{j=0}^d f_j \in I(X)$ where $f_d \neq 0$, hence $\operatorname{gr} f = f_d$. If we set $f_\lambda := \sum_{i=0}^d \lambda^{d-i} f_i$ for $\lambda \in \mathbb{C}^*$, then it follows that $f_{\lambda_i}(\lambda_i x_i) = \lambda_i^d f(x_i) = 0$, and thus

$$0 = \lim_{i\to\infty} f_{\lambda_i}(\lambda_i x_i) = (\lim_{i\to\infty} f_{\lambda_i})(\lim_{i\to\infty} \lambda_i x_i) = (\operatorname{gr} f)(z).$$

This finishes the proof of the proposition. $\square$

Corollary 5.3.6. *One has $\dim \mathcal{N}_V \geq \dim F$ for every fiber F of π. In particular, π is equidimensional if and only if $\dim \mathcal{N}_V$ is minimal, i.e. if and only if*

$$\dim \mathcal{N}_V = \dim V - \dim V/\!\!/G.$$

(This follows directly from Proposition 5.3.5 above and Proposition 5.3.4.)

5.4. The coordinate ring of the associated cone. Now we would like to compare the coordinate rings of X and of $\mathcal{C}X$. As before, we set $R := \mathcal{O}(V) = \bigoplus_{i\geq 0} R_i$. Then we have

$$\mathcal{O}(X) = R/I(X) \text{ and } \mathcal{O}(\mathcal{C}X) = R/\sqrt{\operatorname{gr} I(X)}.$$

Suppose $\mathfrak{a} \subseteq R$ is an ideal, and set $\bar{R} := R/\mathfrak{a}$. Define $R_{\leq j} := \bigoplus_{i \leq j} R_i$ (hence $R_{\leq j} = \{0\}$ for $j < 0$). Setting

$$\bar{R}^{(j)} := (R_{\leq j} + \mathfrak{a})/\mathfrak{a} \subseteq R/\mathfrak{a},$$

we get an ascending filtration

$$\bar{R}^{(0)} := \mathbb{C} \subseteq \bar{R}^{(1)} \subseteq \bar{R}^{(2)} \subseteq \cdots \subseteq \bar{R}$$

such that $\bar{R}^{(i)} \cdot \bar{R}^{(j)} \subseteq \bar{R}^{(i+j)}$. Therefore, the multiplication in $\bar{R}$ defines a $\mathbb{C}$-algebra structure on

$$\operatorname{gr} \bar{R} := \bigoplus_{i=0}^{\infty} \bar{R}^{(i)}/\bar{R}^{(i-1)}.$$

Lemma 5.4.1. (1) *There is a canonical isomorphism $R/\operatorname{gr}\mathfrak{a} \xrightarrow{\sim} \operatorname{gr}(R/\mathfrak{a})$.*
(2) *If $\mathfrak{a}$ is G-stable, then $R/\mathfrak{a}$ and $R/\operatorname{gr}\mathfrak{a}$ are isomorphic G-modules.*

Proof. (1) For $f_i \in R_i$ we denote by $\bar{f}_i$ the image of f_i in

$$\bar{R}^{(i)}/\bar{R}^{(i-1)} = (R_{\leq i} + \mathfrak{a})/(R_{\leq i-1} + \mathfrak{a}).$$

We get a homogeneous surjective $\mathbb{C}$-algebra homomorphism

$$\rho\colon R \to \operatorname{gr}(\bar{R}), \quad f = \sum_{i \geq 0} f_i \mapsto \sum_{i \geq 0} \bar{f}_i.$$

We find

$$\begin{aligned} (\ker\rho) \cap R_i &= (R_{\leq i-1} + \mathfrak{a}) \cap R_i \\ &= \{\operatorname{gr} f \mid f \in \mathfrak{a}, \deg f = i\} \\ &= (\operatorname{gr}\mathfrak{a})_i \end{aligned}$$

and thus $\ker\rho = \operatorname{gr}\mathfrak{a}$.

(2) If $\mathfrak{a}$ is G-stable, then so are all the $\bar{R}^{(i)}$. Because of the semisimplicity of the G-modules, there is a G-stable complement E^i of $\bar{R}^{(i-1)}$ in $\bar{R}^{(i)}$:

$$\bar{R}^{(i)} = E^i \oplus \bar{R}^{(i-1)} \text{ for every } i \in \mathbb{N}.$$

Thus $\operatorname{gr}(\bar{R}) = \bigoplus_{i=0}^{\infty} \bar{R}^{(i)}/\bar{R}^{(i-1)}$ is isomorphic, as a G-module, to $\bigoplus_{i=0}^{\infty} E^i = \bar{R}$. Since the isomorphism $R/\operatorname{gr}\mathfrak{a} \xrightarrow{\sim} \operatorname{gr}\bar{R}$ constructed in (1) is also G-equivariant, we finally see that $R/\operatorname{gr}\mathfrak{a}$ and $\bar{R}$ are isomorphic as G-modules. □

If X is an affine G-variety such that $\mathcal{O}(X)^G = \mathbb{C}$, then the multiplicities of the simple modules in $\mathcal{O}(X)$ are finite, by Hilbert's Finiteness Theorem 2.3.1 (cf. 5.5.2); they are denoted by $m_\lambda(X)$ for $\lambda \in \Lambda_G$.

Proposition 5.4.2. *If X is a G-stable closed subset of V, then the multiplicities satisfy the following inequalities:*

$$m_\lambda(X) \geq m_\lambda(\mathcal{C}X) \text{ for every } \lambda \in \Lambda_G.$$

Proof. We have seen in Lemma 5.4.1 above that $\mathcal{O}(X) = R/I(X)$ is G-isomorphic to $R/\operatorname{gr} I(X)$. The claim now follows from the surjectivity of the canonical G-equivariant homomorphism

$$R/\operatorname{gr} I(X) \twoheadrightarrow R/\sqrt{\operatorname{gr} I(X)} = \mathcal{O}(\mathcal{C}X).$$

□

5.5. Reducedness and normality. The main result of this section is the following theorem. We look again at the quotient map $\pi\colon V \to V/\!\!/G$ where V is a G-module.

THEOREM 5.5.1. *If the null fiber $\mathcal{N}_V$ is reduced and irreducible of dimension $\dim V - \dim V/\!\!/G$, then all fibers of π are reduced and irreducible, and π is equidimensional. If, in addition, $\mathcal{N}_V$ is normal, then so are all the fibers of π.*

PROOF. (a) For $z \in V/\!\!/G$ let $\mathfrak{m}_z \subseteq R^G$ denote the maximal ideal corresponding to w. Then one has $\mathfrak{m}_0 := \bigoplus_{i>0} R_i^G \subseteq \operatorname{gr}\mathfrak{m}_z$. In fact, for a homogeneous $f \in \mathfrak{m}_0$ one has $f - f(z) \in \mathfrak{m}_z$, hence $f = \operatorname{gr}(f - f(z)) \in \operatorname{gr}\mathfrak{m}_z$.

Suppose now that C is an irreducible component of $F := \pi^{-1}(z)$. Then it follows that $\mathfrak{m}_z R \subseteq I(C)$, and $\mathcal{C}C = \mathcal{N}_V$. In fact, $\mathcal{N}_V$ is irreducible and $\dim \mathcal{N}_V = \dim \mathcal{C}C$ by Corollary 5.3.6. Since $\mathcal{N}_V$ is reduced, one even has $I(\mathcal{N}_V) = \mathfrak{m}_0 R$. Altogether we get

$$\mathfrak{m}_0 R \subseteq \operatorname{gr}\mathfrak{m}_z R \subseteq \operatorname{gr} I(C) \subseteq \sqrt{\operatorname{gr} I(C)} = I(\mathcal{N}_V) = \mathfrak{m}_0 R,$$

and thus $\operatorname{gr}\mathfrak{m}_z R = \operatorname{gr} I(C)$. Since $\mathfrak{m}_w R \subseteq I(C)$ we finally get $\mathfrak{m}_w R = I(C)$ by Lemma 5.3.2. Thus $\mathfrak{m}_w R = I(F)$, i.e. the fiber F is reduced and irreducible.

(b) Suppose now that $\mathcal{N}_V$ is also normal. If we set $\bar{R} = \mathcal{O}(F) = R/I(F)$, then it follows from the above and Lemma 5.4.1(1) that

$$\operatorname{gr}\bar{R} \simeq R/\operatorname{gr} I(F) = \mathcal{O}(\mathcal{N}_V),$$

i.e. $\operatorname{gr}\bar{R}$ is a normal integral domain. We want to conclude from this that $\bar{R}$ $\bar{R}$ is also normal. For $f \in \bar{R}$ set

$$\deg f := \begin{cases} d & \text{if } f \neq 0 \text{ and } f \in \bar{R}^{(d)} \setminus \bar{R}^{(d-1)} \\ -\infty & \text{if } f = 0 \end{cases}$$

(We are using the notation which was introduced above in section 5.4.) Since $\operatorname{gr}\bar{R}$ is an integral domain one has

$$\deg(fg) = \deg f + \deg g \quad \text{for every } f, g \in \bar{R}.$$

Now let $K := \operatorname{Quot}(\bar{R})$ be the quotient field of $\bar{R}$, and let $t = f/g \in K$. Then

$$\deg t := \deg f - \deg g$$

is well-defined, i.e. independent of the representation of t as a quotient in K. On K we get the filtration

$$\cdots \subseteq K^{(i)} \subseteq K^{(i+1)} \subseteq K^{(i+2)} \subseteq \cdots \qquad i \in \mathbb{Z},$$

where $K^{(i)} := \{t \in K \mid \deg t \leq i\}$. Then we get

(i) $K^{(i)} \cap \bar{R} = \bar{R}^{(i)}$;
(ii) $K^{(i)} \cdot K^{(j)} \subseteq K^{(i+j)}$;
(iii) $\deg(rs) = \deg r + \deg s$ for $r, s \in K$.

Because of (ii), we see that $\operatorname{gr} K := \bigoplus_{i\in\mathbb{Z}} K^{(i)}/K^{(i-1)}$ is a $\mathbb{C}$-algebra, and it follows from (i) and (iii) that $\operatorname{gr} K$ has no zero divisors and that $\operatorname{gr}\bar{R} \subseteq \operatorname{gr} K$. We claim that $\operatorname{gr} K$ is contained in the quotient field of $\operatorname{gr}\bar{R}$:

$$\operatorname{gr}\bar{R} \subseteq \operatorname{gr} K \subseteq \operatorname{Quot}(\operatorname{gr}\bar{R}).$$

In fact, if $s \in \operatorname{gr} K$, $s \in K^{(i)}/K^{(i-1)}$, then $s = f/h + K^{(i-1)}$ where $f, h \in \bar{R}$, with $d := \deg h = \deg f - 1$. Thus $(\operatorname{gr} h) \cdot s = (h + K^{(d-1)}) \cdot (f/h + K^{(i-1)}) = f + K^{(d+i-1)} = \operatorname{gr} f$, and hence

$$s = \frac{(\operatorname{gr} f)}{(\operatorname{gr} h)} \in \operatorname{Quot}(\operatorname{gr}\bar{R}).$$

Now suppose that $S \subseteq K$ is the integral closure of $\bar{R}$ in K. Then S inherits the filtration from K, i.e. $S^{(i)} := S \cap K^{(i)}$, and one has

$$\operatorname{gr} \bar{R} \subseteq \operatorname{gr} S \subseteq \operatorname{gr} K \subseteq \operatorname{Quot}(\operatorname{gr} \bar{R}).$$

Now the integral closure S of $\bar{R}$ in K is a finitely generated $\bar{R}$-module (Proposition A.5.2.1). Hence, there is an $r \in \bar{R}$, $r \neq 0$, with $rS \subseteq \bar{R}$ Because there are no zero divisors in $\operatorname{gr} S$, it thus follows that $\operatorname{gr}(rS) = (\operatorname{gr} r)(\operatorname{gr} S) \subseteq \operatorname{gr} \bar{R}$. Since $\operatorname{gr} \bar{R}$ is noetherian, $\operatorname{gr} S$ is a finitely generated $(\operatorname{gr} \bar{R})$-module and hence is integral over $\operatorname{gr} \bar{R}$. By assumption we have $\operatorname{gr} S = \operatorname{gr} \bar{R}$, and therefore $\bar{R} = S$ by Lemma 5.3.2(3). i.e. $\bar{R}$ is normal. □

Corollary 5.5.2. *If*

$$\mathcal{N}'_V := \{v \in \mathcal{N}_V \mid (d\pi)_v \colon V \to T_{\pi(0)}(V/\!\!/G) \text{ is surjective}\}$$

is not empty and $\operatorname{codim}_{\mathcal{N}_V} \overline{\mathcal{N}_V} \setminus \mathcal{N}_{V'} \geq 2$, *then all fibers of* π *are reduced and normal, the quotient map* π *is equidimensional, and the quotient* $V/\!\!/G$ *is an affine space.*

Proof. If $v \in \mathcal{N}'_V$, then

$$\begin{aligned} \dim_v \mathcal{N}_V &\leq \dim T_v(\mathcal{N}_V) \leq \dim \ker(d\pi)_v = \dim V - \dim T_{\pi(0)}(V/\!\!/G) \\ &\leq \dim V - \dim_{\pi(0)} V/\!\!/G = \dim V - \dim V/\!\!/G \leq \dim_v \mathcal{N}_V\,. \end{aligned}$$

Thus $\dim T_{\pi(0)}(V/\!\!/G) = \dim V/\!\!/G$, i.e. $\pi(0)$ is a regular point of $V/\!\!/G$, and $\mathcal{N}_V$ is smooth in $\mathcal{N}'_V$, hence $\mathcal{N}_V$ is reduced and normal, by Serre's Criterion A.5.7.5. Now the theorem above implies the first two claims.

It remains to see that the quotient is an affine space. Since $\mathcal{O}(V/\!\!/G) = \mathcal{O}(X)^G$ is positively graded, with homogeneous maximal ideal $\mathfrak{m}_{\pi(0)}$, this is Example III.3.1.3. □

Example 5.5.3 (First Fundamental Theorem). Suppose $\dim U + \dim W \leq \dim V$. Then all fibers of the quotient map

$$\pi \colon L(U,V) \times L(V,W) \to L(U,W), \quad (\alpha,\beta) \mapsto \beta \circ \alpha$$

are reduced and normal.

Proof. This follows immediately from Proposition 4.7.4. We give here a direct proof using the corollary above. The null fiber is given by $\mathcal{N} := \{(\alpha,\beta) \mid \beta \circ \alpha = 0\}$. Hence, it is given by $\dim U \cdot \dim W$ equations. Since $\dim U \cdot \dim W$ is the dimension of the quotient it follows that π is equidimensional. It is easy to see that the tangent in $(\alpha,\beta) \in \mathcal{N}$ is surjective, if either α is injective or β is surjective.

□

Remark 5.5.4. Results analogous to our Theorem 5.5.1 can be proved for other properties of the null fiber, e.g. for the *Cohen-Macaulay property* or the property of having *rational singularities.*

6. The Variety of Representations of an Algebra

6.1. The variety Mod_A^n**.** In the following let A be a finitely generated associative unitary $\mathbb{C}$-algebra, and let $\{a_1, \dots, a_s\}$ be a set of generators of A. A *finite dimensional A-module M* is a finite dimensional $\mathbb{C}$-vector space V together with an action of A on V given by a homomorphism $\rho \colon A \to \operatorname{End}(V)$ of $\mathbb{C}$-algebras. Therefore, the *isomorphism classes* of finite dimensional A-modules correspond in a unique way to the equivalence classes of finite dimensional representations of the algebra A. Now fix $n \in \mathbb{N}$ and define

$$\mathrm{Mod}_A^n := \{\rho\colon A \to \mathrm{M}_n(\mathbb{C}) \mid \rho \text{ a } \mathbb{C}\text{-algebra homomorphism}\}.$$

For $\rho \in \mathrm{Mod}_A^n$ we denote by M_ρ the corresponding A-module with underlying vector space $\mathbb{C}^n$.

Proposition 6.1.1. *The set Mod_A^n of representations of A on $\mathbb{C}^n$ has a natural structure of an affine variety with an action of GL_n. The orbits are the equivalence classes of representations, and they correspond bijectively to the isomorphism classes of n-dimensional A-modules.*

Proof. The homomorphism $\rho\colon A \to \mathrm{M}_n$ is determined by the images of the generators $a_1, \dots, a_s$ of A. Thus we get an embedding

$$\iota = \iota_{\{a_1,\dots,a_s\}}\colon \mathrm{Mod}_A^n \hookrightarrow (\mathrm{M}_n)^s, \quad \rho \mapsto (\rho(a_1), \dots, \rho(a_s)).$$

The image of Mod_A^n is the closed subvariety of the vector space $(\mathrm{M}_n)^s$ defined by the same equations in the matrices $\rho(a_i)$ as those which are satisfied by the generators a_i in A. In order to see that this does not depend on the generators, it suffices to consider a second set of generators of the form $\{a_1, \dots, a_s, b\}$ with an arbitrary element $b \in A$. Then $b = p(a_1, \dots, a_s)$ where p is a linear combination of monomials in the a_i, and we get the following commutative diagram

$$\begin{tikzcd}
(\mathrm{M}_n)^s \arrow[r, "\psi"] & (\mathrm{M}_n)^{s+1} \arrow[r, "\mathrm{pr}"] & (\mathrm{M}_n)^s \\
\mathrm{Mod}_A^n \arrow[u, "\subseteq"', "\iota_{(a_1,\dots,a_s)}"] \arrow[r, equal] & \mathrm{Mod}_A^n \arrow[u, "\subseteq"', "\iota_{(a_1,\dots,a_s,b)}"] \arrow[r, equal] & \mathrm{Mod}_A^n \arrow[u, "\subseteq"', "\iota_{(a_1,\dots,a_s)}"]
\end{tikzcd}$$

where $\psi(A_1, \dots, A_s) := (A_1, \dots, A_s, p(A_1, \dots, A_s))$. Since $\mathrm{pr} \circ \psi = \mathrm{id}$ it follows that ψ induces an isomorphism $\iota_{\{a_1,\dots,a_n\}}(\mathrm{Mod}_A^n) \xrightarrow{\sim} \iota_{\{a_1,\dots,a_n,b\}}(\mathrm{Mod}_A^n)$.

We let GL_n act by conjugation on $(\mathrm{M}_n)^s$. Clearly, the image of Mod_A^n is stable under this action. In fact, along with $\rho\colon A \to \mathrm{M}_n$ the map $g\rho\colon A \to \mathrm{M}_n$, $a \mapsto g\rho(a)g^{-1}$, is also an algebra homomorphism. The remaining statements are now obvious. □

For $\rho \in \mathrm{Mod}_A^n$ we will denote by C_ρ the orbit of ρ under the action of GL_n. By the above every n-dimensional A-module M defines an orbit in Mod_A^n which will be denoted by C_M.

Example 6.1.2. Let $A = \mathbb{C}[x]$, the polynomial ring in one variable x. Then $\mathrm{Mod}_A^n \simeq \mathrm{M}_n$, and the isomorphism classes of the n-dimensional A-modules are in one-to-one correspondence with the conjugacy classes in M_n.

Remark 6.1.3. For the stabilizer $(\mathrm{GL}_n)_\rho$ of a representation $\rho \in \mathrm{Mod}_A^n$ we have in a canonical way

$$(\mathrm{GL}_n)_\rho = \mathrm{Aut}_A(M_\rho).$$

Proof. If $g\colon M_\rho \to M_\rho$ is an isomorphism of A-modules for some $g \in \mathrm{GL}_n$, then one has $g(am) = a(gm)$ for every $a \in A, m \in M_\rho$. By definition, $am = \rho(a)m$ and thus $g(\rho(a)m) = \rho(a)g(m)$ for all $m \in M_\rho$. This implies $g\rho(a) = \rho(a)g$ for every $a \in A$, hence $g\rho = \rho g$, i.e. $g \in (\mathrm{GL}_n)_\rho$. The claim follows easily. □

6.2. Geometric properties. Now one might wonder about the connection between *algebraic properties* of an A-module M and *geometric properties* of the associated orbit C_M. A first result in this direction is the following. Recall that an A-module is *semisimple* if it is a direct sum of simple A-modules.

PROPOSITION 6.2.1. *An A-module M of finite dimension is semisimple if and only if the associated orbit C_M is closed.*

The proposition will be a consequence of a more general result (Theorem 6.3.2). For the proof we need some new tools which we will develop now.

DEFINITION 6.2.2. A *filtration* $\mathcal{F}$ of an A-module M is a finite chain

$$\mathcal{F}\colon M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = \{0\}$$

of submodules. The *associated graded* A-module is defined to be

$$\operatorname{gr}_{\mathcal{F}} M := \bigoplus_{i=0}^{t-1} M_i/M_{i+1}.$$

REMARK 6.2.3. Every finite dimensional A-module M has a *composition series*, i.e. a filtration with all the factors M_i/M_{i+1} being simple. The simple factors which occur, as well as their multiplicities, are independent of the particular series by the famous Theorem of JORDAN-HÖLDER. One calls these simple factors the *composition factors* or the JORDAN-HÖLDER-factors.

LEMMA 6.2.4. *Suppose $\rho, \rho' \in \operatorname{Mod}_A^n$. Then the following are equivalent.*

(i) *There exists a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{GL}_n$ such that*

$$\lim_{t\to 0} \lambda(t)\rho = \rho'.$$

(ii) *There exists a filtration $\mathcal{F}$ of the A-module M_ρ such that $\operatorname{gr}_{\mathcal{F}} M_\rho \simeq M_{\rho'}$ as A-modules.*

PROOF. (i) ⇒ (ii): We decompose the underlying vector space $V = \mathbb{C}^n$ of M_ρ according to its weights with respect to λ:

$$V = \bigoplus_i V_i, \quad V_i := \{v \in V \mid \lambda(t)v = t^i v \text{ for } t \in \mathbb{C}^*\}.$$

Define $M_j := \bigoplus_{i>j} V_i$. We claim that the $M_j \subseteq M_\rho$ are submodules forming a filtration $\mathcal{F}$ of M_ρ such that $\operatorname{gr}_{\mathcal{F}} M_\rho \simeq M_{\rho'}$.

Let $\iota_i\colon V_i \hookrightarrow V$ (resp. $p_i\colon V \twoheadrightarrow V_i$) be the canonical injections (resp. projections) of the weight space decomposition $V = \bigoplus_i V_i$. For $a \in A$ and $\rho(a) \in \operatorname{End}_{\mathbb{C}}(V)$ one has $\rho(a) = (p_k \circ \rho(a) \circ \iota_i) = (\rho(a)_{ik})$ with $\rho(a)_{ik}\colon V_i \to V_k$. We find that $p_k \circ (\lambda(t)\rho)(a) \circ \iota_i = p_k \circ (\lambda(t)\varphi\lambda(t)^{-1}) \circ \iota_i = p_k t^k \rho(a) t^{-i}\iota_i = t^{k-i}\rho(a)_{ik}$:

$$\begin{tikzcd} V_i \arrow[r, "\rho(a)_{ik}"] & V_k \arrow[d, "\lambda(t)"] \\ V_i \arrow[u, "\lambda(t)^{-1}"] \arrow[r, "t^{k-i}\rho(a)_{ik}"] & V_k \end{tikzcd}$$

Since, by assumption, $\lim_{t\to 0}(\lambda(t)\rho)(a)$ exists, this implies the following:

(1) One has $\rho(a)_{ik} = 0$ for $k < i$. In particular, M_j is an A-submodule of M_ρ for every j.
(2) $\rho'(a)_{ik} = \lim_{t\to 0} p_k \circ (\lambda(t)\rho)(a) \circ \iota_i = 0$ for $k > i$.
(3) $\rho'(a)_{ii} = \rho(a)_{ii}$ for every i.

Thus $\rho'(a)$ is given by the matrix

$$\begin{bmatrix} \ddots & & 0 \\ & \rho(a)_{ii} & \\ 0 & & \ddots \end{bmatrix}$$

and the claim follows.

(ii) $\Rightarrow$ (i): Given the filtration $\mathcal{F}\colon M_\rho = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_t = \{0\}$ it suffices to find a suitable one-parameter subgroup λ which induces the filtration $\mathcal{F}$ (in the sense of the first part of the proof). There exist subspaces $V_i \subseteq M_{rho}$, $i = 0, \ldots, t$ such that $V = \bigoplus_{i=0}^{t} V_i$ and $M_j = \bigoplus_{i=j}^{t} V_i$. Now define λ to be $\lambda(t) := t^i \cdot \mathrm{Id}_{V_i}$ on V_i for $i = 0, \ldots, t$. Then it is easy to see that this one-parameter subgroup satisfies the stated conditions, i.e. the coresponding filtration is $\mathcal{F}$. $\square$

Example 6.2.5. The matrix $\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$ defines to a $\mathbb{C}[x]$-module structure M on $\mathbb{C}^2$. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$. Clearly, M contains the submodule $\mathbb{C}e_1$. The graded $\mathbb{C}[x]$-module associated to the filtration $\mathcal{F}\colon M = \mathbb{C}^2 \supseteq \mathbb{C}e_1 \supseteq \{0\}$ is $\mathbb{C}e_1 \oplus \mathbb{C}^2/\mathbb{C}e_1 \simeq \mathbb{C} \oplus \mathbb{C}$ where the action of x induced on this is given by the matrix $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$. Moreover, for $t \in \mathbb{C}^*$ one has

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} \alpha & t^2 \\ 0 & \alpha \end{bmatrix} \xrightarrow{t \to 0} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

This shows that $\mathcal{F}$ is the filtration associated to the 1-PSG $\lambda\colon t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$.

6.3. Degenerations. Suppose M, M' are two A-modules of the same dimension. If the orbit $C_{M'}$ of M' lies in the closure of the orbit C_M of M, then one calls M' a *specialization* or a *degeneration* of M. This property will be indicated by

$$\begin{tikzcd} \bullet\, M \arrow[d, no head] \\ \bullet\, M' \end{tikzcd}$$

Remark 6.3.1. The above lemma implies that the graded module $\mathrm{gr}_{\mathcal{F}} M$ associated to the filtration $\mathcal{F}$ of M is a specialization of M. However, not every specialization can be obtained in this way. For a counterexample we refer the reader to [**Kra82**, Chap. II.4.6, remark 2]

The next theorem serves to clarify and extend Proposition 6.2.1.

Theorem 6.3.2. (1) *The closed orbits in* Mod_A^n *are in one-to-one correspondence with the semisimple A-modules of dimension n.*

(2) *If M is an A-module of dimension n, then the closure $\overline{C_M}$ of the corresponding orbit contains exactly one closed orbit. This corresponds to the module* $\mathrm{gr}_{\mathcal{F}} M$ *where $\mathcal{F}$ is a composition series.*

$$\begin{tikzcd} \bullet\, M \arrow[d, no head] \\ \bullet\, \mathrm{gr}_{\mathcal{F}} M \end{tikzcd}$$

Proof. (1) Suppose M is an A-module whose orbit $C_M \subseteq \mathrm{Mod}_A^n$ is closed, and let $\mathcal{F}$ be a composition series of M. By Lemma 6.2.4 one has $C_{\mathrm{gr}_{\mathcal{F}} M} \subseteq \overline{C_M} = C_M$. Thus $\mathrm{gr}_{\mathcal{F}} M \simeq M$, and so M is semisimple.

Conversely suppose M is a semisimple module. In order to show that C_M is closed we make use of the HILBERT Criterion which we are going to prove later on in V.3. It says that for every closed orbit $C_N \subseteq \overline{C_M}$ and every $\rho \in C_M$ there exists a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{GL}_n$ such that $\lim_{t\to 0} \lambda(t)\rho \in C_N$. By the above lemma this means that for a suitable filtration $\mathcal{F}$ of M the module N is isomorphic to $\mathrm{gr}_{\mathcal{F}} M$. Since M is semisimple, we have $\mathrm{gr}_{\mathcal{F}} M \simeq M$ and thus $C_N = C_M$.

(2) The remaining point is the uniqueness of the closed orbit in $\overline{C_M}$, but this is just the JORDAN-HÖLDER Theorem, see Remark 6.2.4. □

REMARK 6.3.3. We already know that the closure of an orbit contains a unique closed orbit, see Corollary 2.6.5. In the setting above, this give a geometric proof of the JORDAN-HÖLDER Theorem.

EXAMPLE 6.3.4. Suppose $A = \mathbb{C}[x]$. An A-module M of dimension n is semisimple if and only if the associated matrix in M_n is semisimple. Thus the above claim was already verified in I.5.5.2. If the associated matrix is given in JORDAN normal form, then the transformation $M \mapsto \mathrm{gr}_{\mathcal{F}} M$ corresponds to "setting to zero" the elements of the matrix which lie above the diagonal.

EXAMPLE 6.3.5. For $A = \mathbb{C}[\varepsilon]$ we get $\mathrm{Mod}_A^n = \{N \in \mathrm{M}_n \mid N^2 = 0\}$. This is the closure of a single conjugacy class, namely of the class of the nilpotent matrix with partition $p = (2, 2, \ldots, 2)$ in case n is even and $p = (2, 2, \ldots, 2, 1)$ in case n is odd. This follows from Proposition I.4.4.3. In particular, Mod_A^n is normal and Cohen-Macaulay with rational singularities, as a consequence of the main result in [**KP79**].

6.4. Tangent spaces and extensions. To conclude this section we would like to give a module theoretic interpretation of the *tangent spaces* of Mod_A^n, or, more precisely, of the *normal spaces* of the orbits C_M

If M and N are two A-modules, then by an *extension of N by M* one means a short exact sequence of A-modules of the form

$$\zeta\colon 0 \to M \to P \to N \to 0.$$

If $\zeta'\colon 0 \to M \to P' \to N \to 0$ is another extension, then we say that ζ and ζ' are *equivalent*, if there exists an isomorphism $\varphi\colon P \xrightarrow{\sim} P'$ which induces the identity on M and N:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \simeq\big\downarrow\varphi & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & P' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

The set of equivalence classes of extensions of N by M is denoted by $\mathrm{Ext}_A^1(N, M)$.

We now give another description of $\mathrm{Ext}_A^1(N, M)$. The A-module structures on M and N are given by $\rho\colon A \to \mathrm{End}_{\mathbb{C}}(M)$ and $\sigma\colon A \to \mathrm{End}_{\mathbb{C}}(N)$. In the extension ζ the space P is isomorphic, as a vector space, to $M \oplus N$. Thus one gets the middle term P of ζ by endowing the vector space $M \oplus N$ with an A-module structure $\mu\colon A \to \mathrm{End}_A(M \oplus N)$ such that $M = M_\rho$ is a submodule and $N = N_\sigma$ is a coset module of $P = (M \oplus N)_\mu$. This is exactly the case if μ has the form $\begin{bmatrix} \rho & \tau \\ 0 & \sigma \end{bmatrix}$ for a suitable $\tau\colon A \to \mathrm{Hom}_{\mathbb{C}}(N, M)$. For this one must have $\mu(ab) = \mu(a)\mu(b)$ for every $a, b \in A$, i.e.,

$$(*) \qquad \tau(ab) \quad = \quad \rho(a)\tau(b) + \tau(a)\sigma(b).$$

Define

$$Z(N, M) := \{\tau\colon A \to \mathrm{Hom}_{\mathbb{C}}(N, M) \mid \tau \text{ satisfies } (*)\}$$

which is a subspace of $\mathrm{Hom}(N, m)$. If τ, τ' are elements of $Z(N, M)$ and $\mu := \begin{bmatrix} \rho & \tau \\ 0 & \sigma \end{bmatrix}$, $\mu' := \begin{bmatrix} \rho & \tau' \\ 0 & \sigma \end{bmatrix}$, then $(M \oplus N)_\mu$ and $(M \oplus N)_{\mu'}$ yield equivalent extensions if and only if there is an isomorphism $\varphi\colon (M \oplus N)_\mu \xrightarrow{\sim} (M \oplus N)_{\mu'}$ of the form $\varphi = \begin{bmatrix} \mathrm{id} & \beta \\ 0 & \mathrm{id} \end{bmatrix}$ where $\beta \in \mathrm{Hom}_{\mathbb{C}}(N, M)$. This means that $\varphi(\mu(a)q) = \mu'(a)\varphi(q)$ for $a \in A$ and $q \in M \oplus N$, hence $\tau(a) + \beta\sigma(a) = \rho(a)\beta + \tau'(a)$ for every $a \in A$. This shows that τ and τ' define equivalent extensions if and only if $\tau - \tau' \in B(N, M)$ where

$$B(N, M) := \{\delta\colon A \to \mathrm{Hom}_{\mathbb{C}}(N, M) \mid \delta(a) = \rho(a)\beta - \beta\sigma(a) \text{ for all } a \in A \text{ and some } \beta \in \mathrm{Hom}_{\mathbb{C}}(N, M)\}.$$

Thus we get the following description

$$\mathrm{Ext}^1_A(N, M) = Z(N, M)/B(N, M).$$

From this one sees that $\mathrm{Ext}^1_A(N, M)$ is a finite dimensional vector space and that an extension $\zeta\colon 0 \to M \to P \xrightarrow{p} N \to 0$ is zero if and only if it "splits", i.e., if the projection p has a section and thus $P \xrightarrow{\sim} N \oplus M$ as an A-module.

THEOREM 6.4.1. *For every $\rho \in \mathrm{Mod}^n_A$ there is a natural injection*

$$T_\rho(\mathrm{Mod}^n_A)/T_\rho(C_{M_\rho}) \hookrightarrow \mathrm{Ext}^1_A(M, M).$$

PROOF. If $\xi \in T_\rho(\mathrm{Mod}^n_A)$, then $\rho + \varepsilon\xi\colon A \to \mathrm{M}_n(\mathbb{C}[\varepsilon])$ is an algebra homomorphism. A simple calculation shows that

$$\xi(ab) = \rho(a)\xi(b) + \xi(a)\rho(b) \text{ holds for every } a, b \in A.$$

This means that ξ, as a linear map $A \to \mathrm{M}_n = \mathrm{End}(M_\rho)$, satisfies the condition $(*)$ from above, hence $T_\rho(\mathrm{Mod}^n_A) \subseteq Z(M_\rho, M_\rho)$.

Now we consider the orbit map $\mu\colon \mathrm{GL}_n \to \overline{C_{M_\rho}} \subseteq (\mathrm{M}_n)^s$, $g \mapsto g\rho$. (Here we identify $\rho \in \mathrm{Mod}^n_A$ with $(\rho(a_1), \ldots, \rho(a_s)) \in (\mathrm{M}_n)^s$ where GL_n acts by simultaneous conjugation.) The differential $(d\mu)_e\colon \mathrm{Lie}\,\mathrm{GL}_n \to T_\rho(\overline{C_{M_\rho}})$ is surjective (Lemma III.5.1.5), and for $X \in \mathrm{Lie}\,\mathrm{GL}_n = \mathrm{M}_n$ we have

$$(E + \varepsilon X)\rho(a_i)(E - \varepsilon X) = \rho(a_i) + \varepsilon(X\rho(a_i) - \rho(a_i)X).$$

Therefore,

$$d\mu(X)(a) = X\rho(a) - \rho(a)X \text{ for every } a \in A.$$

It follows that $T_\rho(\overline{C_{M_\rho}}) = B(M_\rho, M_\rho)$, and the claim follows. □

REMARK 6.4.2. The natural map $T_\rho(\mathrm{Mod}^n_A)/T_\rho(C_{M_\rho}) \hookrightarrow \mathrm{Ext}^1_A(M, M)$ might be a strict inclusion. As an example, take the algebra $A = \mathbb{C}[\varepsilon]$. Then $\mathrm{Mod}^1_{\mathbb{C}[\varepsilon]}$ is a single orbit

There are a number of interesting consequences of the theorem. We just mention a few. The interested reader can find a more detailed account of these topics with many examples and references to the literature in [**Kra82**].

COROLLARY 6.4.3. *Let M be an n-dimensional A-module. If $\mathrm{Ext}^1_A(M, M) = 0$, then the orbit C_M is open in Mod^n_A, and thus $\overline{C_M}$ is an irreducible component. In particular, the projective (injective) A-modules in Mod^n_A form a finite union of open orbits.*

The proof is easy and is left as an exercise.

COROLLARY 6.4.4. *Let A be a finitely generated algebra with the property that all finite dimensional A-modules are semisimple. Then there are only finitely many isomorphism classes of A-modules of a given dimension. More precisely, the* Mod_A^n *is a finite union of open and closed orbits for every n.*

This follows from the previous corollary, because $\mathrm{Ext}^1(N, M) = 0$ for all finite dimensional A-modules N, M. Typical examples of such algebras are the enveloping algebras of finite dimensional semisimple Lie algebras.

7. Structure of the Quotient

In this section we gather together a few general results about algebraic quotients $\pi\colon Z \to Z/\!\!/G$ where Z a G-variety and G is a reductive group. Some of these we will either be prove or have already been proved. But our methods are not sufficient for all of them, and in those cases we refer to the literature and might give some ideas for the proof.

7.1. Inheritance properties. We start by recalling some properties which are carried over from Z to the quotient $Z/\!\!/G$, see section 3.

(1) If Z is *irreducible* or *normal*, then so is $Z/\!\!/G$ (Proposition 3.1.1).
(2) If Z is *factorial* and G is *semisimple*, i.e. connected with trivial character group, then $Z/\!\!/G$ is factorial (Proposition 3.1.4 and Remarl 3.1.5).
(3) If Z is *smooth*, then Y has the COHEN-MACAULAY *property.*
(This result is due to HOCHSTER-ROBERTS [**HR74**]; the proof is very complicated and was later simplified by KEMPF.)
(4) If Z has *rational singularities*, then so does $Z/\!\!/G$.
(Theorem of BOUTOT [**Bou87**]; this generalizes the result of HOCHSTER-ROBERTS mentioned above.)

7.2. Singularities in the quotient. Suppose V is a vector space with a linear G-action and $\pi\colon V \to Y := V/\!\!/G$ is the quotient.

LEMMA 7.2.1. *If $\pi(0) \in Y$ is a smooth point, then $Y \simeq \mathbb{C}^m$ for some $m \in \mathbb{N}$.*

PROOF. (Cf. Example 3.1.3) This follows directly from Lemma 2.3.3: The algebra $A = \mathcal{O}(Y)$ is graded with $A_0 = \mathbb{C}$, and $\mathfrak{n} := \mathfrak{m}_{\pi(0)}$ is the homogeneous maximal ideal. If we now choose homogeneous $a_1, \dots, a_m \in \mathfrak{n}$ with the property that the images $\overline{a_i} \in \mathfrak{n}/\mathfrak{n}^2$ form a basis, then $A = \mathbb{C}[a_1, \dots, a_m]$. Since $\dim A = \dim \mathfrak{n}/\mathfrak{n}^2 = m$ we see that the a_i are algebraically independent. □

This result can also be viewed within a more general framework. First we note that the scalar multiplication on V induces a $\mathbb{C}^*$-action on the quotient $V/\!\!/G$:

$$\lambda\pi(v) := \pi(\lambda v).$$

(The proof that this is indeed a $\mathbb{C}^*$-action is left as an exercise.) This $\mathbb{C}^*$-action is the geometric interpretation of the grading of the invariant ring $\mathcal{O}(V)^G$, a property which we have already used before more than once.

PROPOSITION 7.2.2. *Let $Z \subseteq V$ be a G-stable closed cone and $\pi\colon Z \to Z/\!\!/G$ the quotient. Then the singular (resp. the non-normal) points in $Z/\!\!/G$ form a closed cone. In particular, $Z/\!\!/G$ is smooth (resp. normal) if and only if $\pi(0)$ is a smooth (resp. normal) point of $Z/\!\!/G$.*

(Here we are calling a subset of V resp. $V/\!\!/G$ a *cone* if it is $\mathbb{C}^*$-stable. It is then clear that the singularities resp. the non-normal points of $Z/\!\!/G$ form a cone; the closedness follows from Corollary A.4.10.6 resp. from Proposition A.5.2.6.)

The $\mathbb{C}^*$-action on the quotient $V/\!\!/G$ is a special case of the following result.

LEMMA 7.2.3. *Suppose Z is a G-variety on which another group H acts compatibly with G. Then H also acts on the quotient $Z/\!\!/G$ and $\pi\colon Z \to Z/\!\!/G$ is H-equivariant.*

PROOF. Since the H-action commutes with the G-action it follows that the H-action on $\mathcal{O}(Z)$ respects the isotypic components. In particular, H induces an action on the invariants $\mathcal{O}(Z)^G$, hence on $Z/\!\!/G$. $\square$

We now give another criterion in order for a quotient to have no singularities. For the sake of simplicity we again consider a vector space V together with a linear G-action.

PROPOSITION 7.2.4. *If the null fiber $\mathcal{N}_V$ has dimension $\dim V - \dim V/\!\!/G$ and if $\mathcal{N}_V$ is reduced at some smooth point $z \in \mathcal{N}_V$, then the quotient $V/\!\!/G$ is an affine space.*

OUTLINE OF THE PROOF. By assumption, the differential $d\pi$ has maximal rank in $z \in \mathcal{N}_V$ and thus also in some neighborhood U of z in V. This implies that $\pi|_U\colon U \to \pi(U)$ is a smooth mapping, i.e. it looks locally in the analytic sense like the projection of a vector space onto a subspace. Since the quotient $V/\!\!/G$ is normal, and, in particular, unibranched, the image $\pi(U)$ is an open subset of $V/\!\!/G$ which contains $\pi(0)$. The result now follows with the help of Lemma **??**. $\square$

REMARK 7.2.5. It is conjectured that the dimension assumption, i.e., the equidimensionality of the quotient map π (see Corollary 5.3.6), is enough to imply that the quotient is an affine space. This came to be known as the "Russian Conjecture". The conjecture holds for irreducible representations of simple groups as a consequence of the classification results of G. SCHWARZ [**Sch78, Sch79**], and also for irreducible representations of semisimple groups due to the classification results of P. LITTELMANN [**Lit89**].

7.3. Smooth quotients. We have already established that a one-dimensional quotient $V/\!\!/G$ is isomorphic to the affine line $\mathbb{C}$ (Remark 3.2.5). If $V/\!\!/G$ is two-dimensional, then it follows from the normality of $V/\!\!/G$ and the lemma above that either $\pi(0) \in V/\!\!/G$ is an *isolated singularity* or $V/\!\!/G$ is the *affine plane* $\mathbb{C}^2$.

The example $\mathbb{C}^2/\!\!/(\mathbb{Z}/2)$ where $\mathbb{Z}/2$ acts by $\pm\,\mathrm{id}$ shows that $V/\!\!/G$ can indeed be singular. However, for a semisimple group G this does not happen, as conjectured by V.L. POPOV.

PROPOSITION 7.3.1 (G. KEMPF [**Kem80**]). *If G is semisimple and V is a G-module with $\dim V/\!\!/G = 2$, then $V/\!\!/G$ is isomorphic to $\mathbb{C}^2$.*

Remark 3: In this connection there are a number of classification results: (V. Kac, V.L. Popov, E.B. Vinberg, G. Schwarz, M. Sato, T. Kimura,... [1]; cf. [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]). Among other things one can find the list of all representations V of *simple*[2] groups G in [**?**] with the property that $V/\!\!/G$ is an affine space (*coregular representations*) [3] and in [**?**] the list of all *irreducible* representations of *semi–simple* groups with the property that V° only contains a finite number of orbits (*observable representations*, cf. [**?**]).

[1] Popov, p.145: cf. O.M. ADAMOVICH - E.O. GOLOVINA [**?**], O.M. ADAMOVICH [**?**]–Footnote of the Russian editor.

[2] Popov, p.145: connected–footnote of the Russian editor.

[3] Popov, p.145: The general method of determining such representations such that $V/\!\!/G$ is an affine space (i.e. the algebra of invariants is free), was first worked out by V. Kac, V.L. Popov and E.B. Vinberg in [**?**], where this was found with the help of irreducible representations of connected simple groups. In [**?**] G. Schwarz, by applying this method, analyzed the reducible case also. Simultaneously and independently this case was investigated by O.M. ADAMOVICH - E.O. GOLOVINA [**?**].–footnote of the Russian editor.

7.4. Semi-continuity statements. In studying the quotient map $\pi : Z \to Z/\!\!/G$ one is often interested in whether the set of points $y \in Z/\!\!/G$ for which the fiber $\pi^{-1}(y)$ has a certain property is *open* in $Z/\!\!/G$. We now want to discuss this problem.

LEMMA 7.4.1. *The function $d : Z/\!\!/G \to \mathbb{N},\ y \mapsto \dim\ \pi^{-1}(y)$ is upper semi–continuous.*

Proof: We have to show that for every $n \in \mathbb{N}$ the set

$$Y' := \{y \in Z/\!\!/G \mid \dim\ \pi^{-1}(y) \geq n\}$$

is closed in $Z/\!\!/G$. By the Theorem of Chevalley (**??**) the set

$$Z' := \{z \in Z \mid \dim_z\ \pi^{-1}(\pi(z)) \geq n\}$$

is closed in Z. As well Z' is G–stable and $Y' = \pi(Z')$. The result now follows from the G–closedness of quotients (**??**).

As an application we have the following result.

THEOREM 7.4.2. *Suppose $\pi : Z \to Z/\!\!/G$ is a quotient. Then the set*

$$\{y \in Z/\!\!/G \mid \pi^{-1}(y) \text{ consists of finitely many orbits}\}$$

is open in $Z/\!\!/G$.

Proof: Suppose $S \subseteq Z$ is a stratum (**??**; w.l.o.g. assume G is connected) made up of orbits of dimension n and let $\overline{S}$ be its closure. Then $\pi(\overline{S}) \subseteq Z/\!\!/G$ is closed and $\pi' : \overline{S} \to \pi(\overline{S})$ is a quotient (**??**). Suppose $y \in \pi(\overline{S})$. If $\dim\ \pi'^{-1}(y) > n$, then $\pi^{-1}(y)$ contains infinitely many orbits of dimension n. Conversely, if $\pi^{-1}(y)$ contains infinitely many orbits of dimension n, then $\dim\ \pi'^{-1}(y) > n$. The complement of the set which is given in the statement of the theorem is thus the union of the sets

$$\{y \in \pi(\overline{S}) \mid \dim(\pi^{-1}(y) \cap \overline{S}) > n_S\},$$

where S runs through all strata and n_S is the orbit dimension of S. By Lemma 7.4.1 these are all closed and the result follows.

THEOREM 7.4.3. *Under the same assumptions as in Theorem 7.4.2 the set*

$$\{y \in Z/\!\!/G \mid \text{the fiber } \pi^{-1}(y) \ \text{ is reduced and normal}\}$$

is open in $Z/\!\!/G$.

Outline of the Proof: This assertion rests on the following result ([**?**, IV, 12.1.7]): If $\eta : Z \to Y$ is a morphism, then the set

$$Z' := \{z \in Z \mid \eta^{-1}(\eta(z)) \quad \text{is reduced and normal at } z\}$$

is open in Z. In the above situation the set Z' is thus G–stable and the assertion follows from the G–closedness of quotients (**??**) applied to $Z - Z'$.

Remark 4: There is a whole list of other properties for which the analogue to Theorem 7.4.3 holds, e.g. *reduced, no singularities, Cohen–Macaulay, rational singularities, etc.* (cf. [**?**, IV, 12.1.7] and [**?**]).

7.5. Generic fiber. Here it is a question of which properties of a general fiber carried over *a priori* via the quotient map.

THEOREM 7.5.1 (Luna, Popov). *Suppose G is semi–simple, V is a G–module and $\pi : V \to V/\!\!/G$ is the quotient. Then the generic fiber contains a dense orbit.*[4] *Moreover, the generic orbit is closed if and only if it is affine.*

COROLLARY 7.5.2. *If the generic stabilizer is finite, i.e. the maximal orbit dimension is equal to* $\dim\ G$, *then the generic fiber of π is a closed orbit.*

Outline of the Proof: For the first assertion of the theorem we refer to the literature ([**?**, III.4]; cf. **??** E). Concerning the second assertion[5], if the generic fiber is closed, then it is naturally affine. Conversely, suppose the generic orbit is affine. If it were not closed, then the complement would be of codimension one in its closure. Then the union of these complements would have a G–stable hypersurface H in V as closure and this would be the zero set of an invariant function f. This is clearly a contradiction, since $\pi(H)$ is dense in $V/\!\!/G$. The result is now clear.

Various results can be found in the literature about generic orbits, stabilizers and fibers (E.M. Andreev, E.B. Vinberg, A.G. Elashvili, A.M. Popov, ... cf. [**?**], [**?**]).[6]

E. Invariant Rational Functions

An important result of Rosenlicht[7] says that in every irreducible G–variety Z there is an *open, dense G–stable subset Z'* whose orbits *are separated by the G–invariant rational functions* defined on Z', i.e. there is a morphism

$$\varphi : Z' \to Y^{8},$$

whose fibers are exactly the orbits. (Note that Z' is, in general, not affine.) In particular, the transcendence degree of $\mathbb{C}(Z)^G$ is equal to the *"dimension" of the family of orbits of maximal dimension*:

$$\mathrm{tr\ deg}_{\mathbb{C}}\ \mathbb{C}(Z)^G \quad = \quad \dim\ Z - \max_{z\in Z}(\dim\ Gz)$$

(cf. [**?**, III.4]). We would now like to give a general proof for a special case of this result. Note that we already know this in the setting of tori (**??** Theorem **??**).

THEOREM 7.5.3. *Suppose Z is an irreducible G–variety. Then $\mathbb{C}(Z)^G = \mathbb{C}$ if and only if Z contains a dense orbit.*

Proof: (by D. Luna) If $Gz \subseteq Z$ is a dense orbit, then every rational invariant function is constant on Gz and thus on Z. Hence one direction of the proof is clear. For the converse consider the map

$$\varphi : G \times Z \to Z \times Z,\ (g, z) \mapsto (gz, z).$$

We want to show that φ is dominant, i.e. that

$$\varphi^* : \mathcal{O}(Z) \otimes \mathcal{O}(Z) \to \mathcal{O}(G) \otimes \mathcal{O}(Z)$$

[4] Popov, p.147: To say that some property is satisfied generically in a variety X means that in X there is a dense, open subset Ω, which depends on the condition under consideration, such that for every point $x \in \Omega$ the condition holds. In the given case this means that there is an open, dense subset Ω in $V/\!\!/G$ such that for every $\xi \in \Omega$ the fiber $\pi^{-1}(\xi)$ contains a dense orbit. The notions of generic orbit and generic stabilizer have analogous interpretations.–footnote of the Russian translator.

[5] Popov, p.147: See V.L. Popov [**?**].–Footnote of the Russian translator.

[6] Popov, p.147: See also the works [**?**],[**?**],[**?**],[**?**]–footnote of the Russian editor.

[7] Popov, p. 147: cf. [**?**]–footnote of the Russian editor.

[8] Popov, p.147: Here Y is a certain variety which is different, generally speaking, from $Z/\!\!/G$.–footnote of the Russian editor.

is injective. By definition one has

$$\varphi^*(f \otimes h)(g,z) \;=\; f(gz)\cdot h(z) \;=\; ((g^{-1}f)\cdot h)(z).$$

Now suppose $\varphi^*\left(\sum_{i=1}^s f_i \otimes h_i\right) = 0$, where w.l.o.g. we may assume that the $f_1, ..., f_s$ are linearly independent over $\mathbb{C}$. Then

$$(*) \qquad \sum_{i=1}^{s} (gf_i)\cdot h_i \;=\; 0 \text{ for every } \quad g \in G.$$

Suppose $V = \langle f_1, ..., f_s\rangle \subseteq \mathbb{C}(Z)$. Since $\mathbb{C}$ is the fixed field of G in $\mathbb{C}(Z)$, the theorem of Artin (cf. [**?**, Chap. V, §7, théorème 1)]) asserts that there exist $s := \dim\ V$ elements $g_1, ..., g_s \in G$ whose restrictions $g_i|_V : V \to \mathbb{C}(Z)$ are linearly independent over $\mathbb{C}(Z)$. This means that the matrix $(g_j f_i)_{i,j=1}^s$ has rank s.[9] Hence in (*) one has $h_i = 0$ for every i and thus φ^* is injective and φ is dominant.
Now for $g \in G$ and $z \in Z$

$$\varphi^{-1}(\varphi(g,z)) \;=\; \{\ (h,z) \mid hz = gz\} \xrightarrow{\sim} G_z,$$

where the isomorphism is given by $(h,z) \mapsto g^{-1}h$. By the dimension formula for fibers (**??**)[10] one thus has

$$\min_{z\in Z}\ \dim G_z \;=\; \dim\ G - \dim\ Z$$

and thus

$$\max_{z\in Z}\ \dim\ Gz \;=\; \dim\ Z,$$

i.e., Z has a dense orbit.

7.6. A finiteness theorem. In conclusion we give a result of Hilbert ([**?**, Kap.I, §4]). This shows how information about the zero fiber V° can lead to information about the ring of invariants.

THEOREM 7.6.1. *Suppose G is linear reductive and connected and V is a G–module. If $f_1, ..., f_t$ are homogeneous invariant functions, which define the zero fiber V°, i.e. $\mathcal{V}(f_1, ..., f_t) = V^\circ$, then $\mathcal{O}(V)^G$ is a finite module over $\mathbb{C}[f_1, ..., f_t]$, namely the integral closure of $\mathbb{C}[f_1, ..., f_t]$ in $\mathcal{O}(V)$.*[11]

Proof: We set

$$R \;:=\; \mathcal{O}(V)^G = \bigoplus_{i\geq 0} R_i, \quad \mathfrak{m} \;:=\; \bigoplus_{i>0} R_i.$$

By the Nullstellensatz (**??**) one has $\sqrt{\sum_i Rf_i} = \mathfrak{m}$. Thus $\mathfrak{m}^N \subseteq \sum_{i=1}^t Rf_i$ for some $N > 0$. Letting $d_i := \deg\ f_i$ one then gets from this

$$R_n \subseteq \sum_{i=1}^{t} f_i R_{n-d_i} \quad \text{for} \quad n \geq N.$$

Therefore we consider the finite dimensional vector space $B := \bigoplus_{i=0}^{N-1} R_i$, and by induction on n we get

$$R_n \subseteq \mathbb{C}[f_1, ..., f_t]\cdot B \quad \text{for every} \quad n$$

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[9] Popov, p.148: over the field $\mathbb{C}(Z)$.–footnote of the Russian editor.

[10] Popov, p.149: In view of the dominance of the morphism φ the minimum dimension of the fiber of this morphism is equal to $\dim(G\times Z) - \dim(Z \times Z) = \dim\ G - \dim\ Z$.–Footnote of the Russian editor.

[11] Popov, p.149: The converse is also true: If $\mathcal{O}(V)^G$ is integral over a subalgebra generated by a system I of homogeneous non–constant invariants, then

$$V^\circ = \{v \in V \mid f(v) = 0 \text{ for every } f \in I\}$$

–Footnote of the Russian editor.

and the result follows from this. (cf. **??** Remark 2).

8. Quotients for Non-Reductive Groups

(Separating morphisms, Rosenlicht, generic fibers, ...)

Exercises

For the convenience of the reader we collect here all exercises from Chapter VI.

CHAPTER V

Representation Theory and U-Invariants

Contents

1. Representations of Linearly Reductive Groups

1.1. Commutative and Diagonalizable Groups. Recall our previous notation: $B_n \subseteq \mathrm{GL}_n(\mathbb{C})$ denotes the subgroup of upper triangular matrices, $T_n \subseteq B_n$ the subgroup of diagonal matrices and $U_n \subseteq B_n$ the subgroup of unipotent matrices (with 1's along the diagonal).

$$B_n := \left\{ \begin{bmatrix} * & * & \cdots & * \\ & * & \cdots & * \\ & & \ddots & \\ & & & * \end{bmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\}, \qquad T_n := \left\{ \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\},$$

$$U_n := \left\{ \begin{bmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & * \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\},$$

We have an obvious surjective homomorphism $p\colon B_n \to T_n$ which is the identity on T_n and has kernel U_n, and so $B_n = T_nU_n = U_nT_n$ and the multiplication $T_n \times U_n \to B_n$ is an isomorphism of varieties.

LEMMA 1.1.1. *Let $H \subseteq \mathrm{GL}_n(\mathbb{C})$ be an arbitrary commutative subgroup.*

(1) *There is a $g \in \mathrm{GL}_n(\mathbb{C})$ such that $gHg^{-1} \subseteq B_n$.*

(2) *If H consists of semisimple elements, then there is a $g \in \mathrm{GL}_n(\mathbb{C})$ such that $gHg^{-1} \subseteq T_n$.*

Proof. $\square$

Proposition 1.1.2. *Let H be a commutative algebraic group. Define*

$$H_u := \{h \in H \mid h \text{ unipotent}\}, \quad H_s := \{h \in H \mid h \text{ semisimple}\}.$$

Then $H_u, H_s \subseteq H$ are closed subgroups and the multiplication induces an isomophism $H_s \times H_u \xrightarrow{\sim} H_s H_u = H$.

Proof. $\square$

Recall the definition of the character group of an algebraic group G:

$$X(G) := \{\chi \colon G \to \mathbb{C}^* \mid \chi \text{ is a homomorphism}\} \subseteq \mathcal{O}(G)^*.$$

Lemma 1.1.3. *The subset $X(G) \subseteq \mathcal{O}(G)$ is linearly independent and the linear span $\langle X(G) \rangle \subseteq \mathcal{O}(G)$ is the group algebra of $X(G)$.*

Exercise 1.1.4. If G is connected, then $X(G)$ is torsion free.

Exercise 1.1.5. Show that $X(G)$ is a finitely generated abelian group of rank $\leq \dim G$.

It is easy to see that $X(G)$ is a contravariant functor from algebraic groups to abelian groups and that $X(G \times H) = X(G) \times X(H)$. Moreover, the comultiplication $\mu^* \colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ induces the diagonal map on $X(G) \to X(G) \times X(G)$: $\mu^*(\chi) = \chi \otimes \chi$.

Proposition 1.1.6. *An algebraic group H is diagonalizable, i.e. isomorphic to a closed subgroup of T_n, if and only if $\mathcal{O}(H) = \langle X(H) \rangle$.*

Theorem 1.1.7. *The functor X defines an anti-equivalence between the diagonalizable groups and the finitely generated abelian groups. This means that every finitely generated abelian group is isomorphic to the character group of a diagonalizable group and that the natural map $\mathrm{Hom}(D, E) \xrightarrow{\sim} \mathrm{Hom}(X(E), X(D))$ is an isomorphism of groups. Moreover, a sequence of diagonalizable group*

$$1 \to D' \to D \to D'' \to 1$$

is exact if and only if the induced sequence $0 \to X(D'') \to X(D) \to X(D') \to 0$ is exact.

1.2. Unipotent Groups.

1.3. Solvable Groups.

1.4. Representation theory of GL_n.

1.5. Representation theory of reductive groups.

2. Characterization of Reductive Groups

2.1. Definitions.

Definition 2.1.1. An algebraic group G is called *linearly reductive* if every representation of G is completely reducible.

There is also the notion of a *reductive group* which we will not introduce here. In our situation where the base field has characteristic zero the two definitions are equivalent. This allows us to use the shorter notion "reductive" instead of "linearly reductive".

EXAMPLE 2.1.2. (1) The multipliative group $\mathbb{C}^*$ is reductive. More generally, every diagonalizable D group is reductive, because every representation of D is diagonalizable (see Proposition 3.4.5(3)).
(2) The famous Theorem of Maschke says that every finite G group is reductive. (One shows that for every representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ there exists an G-invariant unitarien scalar product on V.)
(3) The additive group $\mathbb{C}^+ \simeq U_2$ is not reductive. (The standard representation on $\mathbb{C}^2$ is not completely reducible, because $\mathbb{C}e_1$ is the only U_2-stable subspace.)

The next lemma is a very useful criterion for reductivity. We say that a locally finite and rational representation of G on W is completely reducible if every finite dimensional G-stable subspace of W is semisimple. Clearly, if G is reductive and X a G-variety, then the regular representation of G on $\mathcal{O}(X)$ is completley reducible.

LEMMA 2.1.3. *Let G be an algebraic group and V a faithful G-module, i.e.* $\mathrm{C}_G(V) = \{e\}$. *The following assertions are equivalent:*
(i) *G is reductive.*
(ii) *The regular representation of G on $\mathcal{O}(G)$ is completely reducible.*
(iii) *For all $n \geq 1$ the G-module $V^{\otimes n} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$ is semisimple.*

PROPOSITION 2.1.4. (1) *Let G be a reductive group and N a closed normal subgroup. Then N is reductive.*
(2) *Let $1 \to N \to G \to H \to 1$ be an exact sequence of algebraic groups. Then G is reductive if and only if N and H are both reductive.*
(3) *A product of algebraic groups is reductive if and only if each factor is reductive.*

EXERCISE 2.1.5. A connected solvable algebraic group is reductive if and only if it is isomorphic to a torus T_n.

2.2. Images and kernels. We first study the behavior of linearly reductive groups under homomorphisms.

PROPOSITION 2.2.1. *Let G be an algebraic group, $\varphi\colon G \to H$ a homomorphism, and $N \subseteq G$ a closed normal subgroup.*
(1) *If G is linearly reductive, then so are N and $\varphi(G)$.*
(2) *If $\ker\varphi$ and $\varphi(G)$ are both linearly reductive, then so is G.*

PROOF. (a) It is clear that every homomorphic image of G is again linearly reductive. Suppose $H \subseteq G$ is a normal subgroup. The restriction map $\mathcal{O}(G) \to \mathcal{O}(H)$ is a surjective H-homomorphism. Because of Proposition 2.1.6 above it suffices to prove that $\mathcal{O}(G)$ is a semisimple H-module. To do this consider the socle $S := \mathcal{O}(V)_{\mathrm{soc}} \subseteq \mathcal{O}(G)$, i.e. the sum of all simple H-submodules of $\mathcal{O}(G)$. Since H is normal in G, the socle is G-stable. In fact, if V is a simple H-submodule of $\mathcal{O}(G)$, then so is gV, for every $g \in G$. It follows that S has a G-stable complement S' (Exercise 1.3.4). Hence $S = \mathcal{O}(G)$ and the assertion is proved.

(b) Let $N := \ker\varphi$ and $\bar{G} := \varphi(G)$. It suffices to show that for every surjective G-homomorphism $\varphi\colon V \to W$ the fixed points are also surjectively mapped onto themselves. By assumption $\varphi^N\colon V^N \to W^N$ is surjective, and this is also a homomorphism of $\bar{G}$-modules (Corollary III.2.3.9). Because of $V^G = (V^H)^{\bar{G}}$ and $W^G = (W^H)^{\bar{G}}$, the result now follows. ☐

COROLLARY 2.2.2. *An algebraic group G is linearly reductive if and only if its connected component of the identity G° is linearly reductive.*

Corollary 2.2.3. *A commutative algebraic group G is linearly reductive if and only if G is diagonalizable.*

Proof. We already know that a diagonalizable group is linearly reductive (Example 2.1.2). For the other implication we recall that a commutative algebraic group G is a product $G = G_sG_u$ where G_s is diagonalizable and G_u unipotent (Proposition III.4.3.4). If $G_u \neq \{0\}$, then G contains a normal subgroup isomorphic to $\mathbb{C}^+$, contradicting the linear reductivity. □

Corollary 2.2.4. *Let G be a linearly reductive group. If G is solvable, then G° is a torus.*

Recall that a group G is *solvable* if there is a normal series $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{m-1} \supseteq G_m = \{e\}$ such that G_i/G_{i+1} is commutative. In particular, (G, G) is a strict subgroup of G. We will discuss solvable algebraic groups in detail in chapter V.

Proof. We can assume that G is connected. We proceed by induction on $\dim G$. If $\dim G = 1$ then G is commutative (1.4.8), and we are done by the previous corollary. In general, (G, G) is a connected normal subgroup which is again solvable. Hence, by induction, (G, G) is a torus. Since $G/(G, G)$ is commutative and reductive it is also a torus. Thus G is a torus, by Corollary 3.4.6. □

2.3. Semisimple groups.

Definition 2.3.1. A linearly reductive group G is *semisimple* if G is connected and has no non-trivial character, i.e. the character group $\mathcal{X}(G)$ is trivial.

We will show in the following section 2.4 that the classical groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ for $n \geq 3$, and $\mathrm{Sp}_{2m}(\mathbb{C})$ are semisimple. They all have a finite center. This is a general fact.

Proposition 2.3.2. *For a connected linearly reductive group G the following statements are equivalent:*

(i) *G is semisimple.*
(ii) *The commutator subgroup satisfies $(G, G) = G$.*
(iii) *The center $\mathrm{Z}(G)$ of G is finite.*

For the proof we need the following lemma.

Lemma 2.3.3. *If G is linearly reductive, then its Lie algebra satisfies*

$$\mathrm{Lie}\, G = [\mathrm{Lie}\, G, \mathrm{Lie}\, G] \oplus \mathfrak{z}(\mathrm{Lie}\, G).$$

Proof. We may assume that G is connected. Now $\mathfrak{a} := [\mathrm{Lie}\, G, \mathrm{Lie}\, G]$ is an ideal in $\mathfrak{g} := \mathrm{Lie}\, G$ and thus is stable under the adjoint representation of G on $\mathfrak{g}$ (see Proposition III.5.3.2). Choose a G-stable splitting $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Then $\mathfrak{b}$ is an ideal of $\mathfrak{g}$, and one has $[\mathfrak{g}, \mathfrak{b}] \subseteq \mathfrak{b} \cap [\mathfrak{g}, \mathfrak{g}] = (0)$. This implies $\mathfrak{b} \subseteq \mathfrak{z}(\mathfrak{g})$ and thus $\mathfrak{g} = \mathfrak{a} + \mathfrak{z}(\mathfrak{g})$. In particular, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{a}, \mathfrak{a}]$.

We still have to prove that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a} = \mathfrak{z}(\mathfrak{a}) = (0)$. Not only is $\mathfrak{a}$ stable under G, but also $\mathfrak{z}(\mathfrak{a})$. Thus there is a G-stable splitting $\mathfrak{a} = \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{c}$. But this implies $\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}] = [\mathfrak{c}, \mathfrak{c}] \subseteq \mathfrak{c}$ and the result follows. □

Remark 2.3.4. One has $\mathrm{Lie}(G, G) \supseteq [\mathrm{Lie}\, G, \mathrm{Lie}\, G]$, since on the one hand $G/(G, G)$ is commutative and on the other hand $[\mathfrak{g}, \mathfrak{g}]$ is the smallest ideal $\mathfrak{a}$ of $\mathrm{Lie}\, G$ so that $\mathrm{Lie}\, G/\mathfrak{a}$ is commutative (Exercise III.5.3.4). Thus the lemma above implies that for a connected linearly reductive G one has $G = (G, G) \cdot \mathrm{Z}(G)^\circ$.

Proof of Proposition 2.3.2. (i) ⇒ (ii): Since $G/(G,G)$ is commutative and linearly reductive by Proposition 2.2.1 above, it is a connected diagonalizable group (Corollary 2.2.3), hence a torus. Thus $G = (G,G)$, because G has no non-trivial characters.

(ii) ⇒ (iii): If $\rho\colon G \to \mathrm{GL}(V)$ is an irreducible representation, then the center $\mathrm{Z}(G)$ acts by scalars on V. On the other hand, because $G = (G,G)$, one has $\rho(G) \subseteq \mathrm{SL}(V)$. This implies $\rho(\mathrm{Z}(G)) \subseteq \mathrm{SL}(V) \cap \mathbb{C}^* \,\mathrm{Id}$ and thus $\rho(\mathrm{Z}(G))$ is finite. Because of the complete reducibility this is true for every representation of G and the assertion follows by considering a faithful representation $G \hookrightarrow \mathrm{GL}_n$.

(iii) ⇒ (i): One has $\mathfrak{z}(\mathrm{Lie}\, G) = \mathrm{Lie}(\mathrm{Z}(G)) = (0)$ (Corollary III.5.3.5) and thus $\mathrm{Lie}\, G = [\mathrm{Lie}\, G, \mathrm{Lie}\, G]$ by Lemma 2.3.3 above. If $\chi\colon G \to \mathbb{C}^*$ is a character, then $d\chi_e : \mathrm{Lie}\, G \to \mathbb{C}$ is a Lie algebra homomorphism with commutative image and so $\mathrm{Lie}\, G = [\mathrm{Lie}\, G, \mathrm{Lie}\, G] \subseteq \ker d\chi_e$. This implies that $d\chi_e = 0$ and hence χ is trivial. (Proposition III.5.2.1). □

Proposition 2.3.5. *Suppose G is linearly reductive and connected. Then*

1. *(G,G) and $G/\mathrm{Z}(G)$ are both semisimple;*
2. *$\mathrm{Z}((G,G)) = \mathrm{Z}(G) \cap (G,G)$ is finite;*
3. *$G = (G,G) \cdot \mathrm{Z}(G)^\circ$;*
4. *$\mathrm{Lie}(G,G) = [\mathrm{Lie}\, G, \mathrm{Lie}\, G]$.*

Proof. Let $G' := (G,G)$.

(1) Then (G',G') is a normal subgroup of G and the quotient group $G/(G',G')$ is solvable and linearly reductive and therefore a torus (Corollary 2.2.4). This implies $(G',G') = G'$ which proves that (G,G) is semi-simple (Proposition 2.3.2).

Because of $G = (G,G) \cdot \mathrm{Z}(G)$ (Remark 2.3.4) one has for $\overline{G} := G/\mathrm{Z}(G)$ the relation $(\overline{G}, \overline{G}) = \overline{G}$, and hence $\overline{G}$ is also semi-simple.

(2) Since $\mathrm{Z}(G) \cap (G,G)$ is the center of (G,G), it is finite (Proposition 2.3.2).

(3) is already stated in Remark 2.3.4.

(4) By (2) we have $\mathrm{Lie}(G,G) \cap \mathrm{Lie}\,\mathrm{Z}(G) = (0)$. Since $\mathrm{Lie}(G,G) \supseteq [\mathrm{Lie}\, G, \mathrm{Lie}\, G]$ (Remark 2.3.4) the claim follows from Lemma 2.3.3. □

2.4. The classical groups. Now we show that the classical groups are all linearly reductive. We begin with two lemmas.

Lemma 2.4.1. *Let G be an algebraic group and V a faithful G-module. If all the tensor powers $V^{\otimes m}$, $m \in \mathbb{N}$, are semisimple, then G is linearly reductive.*

Proof. By assumption, the tensor algebra $T(V) := \bigoplus_{j \geq 0} V^{\otimes j}$ is semisimple as well as its quotient module $S(V)$, the symmetric algebra. Now consider $\mathrm{End}(V)$ as a G-module with respect to the right multiplication: $\mathrm{End}(V) = V \otimes V^* = (V^*)^{\oplus n}$ where $n = \dim V$. Then $\mathcal{O}(\mathrm{End}(V)) = \mathcal{O}((V^*)^{\oplus n}) = S(V)^{\otimes n}$, and thus $\mathcal{O}(\mathrm{End}(V))$ is a semisimple G-module as well.

Since V is a faithful G-module we have an embedding $G \subseteq \mathrm{GL}(V) \subseteq \mathrm{End}(V)$ which is $G \times G$-equivariant. If we denote by $\chi \in \mathcal{X}(G)$ the character induced by $\det\colon \mathrm{GL}(V) \to \mathbb{C}^*$, we have $\mathcal{O}(G) = \mathcal{O}(\mathrm{End}(V))_\chi$. Now let W be any representation of G. Then W occurs in $\mathcal{O}(G)^{\oplus m}$ as a G-submodule (with respect to the right multiplication) for some $m > 0$ (2.4.3). This implies that for a large enough k the module $\mathbb{C}_{k\chi} \otimes W$ occurs in $\mathcal{O}(\mathrm{End}(V))^{\oplus m}$ and thus W is semisimple. □

In the next lemma we use *hermitian forms* $\langle v, w\rangle$ on complex vector spaces V, see [**Art91**, 7.4].

Lemma 2.4.2. *Let V be a G-module, and let $\langle v, w\rangle$ be a hermitian form on V. Assume that there is a map ${}^*\colon G \to G$ such that $\langle gv, w\rangle = \langle v, g^* w\rangle$ for all $g \in G$ and $v, w \in V$. Then the tensor powers $V^{\otimes m}$ are semisimple G-modules.*

PROOF. The hermitian form on V induces a hermitian form on all tensor powers $V^{\otimes m}$ in the usual way,

$$\langle v_1 \otimes v_2 \otimes \cdots \otimes v_m, w_1 \otimes w_2 \otimes \cdots \otimes w_m \rangle := \langle v_1, w_1 \rangle \cdot \langle v_2, w_2 \rangle \cdots \langle v_m, w_m \rangle,$$

which again satisfies $\langle gp, q \rangle = \langle p, g^* q \rangle$ for $g \in G$ and $p, q \in V^{\otimes m}$. Now it follows that for every G-stable subspace $W \subseteq V^{\otimes m}$ the orthogonal complement $W^\perp$ with respect to the hermitian form is G-stable. Hence $V^{\otimes m}$ is semisimple. □

THEOREM 2.4.3. *The classical groups* GL_n, SL_n, O_n, SO_n *and* Sp_{2m} *are linearly reductive.*

PROOF. For $A \in \mathrm{M}_n$ define $A^* := \bar{A}^t$, the conjugate transpose of A. If $\langle v, w \rangle$ is the standard hermitian form on $\mathbb{C}^n$, we have

$$\langle Av, w \rangle = \overline{(Av)}^t w = (\bar{v})^t A^* w = \langle v, A^* w \rangle.$$

Moreover, $(AB)^* = B^* A^*$. Using the two lemmas above it remains to check that for every classical group $G \subseteq \mathrm{M}_n$ we have $g^* \in G$ for any $g \in G$. This is obvious for GL_n. For the others it suffices to show that $g^t \in G$ and that $G \subseteq \mathrm{M}_n$ is defined by equations with real coefficients. This is clear for the SL_n, O_n and SO_n. For Sp_{2m} we remark that the equation $g^t J g = J$ together with $J^{-1} = -J$ implies that $g^t J = (-gJ)^{-1}$ and so $g^t J$ and gJ commute. Thus $gJg^tJ = g^tJgJ = J^2 = -E$ and so $g^t J g = J$. □

COROLLARY 2.4.4. *The classical groups* SL_n, SO_n $(n \geq 3)$ *and* Sp_{2m} *are semisimple.*

In fact, we have seen in II.3 that these groups are connected and have a finite center. Thus the claim follows from Theorem 2.4.3 above and Proposition 2.3.2.

2.5. Reductivity of the classical groups.

3. Hilbert's Criterion

3.1. One-parameter subgroups.

3.2. Torus actions.

3.3. Hilbert's Criterion for GL_n.

3.4. Hilbert's Criterion for reductive groups.

4. U-Invariants and Normality Problems

Exercises

For the convenience of the reader we collect here all exercises from Chapter V.

APPENDIX A

Basics from Algebraic Geometry

In this appendix we gather together some notions and results from algebraic geometry which have been used in the text. We concentrate on *affine* algebraic geometry which simplifies a lot the notational part and makes the subject much easier to access in a first attempt. In the second appendix, we discuss the relation between the Zariski topology and the $\mathbb{C}$-topology. With its help we are able to use certain compactness arguments replacing the corresponding results from projective geometry.

The appendix assumes a basic knowledge in commutative algebra. Although we give complete proofs for almost all statements they are mostly rather short. This was done on purpose. For advanced readers we only wanted to recall briefly the basic facts, while beginners are going to find a more detailed study of algebraic geometry is necessary. We recommend the text books [**Har77**], [**Mum99**], [**Mum95**], [**Sha94a, Sha94b**] and the literature cited below. As a substitute we have presented many examples which should make the new ideas clear and with which one can check the results. In addition, a number of exercises are included. The reader is advised to look at them carefully; some of them will be used in the proofs.

Contents

1. Affine Varieties

1.1. Regular functions. Our base field is the field $\mathbb{C}$ of complex numbers. Every polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ can be regarded as a $\mathbb{C}$-valued function on $\mathbb{C}^n$ in the usual way:

$$a = (a_1, \ldots, a_n) \mapsto p(a) = p(a_1, \ldots, a_n).$$

These functions will be called *regular*. More generally, let V be a $\mathbb{C}$-vector space of dimension $\dim V = n < \infty$.

Definition 1.1.1. A $\mathbb{C}$-valued function $f \colon V \to \mathbb{C}$ is called *regular* if f is given by a polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ with respect to one and hence all bases of V. This means that for a given basis $v_1, \ldots, v_n$ of V we have

$$f(a_1 v_1 + \cdots + a_n v_n) = p(a_1, \ldots, a_n)$$

for a suitable polynomial p. The algebra of regular functions on V will be denoted by .

By our definition, every choice of a basis $(v_1, v_2, \ldots, v_n)$ of V defines an isomorphism $\mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\sim} \mathcal{O}(V)$ by identifying x_i with the i-th coordinate function on V defined by the basis, i.e.,

$$x_i(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) := a_i.$$

Another way to express this is by remarking that the linear functions on V are regular and thus the dual space $V^* := \mathrm{Hom}(V, \mathbb{C})$ is a subspace of $\mathcal{O}(V)$. So if $(v_1, v_2, \ldots, v_n)$ is a basis of V and $(x_1, x_2, \ldots, x_n)$ the dual basis of V^*, then $\mathcal{O}(V) = \mathbb{C}[x_1, x_2, \ldots, x_n]$ and the linear functions x_i are algebraically independent.

Example 1.1.2. Denote by $\mathrm{M}_n = \mathrm{M}_n(\mathbb{C})$ the complex $n \times n$-matrices so that $\mathcal{O}(\mathrm{M}_n) = \mathbb{C}[x_{ij} \mid 1 \leq i, j \leq n]$. Consider $\det(tE_n - X)$ as a polynomial in $\mathbb{C}[t, x_{ij}, i, j = 1, \ldots, n]$ where $X := (x_{ij})$. Developing this as a polynomial in t we find

$$\det(tE_n - X) = t^n - q_1 t^{n-1} + q_2 t^{n-2} - \cdots + (-1)^n q_n$$

with polynomials q_k in the variables x_{ij}. This defines regular functions $q_k \in \mathcal{O}(\mathrm{M}_n)$ which are homogeneous of degree k. Of course, we have $q_1(A) = \mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$ and $q_n(A) = \det(A)$ for any matrix $A \in \mathrm{M}_n$.

The same construction applies to $\mathrm{End}(V)$ for any finite dimensional vector space V and defines regular function $s_k \in \mathcal{O}(\mathrm{End}(V))$.

Example 1.1.3. Consider the the space of unitary polynomials of degree n:

$$P_n := \{t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_n \mid a_1, \cdots, a_n \in \mathbb{C}\} \simeq \mathbb{C}^n.$$

There is a regular function $D_n \in \mathcal{O}(P_n)$, the discriminant, with the following property: $D_n(p) = 0$ for a $p \in P_n$ if and only if p has a multiple root. E.g.

$$D_2(a_1, a_2) = a_1^2 - 4a_2, \quad D_3(a_1, a_2, a_3) = a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 a_3 + 18 a_1 a_2 a_3 - 27 a_3^2.$$

Proof. Expanding $\prod_{i=1}^n (t - y_i) = t^n - s_1(y) t^{n-1} + \cdots + (-1)^n s_n(y)$ we see that the polynomials $s_j(y)$ are the *elementary symmetric polynomials* in n variables $y_1, \ldots, y_n$, i.e.

$$s_k(y) = \sigma_k(y_1, \ldots, y_n) := \sum_{i_1 < i_2 < \cdots < i_k} y_{i_1} y_{i_2} \cdots y_{i_k}.$$

Define $\tilde{D}_n := \prod_{i<j} (y_i - y_j)^2$. Since $\tilde{D}_n$ is symmetric it can be (uniquely) written as a polynomial in the elementary symmetric functions $s_k(y)$ (see Proposition I.2.2.1, cf. [**Art91**, Chap. 14, Theorem 3.4]), $\tilde{D}_n(y_1, \ldots, y_n) = F_n(s_1, s_2, \ldots, s_n)$ with a suitable polynomial F_n. If $\lambda_1, \ldots, \lambda_n$ are the roots of $f \in P_n$, then $a_i = (-1)^i s_i(\lambda_1, \ldots, \lambda_n)$,

and so the regular function $D_n(a_1,\dots,a_n) := F_n(-a_1,a_2,-a_3,\dots,(-1)^n a_n)$ has the required property. $\square$

EXAMPLE 1.1.4. We denote by $\mathrm{Alt}_n \subseteq \mathrm{M}_n$ the subspace of *alternating matrices*:

$$\mathrm{Alt}_n := \{A \in \mathrm{M}_n \mid A^t = -A\}.$$

There is a regular function $\mathrm{Pf} \in \mathcal{O}(\mathrm{Alt}_{2m})$, the *Pfaffian*, with the following property: $\det(A) = \mathrm{Pf}(A)^2$ for all $A \in \mathrm{Alt}_{2m}$. Usually, the sign of the Pfaffian is determined by requiring that $\mathrm{Pf}(\begin{bmatrix} J & & \\ & \ddots & \\ & & J\end{bmatrix}) = 1$ where $J := \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$. E.g.

$$\mathrm{Pf}(\left[\begin{smallmatrix} 0 & x_{12} \\ -x_{12} & 0\end{smallmatrix}\right]) = x_{12}, \qquad \mathrm{Pf}\left(\left[\begin{smallmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0\end{smallmatrix}\right]\right) = x_{14}x_{23} - x_{13}x_{24} + x_{12}x_{34}$$

PROOF. It is well-known that for any alternating matrix A with entries in an arbitrary field K there is a $g \in \mathrm{GL}_n(K)$ such that

$$gAg^t = \begin{bmatrix} J & & & & \\ & \ddots & & & \\ & & J & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix}. \tag{4}$$

Now take $K = \mathbb{C}(x_{ij} \mid 1 \le i < j \le n = 2m)$ and put

$$A := \begin{bmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ -x_{12} & 0 & x_{23} & \cdots & x_{2n} \\ -x_{13} & -x_{23} & 0 & \cdots & x_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ -x_{1n} & -x_{2n} & -x_{3n} & \cdots & 0 \end{bmatrix}).$$

Then there is a $g \in \mathrm{GL}_n(K)$ such that gAg^t has the form given in (4). It follows that the polynomial $\det(A) \in K[x_{ij} \mid 1 \le i < j \le n]$ equals $\det(g)^{-2}$, the square of a rational function, hence the claim. $\square$

EXERCISE 1.1.5. For $a = (a_1,a_2,\dots,a_n) \in \mathbb{C}^n$ denote by $\mathrm{ev}_a\colon \mathcal{O}(\mathbb{C}^n) \to \mathbb{C}$ the *evaluation map* $f \mapsto f(a)$. Then the kernel of ev_a is the maximal ideal

$$\mathfrak{m}_a := (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n).$$

EXERCISE 1.1.6. Let $W \subseteq \mathcal{O}(V)$ a finite dimensional subspace. Then the linear functions $\mathrm{ev}_v|_W$ for $v \in V$ span the dual space W^*.

1.2. Zero sets and Zariski topology.

We now define the basic object of algebraic geometry, namely the zero set of regular functions. Let V be a finite dimensional vector space.

DEFINITION 1.2.1. If $f \in \mathcal{O}(V)$, then we define the *zero set* of f by

$$\mathcal{V}(f) := \{v \in V \mid f(v) = 0\} = f^{-1}(0).$$

More generally, the *zero set* of $f_1, f_2, \dots, f_s \in \mathcal{O}(V)$ or of a subset $S \subseteq \mathcal{O}(V)$ is defined by

$$\mathcal{V}(f_1,f_2,\dots,f_s) := \bigcap_{i=1}^s \mathcal{V}(f_i) = \{v \in V \mid f_1(v) = \cdots = f_s(v) = 0\}$$

or

$$\mathcal{V}(S) := \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}.$$

REMARK 1.2.2. The following properties of zero sets follow immediately from the definition.

(1) Let $S \subseteq \mathcal{O}(V)$ and denote by $\mathfrak{a} := (S) \subseteq \mathcal{O}(V)$ the ideal generated by S. Then $\mathcal{V}(S) = \mathcal{V}(\mathfrak{a})$.
(2) If $S \subseteq T \subseteq \mathcal{O}(V)$, then $\mathcal{V}(S) \supseteq \mathcal{V}(T)$.
(3) For any family $(S_i)_{i \in I}$ of subset $S_i \subseteq \mathcal{O}(V)$ we have

$$\mathcal{V}(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} \mathcal{V}(S_i).$$

EXAMPLE 1.2.3. (1) $\mathrm{SL}_n(\mathbb{C}) = \mathcal{V}(\det -1) \subseteq \mathrm{M}_n(\mathbb{C})$.
(2) $\mathrm{O}_n(\mathbb{C}) = \mathcal{V}(\sum_{\nu=1}^n x_{i\nu}x_{j\nu} - \delta_{ij} \mid 1 \leq i \leq j \leq n)$.
(3) If $f = f(x,y) \in \mathbb{C}[x,y]$ is a nonconstant polynomial in 2 variables, then $\mathcal{V}(f) \subseteq \mathbb{C}$ is called a *plane curve*. In order to visualize a plane curve, we usually draw a real picture $\subseteq \mathbb{R}^2$.

LEMMA 1.2.4. *Let V be a finite dimensional vector space and let $\mathfrak{a}, \mathfrak{b}$ be ideals in $\mathcal{O}(V)$ and $(\mathfrak{a}_i \mid i \in I)$ a family of ideals of $\mathcal{O}(V)$.*

(1) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathcal{V}(\mathfrak{a}) \supseteq \mathcal{V}(\mathfrak{b})$.*
(2) $\bigcap_{i \in I} \mathcal{V}(\mathfrak{a}_i) = \mathcal{V}(\sum_{i \in I} \mathfrak{a}_i)$.
(3) $\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) = \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}(\mathfrak{a} \cdot \mathfrak{b})$.
(4) $\mathcal{V}(0) = V$ *and* $\mathcal{V}(1) = \emptyset$.

PROOF. Properties (1) and (2) follow from Remark 1, and property (4) is easy. So we are left with property (3). Since $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b} \supseteq \mathfrak{a} \cdot \mathfrak{b}$, it follows from (1) that $\mathcal{V}(\mathfrak{a}) \subseteq \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a} \cdot \mathfrak{b})$. So it remains to show that $\mathcal{V}(\mathfrak{a} \cdot \mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b})$. If $v \in V$ does not belong to $\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b})$, then there are functions $f \in \mathfrak{a}$ and $h \in \mathfrak{b}$ such that $f(v) \neq 0 \neq h(v)$. Since $f \cdot h \in \mathfrak{a} \cdot \mathfrak{b}$ and $(f \cdot h)(v) \neq 0$ we see that $v \notin \mathcal{V}(\mathfrak{a} \cdot \mathfrak{b})$, and the claim follows. □

DEFINITION 1.2.5. The lemma shows that the subsets $\mathcal{V}(\mathfrak{a})$ where $\mathfrak{a}$ runs through the ideals of $\mathcal{O}(V)$ form the closed sets of topology on V which is called ZARISKI *topology*. From now on all topological terms like "open", "closed", "neighborhood", "continuous", etc. will refer to the ZARISKI topology.

EXAMPLE 1.2.6. (1) The *nilpotent cone* $N \subseteq \mathrm{M}_n$ consisting of all nilpotent matrices is closed and is a *cone*, i.e. stable under multiplication with scalars. E.g. for $n = 2$ we have

$$N = \mathcal{V}(x_{11} + x_{22}, x_{11}x_{22} - x_{12}x_{21}) \subseteq \mathrm{M}_2\,.$$

(2) The subset $\mathrm{M}_n^{(r)} \subseteq \mathrm{M}_n$ of matrices of rank $\leq r$ are closed cones.
(3) The set of polynomials $f \in P_n$ with a multiple root is closed (see Example 1.1.3).
(4) The closed subsets of $\mathbb{C}$ are the finite sets together with $\mathbb{C}$. So the nonempty open sets of $\mathbb{C}$ are the cofinite sets.

EXERCISE 1.2.7. Show that the subset $A := \{(n,m) \in \mathbb{C}^2 \mid n, m \in \mathbb{Z} \text{ and } m \geq n \geq 0\}$ is ZARISKI-dense in $\mathbb{C}^2$.

DEFINITION 1.2.8. Let $X \subseteq V$ be a closed subset. A *regular function on X* is defined to be the restriction of a regular function on V:

$$\mathcal{O}(X) := \{f|_X \mid f \in \mathcal{O}(V)\}.$$

The kernel of the (surjective) restriction map $\mathrm{res}\colon \mathcal{O}(V) \to \mathcal{O}(X)$ is called the *vanishing ideal of X*, or shortly the *ideal of X*:

$$I(X) := \{f \in \mathcal{O}(V) \mid f(x) = 0 \text{ for all } x \in X\}.$$

Thus we have a canonical isomorphism $\mathcal{O}(V)/I(X) \xrightarrow{\sim} \mathcal{O}(X)$.

EXERCISE 1.2.9. A regular function $f \in \mathcal{O}(V)$ is called *homogeneous of degree d* if $f(tv) = t^d f(v)$ for all $t \in \mathbb{C}$ and all $v \in V$.

(1) A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree d as a regular function on $\mathbb{C}^n$ if and only if all monomials occurring in f have degree d.
(2) Assume that the ideal $\mathfrak{a} \subseteq \mathcal{O}(V)$ is generated by homogeneous functions. Then the zeros set $\mathcal{V}(\mathfrak{a}) \subseteq V$ is a cone.
(3) Conversely, if $X \subseteq V$ is a cone, then the ideal $I(X)$ can be generated by homogeneous functions. More precisely, if $f|_X = 0$, then $f_d|_X = 0$ for every homogeneous component f_d of f.

REMARK 1.2.10. Every finite dimensional $\mathbb{C}$-vector space V carries a natural topology given by the choice of a norm or a hermitian scalar product. We will call it the $\mathbb{C}$-*topology*. Since all polynomials are continuous with respect to the $\mathbb{C}$-topology we see that the $\mathbb{C}$-topology is finer than the ZARISKI topology.

EXERCISE 1.2.11. Show that every non-empty open set in $\mathbb{C}^n$ is dense in the $\mathbb{C}$-topology. (Hint: Reduce to the case $n = 1$ where the claim follows from Example 1.2.6(4).)

REMARK 1.2.12. In the ZARISKI topology the finite sets are closed. This follows from the fact that for any two different points $v, w \in V$ one can find a regular function $f \in \mathcal{O}(V)$ such that $f(v) = 0$ and $f(w) \neq 0$. (One says that the regular functions *separate the points.*) But the ZARISKI topology is not Hausdorff (see the following exercise).

EXERCISE 1.2.13. Let $U, U' \subseteq \mathbb{C}^n$ be two non-empty open sets. Then $U \cap U'$ is non-empty, too. In particular, the ZARISKI topology is not Hausdorff.

EXERCISE 1.2.14. Consider a polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ of the form $f = x_0 - p(x_1, \ldots, x_n)$, and let $X = \mathcal{V}(f)$ be its zero set. Then $I(X) = (f)$ and $\mathcal{O}(X) \simeq \mathbb{C}[x_1, \ldots, x_n]$.

1.3. Hilbert's Nullstellensatz. The famous Nullstellensatz of HILBERT appears in many different forms which are all more or less equivalent. We will give some of them which are suitable for our purposes.

DEFINITION 1.3.1. If $\mathfrak{a}$ is an ideal of an arbitrary ring R, its *radical* $\sqrt{\mathfrak{a}}$ is defined by
$$\sqrt{\mathfrak{a}} := \{r \in R \mid r^m \in \mathfrak{a} \text{ for some } m > 0\}.$$
Clearly, $\sqrt{\mathfrak{a}}$ is an ideal and $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$. Moreover, $\sqrt{\mathfrak{a}} = R$ implies that $\mathfrak{a} = R$. The ideal $\mathfrak{a}$ is called *perfect* if $\mathfrak{a} = \sqrt{\mathfrak{a}}$. The ring R is called *reduced* if $\sqrt{(0)} = (0)$, or, equivalently, if R contains no nonzero nilpotent elements. Also, if $\mathfrak{a} \subseteq \mathcal{O}(V)$ is an ideal, then $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(\sqrt{\mathfrak{a}})$, hence $I(X)$ is perfect for every $X \subseteq V$.

THEOREM 1.3.2 (HILBERT's Nullstellensatz). *Let $\mathfrak{a} \subseteq \mathcal{O}(V)$ be an ideal and $X := \mathcal{V}(\mathfrak{a}) \subseteq V$ its zero set. Then*
$$I(X) = I(\mathcal{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

A first consequence is that every proper ideal has a non-empty zero set, because $X = \mathcal{V}(\mathfrak{a}) = \emptyset$ implies that $\sqrt{\mathfrak{a}} = I(X) = \mathcal{O}(V)$ and so $\mathfrak{a} = \mathcal{O}(V)$.

COROLLARY 1.3.3. *For every ideal $\mathfrak{a} \neq \mathcal{O}(V)$ we have $\mathcal{V}(\mathfrak{a}) \neq \emptyset$.*

Let $\mathfrak{m} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a maximal ideal and $a = (a_1, \ldots, a_n) \in \mathcal{V}(\mathfrak{m})$ which exists by the previous corollary. Then $\mathfrak{m} \subseteq (x_1 - a_1, \ldots, x_n - a_n)$, and so these two are equal.

COROLLARY 1.3.4. *Every maximal ideal $\mathfrak{m}$ of $\mathbb{C}[x_1, \ldots, x_n]$ is of the form*
$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n).$$

Another way to express this is by using the evaluation map $\mathrm{ev}_v \colon \mathcal{O}(V) \to \mathbb{C}$ (see Exercise 1.1.5).

COROLLARY 1.3.5. *Every maximal ideal of $\mathcal{O}(V)$ equals the kernel of the evaluation map $\mathrm{ev}_v \colon \mathcal{O}(V) \to \mathbb{C}$ at a suitable $v \in V$.*

EXERCISE 1.3.6. If $X \subseteq V$ is a closed subset and $\mathfrak{m} \subseteq \mathcal{O}(X)$ a maximal ideal, then $\mathcal{O}(X)/\mathfrak{m} = \mathbb{C}$. Moreover, $\mathfrak{m} = \ker(\mathrm{ev}_x \colon f \mapsto f(x))$ for a suitable $x \in X$.

PROOF OF THEOREM 1.3.2. We first prove Corollary 1.3.4 which implies Corollary 1.3.5 as we have seen above. It also implies Corollary 1.3.3, because every proper ideal is contained in a maximal ideal.

Let $\mathfrak{m} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a maximal ideal and denote by $K := \mathbb{C}[x_1, \dots, x_n]/\mathfrak{m}$ its residue class field. Then K contains $\mathbb{C}$ and has a countable $\mathbb{C}$-basis, because $\mathbb{C}[x_1, \dots, x_n]$ does. If $K \neq \mathbb{C}$ and $p \in K \setminus \mathbb{C}$, then p is transcendental over $\mathbb{C}$. It follows that the elements $(\frac{1}{p-a} \mid a \in \mathbb{C})$ from K form a non-countable set of linearly independent elements over $\mathbb{C}$. This contradiction shows that $K = \mathbb{C}$. Thus $x_i + \mathfrak{m} = a_i + \mathfrak{m}$ for a suitable $a_i \in \mathbb{C}$ (for $i = 1, \dots, n$), and so $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$. This proves Corollary 1.3.4.

To get the theorem, we use the so-called trick of RABINOWICH. Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal and assume that the polynomial f vanishes on $\mathcal{V}(\mathfrak{a})$. Now consider the polynomial ring $R := \mathbb{C}[x_0, x_1, \dots, x_n]$ in $n+1$ variables and the ideal $\mathfrak{b} := (\mathfrak{a}, 1 - x_0 f)$ generated by $1 - x_0 f$ and the elements of $\mathfrak{a}$. Clearly, $\mathcal{V}(\mathfrak{b}) = \emptyset$ and so $1 \in \mathfrak{b}$, by Corollary 1.3.3. This means that we can find an equation of the form

$$\sum_i h_i f_i + h(1 - x_0 f) = 1$$

where $f_i \in \mathfrak{a}$ and $h_i, h \in R$. Now we substitute $\frac{1}{f}$ for x_0 and find

$$\sum_i h_i(\frac{1}{f}, x_1, \dots, x_n) f_i = 1.$$

Clearing denominators finally gives $\sum_i \tilde{h}_i f_i = f^m$, i.e., $f^m \in \mathfrak{a}$, and the claim follows. $\square$

COROLLARY 1.3.7. *For any ideal $\mathfrak{a} \subseteq \mathcal{O}(V)$ and its zero set $X := \mathcal{V}(\mathfrak{a})$ we have $\mathcal{O}(X) = \mathcal{O}(V)/\sqrt{\mathfrak{a}}$.*

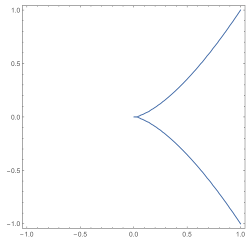
EXERCISE 1.3.8. Let $\mathfrak{a} \subseteq R$ be an ideal of a (commutative) ring R. Then $\mathfrak{a}$ is perfect if and only if the residue class ring $R/\mathfrak{a}$ has no nilpotent elements different from 0.

EXAMPLE 1.3.9. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be an arbitrary polynomial and consider its decomposition into irreducible factors: $f = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$. Then $\sqrt{(f)} = (p_1 p_2 \cdots p_s)$ and so the ideal (f) is perfect if and only if the polynomial f it is square-free. In particular, if $f \in \mathbb{C}[x_1, \dots, x_n]$ is irreducible, then $\mathcal{O}(\mathcal{V}(f)) \simeq \mathbb{C}[x_1, \dots, x_n]/(f)$. A closed subset of the form $\mathcal{V}(f)$ is called a *hypersurface*.

EXAMPLE 1.3.10. We have $\mathcal{O}(\mathrm{SL}_n(\mathbb{C})) \simeq \mathcal{O}(\mathrm{M}_n)/(\det -1)$, because the polynomial $\det -1$ is irreducible.

PROOF. For a fixed i_0, the polynomial $\det -1$ is linear in the $x_{i_0 1}, \dots, x_{i_0 n}$. Thus, if $\det -1 = f_1 \cdot f_2$, then all of them appear in one factor and none in the other. The same argument applied to $x_{1 j_0}, \dots, x_{n j_0}$ finally shows that one of the factors is a constant. $\square$

EXAMPLE 1.3.11. Consider the plane curve $C := \mathcal{V}(y^2 - x^3)$ which is called NEIL'S *parabola*. Then $\mathcal{O}(C) \simeq \mathbb{C}[x, y]/(y^2 - x^3) \xrightarrow{\sim} \mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t]$ where the second isomorphism is given by $\rho \colon x \mapsto t^2, y \mapsto t^3$.



Proof. Clearly, $y^2 - x^3 \in \ker\rho$. For any $f \in \mathbb{C}[x,y]$ we can write $f = f_0(x) + f_1(x)y + h(x,y)(y^2 - x^3)$. If $f \in \ker\rho$, then $0 = \rho(f) = f_0(t^2) + f_1(t^2)t^3$ and so $f_0 = f_1 = 0$. This shows that $\ker\rho = (y^2 - x^3)$, and the claim follows. $\square$

Exercise 1.3.12. Let $C \subseteq \mathbb{C}^2$ be the plane curve defined by $y - x^2 = 0$. Then $I(C) = (y - x^2)$ and $\mathcal{O}(C)$ is a polynomial ring in one variable.

Exercise 1.3.13. Let $D \subseteq \mathbb{C}^2$ be the zero set of $xy - 1$. Then $\mathcal{O}(D)$ is not isomorphic to a polynomial ring, but there is an isomorphism $\mathcal{O}(D) \xrightarrow{\sim} \mathbb{C}[t, t^{-1}]$.

Exercise 1.3.14. Consider the "parametric curve"

$$Y := \{(t, t^2, t^3) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}.$$

Then Y is closed in $\mathbb{C}^3$. Find generators for the ideal $I(Y)$ and show that $\mathcal{O}(Y)$ is isomorphic to the polynomial ring in one variable.

Another important consequence of the "Nullstellensatz" is a one-to-one correspondence between closed subsets of $\mathbb{C}^n$ and perfect ideals of the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$.

Corollary 1.3.15. *The map $X \mapsto I(X)$ defines a inclusion-reversing bijection*

$$\{X \subseteq V \text{ closed}\} \xrightarrow{\sim} \{\mathfrak{a} \subseteq \mathcal{O}(V) \text{ perfect ideal}\}$$

whose inverse map is given by $\mathfrak{a} \mapsto \mathcal{V}(\mathfrak{a})$. Moreover, for any finitely generated reduced $\mathbb{C}$-algebra R there is a closed subset $X \subseteq \mathbb{C}^n$ for some n such that $\mathcal{O}(X)$ is isomorphic to R

Proof. The first part is clear since $\mathcal{V}(I(X)) = X$ and $I(\mathcal{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for any closed subset $X \subseteq V$ and any ideal $\mathfrak{a} \subseteq \mathcal{O}(V)$.

If R is a reduced and finitely generated $\mathbb{C}$-Algebra, $R = \mathbb{C}[f_1, \ldots, f_n]$, then $R \simeq \mathbb{C}[x_1, x_2, \ldots, x_n]/\mathfrak{a}$ where $\mathfrak{a}$ is the kernel of the homomorphism defined by $x_i \mapsto f_i$. Since R is reduced we have $\sqrt{\mathfrak{a}} = \mathfrak{a}$ and so $\mathcal{O}(\mathcal{V}(\mathfrak{a})) \simeq \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{a} \simeq R$. $\square$

Exercise 1.3.16. Let $X \subseteq V$ be a closed subset and $f \in \mathcal{O}(X)$ a regular function such that $f(x) \neq 0$ for all $x \in X$. Then f is invertible in $\mathcal{O}(X)$, i.e. the $\mathbb{C}$-valued function $f^{-1} \colon x \mapsto f(x)^{-1}$ is regular on X.

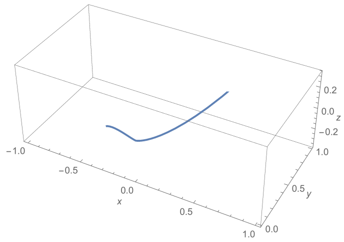
Exercise 1.3.17. Every closed subset $X \subseteq \mathbb{C}^n$ is quasi-compact, i.e., every covering of X by open sets contains a finite covering. Is this also true for open or even locally closed subsets of $\mathbb{C}^n$?

Exercise 1.3.18. Let $X \subseteq \mathbb{C}^n$ be a closed subset. Assume that there are no nonconstant invertible regular function on X. Then every nonconstant $f \in \mathcal{O}(X)$ attains all values in $\mathbb{C}$, i.e. $f \colon X \to \mathbb{C}$ is surjective.

Exercise 1.3.19. Consider the curve

$$Y := \{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$$

cf. Exercise 1.3.14. Then Y is closed in $\mathbb{C}^3$. Find generators for the ideal $I(Y)$ and show that $I(Y)$ cannot be generated by two polynomials.



(Hint: Define the *weight* of a monomial in x, y, z by $\mathrm{wt}(x) := 3, \mathrm{wt}(y) := 4, \mathrm{wt}(z) := 5$. Then the ideal $I(Y)$ is linearly spanned by the differences $m_1 - m_2$ of two monomials of the same weight. This occurs for the first time for the weight 8, and then also for the weights 9 and 10. Now show that for any generating system of $I(Y)$ these three differences have to occur in three different generators.)

1.4. Affine varieties. We have seen in the previous section that every closed subset $X \subseteq V$ (or $X \subseteq \mathbb{C}^n$) is equipped with an algebra of $\mathbb{C}$-valued functions, namely the coordinate ring $\mathcal{O}(X)$. We first remark that $\mathcal{O}(X)$ determines the topology of X. In fact, define for every ideal $\mathfrak{a} \subseteq \mathcal{O}(X)$ *the zero set in* X by

$$\mathcal{V}_X(\mathfrak{a}) := \{x \in X \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

Clearly, we have $\mathcal{V}_X(\mathfrak{a}) = \mathcal{V}(\tilde{\mathfrak{a}}) \cap X$ where $\tilde{\mathfrak{a}} \subseteq \mathcal{O}(V)$ is an ideal which maps surjectively onto $\mathfrak{a}$ under the restriction map. This shows that the sets $\mathcal{V}_X(\mathfrak{a})$ are the closed sets of the topology on X induced by the Zariski topology of V. Moreover, the coordinate ring $\mathcal{O}(X)$ also determines the points of X since they are in one-to-one correspondence with the maximal ideals of $\mathcal{O}(X)$:

$$x \in X \mapsto \mathfrak{m}_x := \ker \mathrm{ev}_x \subseteq \mathcal{O}(X)$$

where $\mathrm{ev}_x : \mathcal{O}(X) \to \mathbb{C}$ is the evaluation map $f \mapsto f(x)$. This leads to the following definition of an affine variety.

Definition 1.4.1. A set Z together with a $\mathbb{C}$-algebra $\mathcal{O}(Z)$ of $\mathbb{C}$-valued functions on Z is called an *affine variety* if there is a closed subset $X \subseteq \mathbb{C}^n$ for some n and a bijection $\varphi : Z \xrightarrow{\sim} X$ which identifies $\mathcal{O}(X)$ with $\mathcal{O}(Z)$, i.e., $\varphi^* : \mathcal{O}(X) \to \mathcal{O}(Z)$ given by $f \mapsto f \circ \varphi$, is an isomorphism.

The functions from $\mathcal{O}(Z)$ are called *regular*, and the algebra $\mathcal{O}(Z)$ is called *coordinate ring of* Z or *algebra of regular functions on* Z.

The affine variety Z has a natural topology, also called Zariski *topology*, the closed sets being the *zero sets*

$$\mathcal{V}_Z(\mathfrak{a}) := \{z \in Z \mid f(z) = 0 \text{ for all } f \in \mathfrak{a}\}$$

where $\mathfrak{a}$ runs through the ideals of $\mathcal{O}(Z)$. If follows from what we said above that the bijection $\rho : Z \xrightarrow{\sim} X$ is a homeomorphism with respect to the Zariski topology.

Clearly, every closed subset $X \subseteq V$ or $X \subseteq \mathbb{C}^n$ together with its coordinate ring $\mathcal{O}(X)$ is an affine variety. More generally, if X is an affine variety and $Y \subseteq X$ a closed subset, then Y together with the restrictions $\mathcal{O}(Y) := \{f|_Y \mid f \in \mathcal{O}(X)\}$ is an affine variety, called a *closed subvariety.* Less trivial examples are the following.

Example 1.4.2. Let M be a finite set and define $\mathcal{O}(M) := \mathbb{C}^M = \mathrm{Maps}(M, \mathbb{C})$ to be the set of all $\mathbb{C}$-valued functions on M. Then $(M, \mathcal{O}(M))$ is an affine variety and $\mathcal{O}(M)$ is isomorphic to a product of copies of $\mathbb{C}$. This follows immediately from the fact that any finite subset $N \subseteq \mathbb{C}^n$ is closed and that $\mathcal{O}(N) = \mathrm{Maps}(N, \mathbb{C})$.

Example 1.4.3. Let X be a set and define the *symmetric product* $\mathrm{Sym}_n(X)$ to be the set of unordered n-tuples of elements from X, i.e.,

$$\mathrm{Sym}_n(X) = X \times X \times \cdots \times X/\sim$$

where $(a_1, a_2, \ldots, a_n) \sim (b_1, b_2, \ldots, b_n)$ if and only if one is a permutation of the other.

In case $X = \mathbb{C}$ we define $\mathcal{O}(\mathrm{Sym}_n(\mathbb{C}))$ to be the symmetric polynomials in n variables and claim that $\mathrm{Sym}_n(\mathbb{C})$ is an affine variety.

To see this consider the map

$$\Phi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad a = (a_1, \ldots, a_n) \mapsto (\sigma_1(a), \sigma_2(a), \ldots, \sigma_n(a))$$

where $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric polynomials (see Example 1.1.3). It is easy to see that Φ is surjective and that $\Phi(a) = \Phi(b)$ if and only if $a \sim b$. Thus, Φ defines a bijection $\varphi\colon \mathrm{Sym}_n(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^n$, and the pull-back of the regular functions on $\mathbb{C}^n$ are the symmetric polynomials: $\varphi^*\colon \mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\sim} \mathcal{O}(\mathrm{Sym}_n(\mathbb{C}))$.

In general, one defines

$$\mathcal{O}(\underbrace{X \times X \times \cdots \times X}_{n \text{ copies}}) := \mathbb{C}[f_1 f_2 \cdots f_n \mid f_i \in \mathcal{O}(X)]$$

and

$$\mathcal{O}(\mathrm{Sym}_n(X) := \{f \in \mathcal{O}(X \times X \times \cdots \times X) \mid f \text{ symmetric}\}.$$

Exercise 1.4.4. Let Z be an affine variety with coordinate ring $\mathcal{O}(Z)$. Then every polynomial $f \in \mathcal{O}(Z)[t]$ with coefficients in $\mathcal{O}(Z)$ defines a function on the product $Z \times \mathbb{C}$ in the usual way:

$$f = \sum_{k=0}^{m} f_k t^k \colon (z, a) \mapsto \sum_{k=0}^{m} f_k(z) a^k \in \mathbb{C}$$

Show that $Z \times \mathbb{C}$ together with $\mathcal{O}(Z)[t]$ is an affine variety.
(Hint: For any ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ there is a canonical isomorphism $\mathbb{C}[x_1, \ldots, x_n, t]/(\mathfrak{a}) \xrightarrow{\sim} (\mathbb{C}[x_1, \ldots, x_n]/\mathfrak{a})[t]$.)

Exercise 1.4.5. For any affine variety Z there is a inclusion-reversing bijection

$$\{A \subseteq Z \text{ closed}\} \xrightarrow{\sim} \{\mathfrak{a} \subseteq \mathcal{O}(Z) \text{ perfect ideal}\}$$

given by $A \mapsto I(A) := \{f \in \mathcal{O}(Z) \mid f|_A = 0\}$ (cf. Corollary 1.3.15).

For the last example we start with a reduced and finitely generated $\mathbb{C}$-algebra R. Denote by the set of maximal ideals of R:

$$\operatorname{spec} R := \{\mathfrak{m} \mid \mathfrak{m} \subseteq R \text{ a maximal ideal}\}.$$

We know from Hilbert's Nullstellensatz (see Exercise 5.8) that $R/\mathfrak{m} = \mathbb{C}$ for all maximal ideals $\mathfrak{m} \in \operatorname{spec} R$. This allows to identify the elements from R with $\mathbb{C}$-valued functions on $\operatorname{spec} R$: For $f \in R$ and $\mathfrak{m} \in \operatorname{spec} R$ we define

$$f(\mathfrak{m}) := f + \mathfrak{m} \in R/\mathfrak{m} = \mathbb{C}.$$

Proposition 1.4.6. *Let R be a reduced and finitely generated $\mathbb{C}$-algebra. Then the set of maximal ideals $\operatorname{spec} R$ together with the algebra R considered as functions on $\operatorname{spec} R$ is an affine variety.*

Proof. We have already seen earlier that every such algebra R is isomorphic to the coordinate ring of a closed subset $X \subseteq \mathbb{C}^n$. The claim then follows by using the bijection $X \xrightarrow{\sim} \operatorname{spec} \mathcal{O}(X)$, $x \mapsto \mathfrak{m}_x = \ker \operatorname{ev}_x$, and remarking that for $f \in \mathcal{O}(X)$ and $x \in X$ we have $f(x) = \operatorname{ev}_x(f) = f + \mathfrak{m}_x$, by definition. $\square$

Exercise 1.4.7. Denote by C_n the n-tuples of complex numbers up to sign, i.e., $C_n := \mathbb{C}^n/\sim$ where $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ if $a_i = \pm b_i$ for all i. Then every polynomial in $\mathbb{C}[x_1^2, x_2^2, \ldots, x_n^2]$ is a well-defined function on C_n. Show that C_n together with these functions is an affine variety.
(Hint: Consider the map $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$, $(a_1, \ldots, a_n) \mapsto (a_1^2, \ldots, a_n^2)$ and proceed like in Example 1.4.3.)

Although every affine variety is isomorphic to a closed subset of $\mathbb{C}^n$ for a suitable n, there are many advantages to look at these objects and not only at closed subsets. In fact, an affine variety can be identified with many different closed subsets of this form (see the following Exercise 1.4.8), and depending on the questions we are asking one of them might be more useful than another. We will even see in the following section that certain open subsets are affine varieties in a natural way.

On the other hand, whenever we want to prove some statements for an affine variety X we can always assume that $X = \mathcal{V}(\mathfrak{a}) \subseteq \mathbb{C}^n$ so that the regular functions on X appear as restrictions of polynomial functions. This will be helpful in the future and quite often simplify the arguments.

EXERCISE 1.4.8. Let X be an affine variety. Show that every choice of a generating system $f_1, f_2, \ldots, f_n \in \mathcal{O}(X)$ of the algebra $\mathcal{O}(X)$ consisting of n elements defines an identification of X with a closed subset $\mathcal{V}(\mathfrak{a}) \subseteq \mathbb{C}^n$.
(Hint: Consider the map $X \to \mathbb{C}^n$ given by $x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$.)

1.5. Special open sets. Let X be an affine variety and $f \in \mathcal{O}(X)$. Define the open set $X_f \subseteq X$ by

$$X_f := X \setminus \mathcal{V}_X(f) = \{x \in X \mid f(x) \neq 0\}.$$

An open set of this form is called a *special open set*.

LEMMA 1.5.1. *The special open sets of an affine variety X form a basis of the topology.*

PROOF. If $U \subseteq X$ is open and $x \in U$, then $X \setminus U$ is closed and does not contain x. Thus, there is a regular function $f \in \mathcal{O}(X)$ vanishing on $X \setminus U$ such that $f(x) \neq 0$. This implies $x \in X_f \subseteq U$. □

Given a special open set $X_f \subseteq X$ we see that $f(x) \neq 0$ for all $x \in X_f$ and so the function $\frac{1}{f}$ is well-defined on X_f.

PROPOSITION 1.5.2. *Denote by $\mathcal{O}(X_f)$ the algebra of functions on X_f generated by $\frac{1}{f}$ and the restrictions $h|_{X_f}$ of regular functions h on X:*

$$\mathcal{O}(X_f) := \mathbb{C}[\frac{1}{f}, \{h|_{X_f} \mid h \in \mathcal{O}(X)\}] = \mathcal{O}(X)|_{X_f}[\frac{1}{f}].$$

Then $(X_f, \mathcal{O}(X_f))$ is an affine variety and $\mathcal{O}(X_f) \simeq \mathcal{O}(X)[t]/(f \cdot t - 1)$.

PROOF. Let $X = \mathcal{V}(\mathfrak{a}) \subseteq \mathbb{C}^n$ and define

$$\tilde{X} := \mathcal{V}(\mathfrak{a}, f \cdot x_{n+1} - 1) \subseteq \mathbb{C}^{n+1}.$$

It is easy to see that the projection $\mathrm{pr}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ onto the first n coordinates induces a bijective map $\tilde{X} \xrightarrow{\sim} X_f$ whose inverse $\varphi\colon X_f \xrightarrow{\sim} \tilde{X}$ is given by

$$\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)^{-1}).$$

The following commutative diagram now shows that $\varphi^*(\mathcal{O}(\tilde{X}))$ is generated by $\varphi^*(x_{n+1}) = \frac{1}{f}$ and the restrictions $h|_{X_f}$ $(h \in \mathcal{O}(X))$.

$$\begin{tikzcd}
\tilde{X} \arrow[r, equal] & \tilde{X} \arrow[r, "\subseteq", "\text{closed}"'] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, "\mathrm{pr}"] \\
X_f \arrow[u, "\varphi", "\simeq"'] \arrow[r, "\subseteq", "\text{open}"'] & X \arrow[r, "\subseteq", "\text{closed}"'] & \mathbb{C}^n
\end{tikzcd}$$

This proves the first claim. For the second, we have to show that the canonical homomorphism $\mathcal{O}(X)[t]/(f \cdot t - 1) \to \mathcal{O}(\tilde{X})$ is an isomorphism. In other words, every function $h = \sum_{i=0}^m h_i t^i \in \mathcal{O}(X)[t]$ which vanishes on $\tilde{X}$ is divisible by $f \cdot t - 1$. Since $f|_{\tilde{X}}$ is invertible we first obtain $\sum_i h_i f^{m-i} = 0$ which implies

$$h = h - t^m \sum_{i=0}^{m} h_i f^{m-i} = \sum_{i=0}^{m-1} h_i t^i (1 - f^{m-i} t^{m-i}),$$

and the claim follows. □

EXAMPLE 1.5.3. The group is a special open set of $\mathrm{M}_n(\mathbb{C})$, hence $\mathrm{GL}_n(\mathbb{C})$ is an affine variety with coordinate ring $\mathcal{O}(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}[\{x_{ij} \mid 1 \leq i,j \leq n\}, \frac{1}{det}]$. In particular, $\mathbb{C}^* := \mathrm{GL}_1 = \mathbb{C} \setminus \{0\}$ is an affine variety with coordinate ring $\mathbb{C}[x, x^{-1}]$.

EXERCISE 1.5.4. Let R be an arbitrary $\mathbb{C}$-algebra. For any element $s \in R$ define $R_s := R[x]/(s \cdot x - 1)$.

(1) Describe the kernel of the canonical homomorphism $\iota\colon R \to R_s$.
(2) Prove the universal property: *For any homomorphism $\rho\colon R \to A$ such that $\rho(s)$ is invertible in A there is a unique homomorphism $\bar{\rho}\colon R_s \to A$ such that $\bar{\rho} \circ \iota = \rho$.*
(3) What happens if s is a zero divisor and what if s is invertible?

1.6. Decomposition into irreducible components. We start with a purely topological notion.

DEFINITION 1.6.1. A topological space T is called *irreducible* if it cannot be decomposed in the form $T = A \cup B$ where $A, B \subsetneq T$ are proper closed subsets. Equivalently, every non-empty open subset is dense.

LEMMA 1.6.2. *Let $X \subseteq \mathbb{C}^n$ be a closed subset. Then the following are equivalent:*

(i) *X is irreducible.*
(ii) *$I(X)$ is a prime ideal.*
(iii) *$\mathcal{O}(X)$ is a domain, i.e., has no zero-divisor.*

PROOF. (i)⇒(ii): If $I(X)$ is not prime we can find two polynomials $f, h \in \mathbb{C}[x_1, \ldots, x_n] \setminus I(X)$ such that $f \cdot h \in I(X)$. This implies that $X \subseteq \mathcal{V}(f \cdot h) = \mathcal{V}(f) \cup \mathcal{V}(h)$, but X is neither contained in $\mathcal{V}(f)$ nor in $\mathcal{V}(h)$. Thus $X = (\mathcal{V}(g) \cap X) \cup (\mathcal{V}(h) \cap X)$ is a decomposition into proper closed subsets, contradicting the assumption.

(ii)⇒(iii): This is clear since $\mathcal{O}(X) = \mathbb{C}[x_1, \ldots, x_n]/I(X)$.

(iii)⇒(i): If $X = A \cup B$ is a decomposition into proper closed subsets there are nonzero functions $f, h \in \mathcal{O}(X)$ such that $f|_A = 0$ and $h|_B = 0$. But then $f \cdot h$ vanishes on all of X and so $f \cdot h = 0$. This contradicts the assumption that $\mathcal{O}(X)$ does not contain zero-divisor. □

EXAMPLE 1.6.3. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then the hypersurface $\mathcal{V}(f)$ is irreducible if and only if f is a power of an irreducible polynomial. This follows from Example 1.3.9 and the fact that the ideal (f) is prime if and only if f is irreducible. More generally, if $f = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ is the primary decomposition, then

$$\mathcal{V}(f) = \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \cdots \cup \mathcal{V}(p_n)$$

where each $\mathcal{V}(p_i)$ is irreducible, and this decomposition is *irredundant*, i.e., no term can be dropped.

THEOREM 1.6.4. *Every affine variety X is a finite union of irreducible closed subsets X_i:*

$$X = X_1 \cup X_2 \cup \cdots \cup X_s. \tag{5}$$

If this decomposition is irredundant, then the X_i's are the maximal irreducible subsets of X and are therefore uniquely determined.

The maximal X_i's are called the *irreducible components* of X and the unique irredundant decomposition of X in the form (5) is called *decomposition into irreducible components*.

For the proof of the theorem above we first recall that a $\mathbb{C}$-algebra R is called *Noetherian* if the following equivalent conditions hold:

(i) *Every ideal of R is finitely generated.*
(ii) *Every strictly ascending chain of ideals of R terminates.*
(iii) *Every non-empty set of ideals of R contains maximal elements.*

(The easy proofs are left to the reader; for the equivalence of (ii) and (iii) one has to use Zorn's Lemma.)

The famous "Basissatz" of Hilbert implies that every finitely generated $\mathbb{C}$-algebra is Noetherian (see [**Art91**, Chap. 12, Theorem 5.18]). In particular, this holds for the coordinate ring $\mathcal{O}(X)$ of any affine variety X. Using the inclusion reversing bijection between closed subsets of X and perfect ideals of $\mathcal{O}(X)$ (see Corollary 1.3.15 and Exercise 1.4.5) we get the following result.

Proposition 1.6.5. *Let X be an affine variety. Then*

(1) *Every closed subset $A \subseteq X$ is of the form $\mathcal{V}_X(f_1, f_2, \dots, f_r)$.*
(2) *Every strictly descending chain of closed subsets of X terminates.*
(3) *Every non-empty set of closed subsets of X contains minimal elements.*

Remark 1.6.6. It is easy to see that for an arbitrary topological space T the properties (2) and (3) from the previous proposition are equivalent. If they hold, then T is called *Noetherian*.

Proof of Theorem 1.6.4. We first show that such a decomposition exists. Consider the following set

$$\mathcal{M} := \{A \subseteq X \mid A \text{ closed and not a finite union of irreducible closed subsets}\}.$$

If $\mathcal{M} \neq \emptyset$, then it contains a minimal element A_0. Since A_0 is not irreducible, we can find proper closed subset $B, B' \subsetneq A_0$ such that $A_0 = B \cup B'$. But then $B, B' \notin \mathcal{M}$ and so both are finite unions of irreducible closed subsets. Hence A_0 is a finite union of irreducible closed subsets, too, contradicting the assumption.

To show the uniqueness let $X = X_1 \cup X_2 \cup \cdots \cup X_s$ where all X_i are irreducible closed subsets and assume that the decomposition is irredundant. Then, clearly, every X_i is maximal. Let $Y \subseteq X$ be a maximal irreducible closed subset. Then $Y = (Y \cap X_1) \cup (Y \cap X_2) \cup \cdots \cup (Y \cap X_s)$ and so $Y = Y \cap X_j$ for some j. It follows that $Y \subseteq X_j$ and so $Y = X_j$ because of maximality. □

Remark 1.6.7. The algebraic counterpart to the decomposition into irreducible components is the following statement about radical ideals in finitely generated algebras R: *Every radical ideal $\mathfrak{a} \subseteq R$ is a finite intersection of prime ideals:*

$$\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_s.$$

If this intersection is irredundant, then the $\mathfrak{p}_i$'s are the minimal prime ideals containing $\mathfrak{a}$. (The easy proof is left to the reader.)

Example 1.6.8. Consider the two hypersurfaces $H_1 := \mathcal{V}(xy - z)$, $H_2 := \mathcal{V}(xz - y^2)$ in $\mathbb{C}^3$ and their intersection $X := H_1 \cap H_2$. Then

$$X = \mathcal{V}(y, z) \cup C \quad \text{where } C := \{(t, t^2, t^3) \mid t \in \mathbb{C}\},$$

and this is the irreducible decomposition.

In fact, it is obvious that the x-axis $\mathcal{V}(y, z)$ is closed and irreducible and belongs to X, and the same holds for the curve C (see Exercise 1.3.14). If $(a, b, c) \in X \setminus \mathcal{V}(y, z)$, then either b or c is $\neq 0$. But then $b \neq 0$ because $ab = c$. Hence $a = cb^{-1}$ and so $b^2 = ac = c^2b^{-1}$ which implies that $c^2 = b^3$. Thus $b = (cb^{-1})^2$ and $c = (cb^{-1})^3$, i.e. $(a, b, c) \in C$.

Another way to see this is by looking at the coordinate ring:

$$\mathbb{C}[x, y, z]/(xy - z, xz - y^2) \xrightarrow{\sim} \mathbb{C}[x, y]/(x^2y - y^2).$$

On the level of ideals we get $(x^2y - y^2) = (y(x - y^2)) = (y) \cap (x - y^2)$, and the ideals (y) and $(x - y^2)$ are obviously prime, with residue class ring isomorphic to a polynomial ring in one variable. This shows that X has two irreducible components, both with coordinate ring isomorphic to $\mathbb{C}[t]$.

EXERCISE 1.6.9. The closed subvariety $X := \mathcal{V}(x^2 - yz, xz - x) \subseteq \mathbb{C}^3$ has three irreducible components. Describe the corresponding prime ideals in $\mathbb{C}[x, y, z]$.

EXAMPLE 1.6.10. The group $\mathrm{O}_2 := \{A \in \mathrm{M}_2 \mid AA^t = E\}$ has two irreducible components, namely $\mathrm{SO}_2 := \mathrm{O}_2 \cap \mathrm{SL}_2$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathrm{SO}_2$, and the two components are disjoint.

In fact, the condition $AA^t = E$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ implies that $\begin{bmatrix} a \\ b \end{bmatrix} = \pm \begin{bmatrix} d \\ -c \end{bmatrix}$. Since $\det \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a^2 + b^2$ we see that $\mathrm{SO}_2 = \{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a^2 + b^2 = 1\}$ is irreducible as well as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathrm{SO}_2 = \{\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \mid a^2 + b^2 = 1\}$, and the claim follows.

EXERCISE 1.6.11. Let $X = X_1 \cup X_2$ where $X_1, X_2 \subseteq X$ are closed and disjoint. Then one has a canonical isomorphism $\mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(X_1) \times \mathcal{O}(X_2)$.

EXERCISE 1.6.12. Let $X = \bigcup_i X_i$ be the decomposition into irreducible components. Let $U_i \subseteq X_i$ be open subsets and put $U := \bigcup_i U_i \subseteq X$.

(1) Show that U is not necessarily open in X.
(2) Find sufficient conditions to ensure that U is open in X.
(3) Show that U is dense in X if and only if all U_i are non-empty.

1.7. Rational functions and local rings. If X is an irreducible affine variety, then $\mathcal{O}(X)$ is a domain by Lemma 1.6.2. Therefore, we can form the field of fractions $\mathrm{Quot}(\mathcal{O}(X))$ of $\mathcal{O}(X)$ which is called *the field of rational functions* on X and will be denoted by . Clearly, if $X = \mathbb{C}^n$, then $\mathbb{C}(X) = \mathbb{C}(x_1, x_2, \ldots, x_n)$, the rational function field. An irreducible affine variety X is called *rational* if its field of rational functions $\mathbb{C}(X)$ is isomorphic to a rational function field.

A *rational function* $f \in \mathbb{C}(X)$ can be regarded as a function "defined almost everywhere" on X. In fact, we say that f *is defined in* $x \in X$ if there are $p, q \in \mathcal{O}(X)$ such that $f = \frac{p}{q}$ and $q(x) \neq 0$.

EXAMPLE 1.7.1. Consider again NEIL'S parabola $C := \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$ from Example 1.3.11 and put $\bar{x} := x|_C$ and $\bar{y} := y|_C$. Then the rational function $f := \frac{\bar{y}}{\bar{x}} \in \mathbb{C}(C)$ is not defined in $(0, 0)$. Note that $f^2 = \bar{x}$. The interesting point here is that f has a continuous extension to all of C with value 0 at $(0, 0)$, even in the $\mathbb{C}$-topology.

PROOF. There is an isomorphism $\mathcal{O}(C) \xrightarrow{\sim} \mathbb{C}[t^2, t^3]$ (see Example 1.3.11) which maps $\bar{x}$ to t^2 and $\bar{y}$ to t^3, and so $f = \frac{\bar{y}}{\bar{x}}$ is mapped to t. Since $t \notin \mathbb{C}[t^2, t^3]$ the first claim follows from Lemma 1.7.3 above. The second part is easy, because the map $\mathbb{C} \to C\colon t \mapsto (t^2, t^3)$ is a homeomorphism even in the $\mathbb{C}$-topology. □

EXERCISE 1.7.2. If $f \in \mathbb{C}(\mathbb{C}^2) = \mathbb{C}(x, y)$ is defined in $\mathbb{C}^2 \setminus \{(0, 0)\}$, then f is regular.

For a rational function f on the irreducible affine variety X we denote by $\mathrm{Def}(f) \subseteq X$ the set of points where f is defined. By definition, $\mathrm{Def}(f) \subseteq X$ is an open set. Moreover, we have the following result.

LEMMA 1.7.3. *$\mathrm{Def}(f) = X$ if and only if f is regular on X.*

PROOF. Consider the "ideal of denominators" $\mathfrak{a} := \{p \in \mathcal{O}(X) \mid p \cdot f \in \mathcal{O}(X)\}$. If $\mathrm{Def}(f) = X$, then $\mathcal{V}(\mathfrak{a}) = \emptyset$. Hence $1 \in \mathfrak{a}$, and so $f \in \mathcal{O}(X)$. □

EXERCISE 1.7.4. Let $f \in \mathbb{C}(V)$ be a nonzero rational function on the vector space V. Then $\mathrm{Def}(f)$ is a special open set in V.

Assume that X is irreducible and let $x \in X$. Define

$$\mathcal{O}_{X,x} := \{f \in \mathbb{C}(X) \mid f \text{ is defined in } x\}.$$

It is easy to see that $\mathcal{O}_{X,x}$ is the *localization* of $\mathcal{O}(X)$ at the maximal ideal $\mathfrak{m}_x$. (For the definition of the localization of a ring at a prime ideal and, more generally, for the construction of rings of fractions we refer to [**Eis95**, I.2.1].) This example motivates the following definition.

DEFINITION 1.7.5. Let X be an affine variety and $x \in X$ an arbitrary point. Then the localization $\mathcal{O}(X)_{\mathfrak{m}_x}$ of the coordinate ring $\mathcal{O}(X)$ at the maximal ideal in x is called the *local ring of X at x*. It will be denoted by $\mathcal{O}_{X,x}$, its unique maximal ideal by $\mathfrak{m}_{X,x}$. ,

We will see later that the local ring of X at x completely determines X in a neighborhood of x (see Proposition 2.3.1(3)).

EXERCISE 1.7.6. If X is irreducible, then $\mathcal{O}(X) = \bigcap_{x \in X} \mathcal{O}_{X,x}$.

EXERCISE 1.7.7. Let X be an affine variety, $x \in X$ a point and $X' \subseteq X$ the union of irreducible components of X passing through x. Then the restriction map induces a natural isomorphism $\mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{X',x}$.

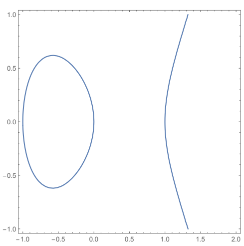
EXERCISE 1.7.8. Let R be an algebra and $\mu \colon R \to R_S$ the canonical map $r \mapsto \frac{r}{1}$ where R_S is the localization at a multiplicatively closed subset $1 \in S \subseteq R$ ($0 \notin S$).

(1) If $\mathfrak{a} \subseteq R$ is an ideal and $\mathfrak{a}_S := R_S\, \mu(\mathfrak{a}) \subseteq R_S$, then
$$\mu^{-1}(\mu(\mathfrak{a})) = \mu^{-1}(\mathfrak{a}_S) = \{b \in R \mid sb \in \mathfrak{a} \text{ for some } s \in S\}.$$
Moreover, $(R/\mathfrak{a})_{\bar{S}} \xrightarrow{\sim} R_S/\mathfrak{a}_S$ where $\bar{S}$ is the image of S in $R/\mathfrak{a}$.
(Hint: For the second assertion use the universal property of the localization.)

(2) If $\mathfrak{m} \subseteq R$ is a maximal ideal and $S := R \setminus \mathfrak{m}$, then $\mathfrak{m}_S$ is the maximal ideal of R_S and the natural maps $R/\mathfrak{m}^k \xrightarrow{\sim} R_S/\mathfrak{m}_S^k$ are isomorphisms for all $k \geq 1$.
(Hint: The image $\bar{S}$ in $R/\mathfrak{m}^k$ consists of invertible elements.)

EXERCISE 1.7.9. Let $p < q$ be positive integers with no common divisor and define $C_{p,q} := \{(t^p, t^q) \mid t \in \mathbb{C}\} \subseteq \mathbb{C}^2$. Then $C_{p,q}$ is a closed irreducible plane curve which is rational, i.e. $\mathbb{C}(C_{p,q}) \simeq \mathbb{C}(t)$. Moreover, $\mathcal{O}(C_{p,q})$ is a polynomial ring if and only if $p = 1$.

EXERCISE 1.7.10. Consider the *elliptic curve* $E := \mathcal{V}(y^2 - x(x^2-1)) \subseteq \mathbb{C}^2$. Show that E is not rational, i.e. that $\mathbb{C}(E)$ is not isomorphic to $\mathbb{C}(t)$.
(Hint: If $\mathbb{C}(E) = \mathbb{C}(t)$, then there are rational functions $f(t), h(t)$ which satisfy the equation $f(t)^2 = h(t)(h(t)^2 - 1)$.)



2. Morphisms

2.1. Morphisms and comorphisms. In the previous sections we have defined and discussed the main objects of algebraic geometry, the affine varieties. Now we have to introduce the “regular maps” between affine varieties which should be compatible with the concept of regular functions.

Definition 2.1.1. Let X, Y be affine varieties. A map $\varphi\colon X \to Y$ is called *regular* or *a morphism* if the pull-back of a regular function on Y is regular on X:

$$f \circ \varphi \in \mathcal{O}(X) \text{ for all } f \in \mathcal{O}(Y).$$

Thus we obtain a homomorphism $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ of $\mathbb{C}$-algebras given by $\varphi^*(f) := f \circ \varphi$, which is called *comorphism* of φ.

A morphism φ is called an *isomorphism* if φ is bijective and the inverse map φ^{-1} is also a morphism. If, in addition, $Y = X$, then φ is called an *automorphism*.

Example 2.1.2. A map $\varphi = (f_1, f_2, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ is regular if and only if the components f_i are polynomials in $\mathbb{C}[x_1, \dots, x_n]$. This is clear, since $\varphi^*(y_j) = f_j$ where $y_1, y_2, \dots, y_m$ are the coordinate functions on $\mathbb{C}^m$.

More generally, let X be an affine variety and a $\varphi = (f_1, \dots, f_m)\colon X \to \mathbb{C}^m$ a map. *Then φ is a morphism if and only if the components f_j are regular functions on X.* (This is clear since $f_j = \varphi^*(y_j)$.)

Example 2.1.3. The morphism $t \mapsto (t^2, t^3)$ from $\mathbb{C} \to \mathbb{C}^2$ induces a bijective morphism $\mathbb{C} \to C := \mathcal{V}(y^2 - x^3)$ which is not an isomorphism (see Example 1.3.11).

Similarly, for the curve $D := \mathcal{V}(y^2 - x^2 - x^3)$ there is a morphism $\psi\colon \mathbb{C} \to D$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. This time ψ is surjective, but not injective since $\psi(1) = \psi(-1) = (0, 0)$.

Exercise 2.1.4. Let $g \in \mathrm{GL}_n$ be an invertible matrix. Then left multiplication $A \mapsto gA$, right multiplication $A \mapsto Ag$ and conjugation $A \mapsto gAg^{-1}$ are automorphisms of M_n.

If a morphism $\varphi = (f_1, f_2, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ maps a closed subset $X \subseteq \mathbb{C}^n$ into a closed subset $Y \subseteq \mathbb{C}^m$, then the induced map $\bar{\varphi}\colon X \to Y$ is clearly a morphism, just by definition. This holds in a slightly more general setting, as claimed in the next exercise.

Exercise 2.1.5. Let $\varphi\colon X \to Y$ be a morphism. If $X' \subseteq X$ and $Y' \subseteq Y$ are closed subvarieties such that $\varphi(X') \subseteq Y'$, then the induced map $\varphi'\colon X' \to Y'$, $x \mapsto \varphi(x)$, is again a morphism. The same holds if X' and Y' are special open sets.

These examples have the following converse which will be useful in many applications.

Lemma 2.1.6. *Let $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ be closed subvarieties and let $\varphi\colon X \to Y$ be a morphism. Then there are polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\Phi:=(f_1,\dots,f_m)"] & \mathbb{C}^m \\
X \arrow[u, "\subseteq"'] \arrow[r, "\varphi"] & Y \arrow[u, "\subseteq"']
\end{tikzcd}$$

Proof. Let $y_1, \dots, y_m$ denote the coordinate functions on $\mathbb{C}^m$. Put $\bar{y}_j := y_j|_Y$ and consider $\varphi^*(\bar{y}_j) \in \mathcal{O}(X)$. Since $X \subseteq \mathbb{C}^n$ is closed there exist $f_j \in \mathbb{C}[x_1, \dots, x_n]$ such that $\bar{f}_j|_X = \varphi^*(\bar{y}_j)$, for $j = 1, \dots, m$. We claim that the morphism $\Phi := (f_1, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ satisfies the requirements of the lemma. In fact, let $a \in X \subseteq \mathbb{C}^n$ and set $\varphi(a) =: b = (b_1, \dots, b_m)$. Then

$$b_j = y_j(b) = \bar{y}_j(b) = \bar{y}_j(\varphi(a)) = \varphi^*(\bar{y}_j)(a) = \bar{f}_j(a) = f_j(a),$$

and so $\varphi(a) = \Phi(a)$. $\square$

EXERCISE 2.1.7. (1) Every morphism $\mathbb{C} \to \mathbb{C}^*$ is constant.
(2) Describe all morphisms $\mathbb{C}^* \to \mathbb{C}^*$.
(3) Every nonconstant morphism $\mathbb{C} \to \mathbb{C}$ is surjective.
(4) An injective morphism $\mathbb{C} \to \mathbb{C}$ is an isomorphism, and the same holds for injective morphisms $\mathbb{C}^* \to \mathbb{C}^*$.

EXERCISE 2.1.8. Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism and define

$$\Gamma_\varphi := \{(a, \varphi(a)) \in \mathbb{C}^{n+m}\}.$$

which is called the *graph* of the morphism φ. Show that Γ_φ is closed in $\mathbb{C}^{n+m}$, that the projection $\mathrm{pr}_{\mathbb{C}^n}\colon \mathbb{C}^{n+m} \to \mathbb{C}^n$ induces an isomorphism $p\colon \Gamma_\varphi \xrightarrow{\sim} \mathbb{C}^n$ and that $\varphi = \mathrm{pr}_{\mathbb{C}^m} \circ p^{-1}$.

PROPOSITION 2.1.9. *Let X, Y be affine varieties. The map $\varphi \mapsto \varphi^*$ induces a bijection*

$$\mathrm{Mor}(X, Y) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{C}}(\mathcal{O}(Y), \mathcal{O}(X)).$$

between the morphisms from X to Y and the algebra homomorphism from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.

REMARK 2.1.10. The mathematical term used in the situation above is that of a *contravariant functor* from the category of affine varieties and morphisms to the category of finitely generated reduced $\mathbb{C}$-algebras and homomorphism, given by $X \mapsto \mathcal{O}(X)$ and $\varphi \mapsto \varphi^*$. In particular, we have $\varphi^*(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{O}(X)}$ and $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ whenever the expressions make sense. The proposition above then says that this functor is an *equivalence*, the inverse functor being $R \mapsto \operatorname{spec} R$ defined in Proposition 1.4.6. It will be helpful to keep this "functorial point of view" in mind although it will not play an important role in the following.

PROOF. (a) If $\varphi_1^* = \varphi_2^*$, then, for all $f \in \mathcal{O}(Y)$ and all $x \in X$, we get

$$f(\varphi_1(x)) = \varphi_1^*(f)(x) = \varphi_2^*(f)(x) = f(\varphi_2(x)).$$

Hence, $\varphi_1(x) = \varphi_2(x)$ since the regular functions separate the points (Remark 1.2.12).

(b) Let $\rho\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ be an algebra homomorphism. We want to construct a morphism $\varphi\colon X \to Y$ such that $\varphi^* = \rho$. For this we can assume that $Y \subseteq \mathbb{C}^m$ is a closed subvariety. Let $\bar{y}_j := y_j|_Y$ be the restrictions of the coordinate functions on $\mathbb{C}^m$ and define $f_j := \rho(\bar{y}_j) \in \mathcal{O}(X)$. Then we get a morphism $\Phi := (f_1, \dots, f_m)\colon X \to \mathbb{C}^m$ such that $\Phi^*(y_j) = f_j$ (see Example 2.1.2). If $h = h(y_1, \dots, y_m) \in I(Y)$, then

$$h(f_1, \dots, f_m) = h(\rho(\bar{y}_1), \dots, \rho(\bar{y}_m)) = \rho(h(\bar{y}_1, \dots, \bar{y}_m)) = 0$$

because $h(\bar{y}_1, \dots, \bar{y}_m) = h|_Y = 0$ by assumption. Therefore $h(\Phi(a)) = 0$ for all $a \in X$ and all $h \in I(Y)$ and so $\Phi(X) \subseteq Y$. This shows that Φ induces a morphism $\varphi\colon X \to Y$ such that $\varphi^*(\bar{y}_j) = \Phi^*(y_j) = f_j = \rho(\bar{y}_j)$, and so $\varphi^* = \rho$. $\square$

EXAMPLE 2.1.11. Let X be an affine variety, V a finite dimensional vector space and $\varphi\colon X \to V$ a morphism. The linear functions on V form a subspace $V^* \subseteq \mathcal{O}(V)$ which generates $\mathcal{O}(V)$. Therefore, the induced linear map $\varphi^*|_{V^*}\colon V^* \to \mathcal{O}(X)$ completely determines φ^*, and we get a bijection

$$\mathrm{Mor}(X, V) \xrightarrow{\sim} \mathrm{Hom}(V^*, \mathcal{O}(X)) \quad \varphi \mapsto \varphi^*|_{V^*}.$$

The second assertion follows from Proposition 2.1.9 and the well-known "Substitution Principle" for polynomials rings (see [**Art91**, Chap. 10, Proposition 3.4]).

EXERCISE 2.1.12. Show that for an affine variety X the morphisms $X \to \mathbb{C}^*$ correspond bijectively to the invertible functions on X.

EXERCISE 2.1.13. Let X, Y be affine varieties and $\varphi\colon X \to Y$, $\psi\colon Y \to X$ morphisms such that $\psi \circ \varphi = \mathrm{Id}_X$. Then $\varphi(X) \subseteq Y$ is closed and $\varphi\colon X \xrightarrow{\sim} \varphi(X)$ is an isomorphism.

2.2. Images, preimages and fibers. It is easy to see that morphisms are continuous. In fact, the ZARISKI topology is the finest topology such that regular functions are continuous, and since morphisms are defined by the condition that the pull-back of a regular function is again regular, it immediately follows that morphisms are continuous. We will get this result again from the next proposition where we describe images and preimages of closed subsets under morphisms.

PROPOSITION 2.2.1. *Let $\varphi\colon X \to Y$ be a morphism of affine varieties.*

(1) *If $B := \mathcal{V}_Y(S) \subseteq Y$ is the closed subset defined by $S \subseteq \mathcal{O}(Y)$, then $\varphi^{-1}(B) = \mathcal{V}_X(\varphi^*(S))$. In particular, φ is continuous.*
(2) *Let $A := \mathcal{V}(\mathfrak{a}) \subseteq X$ be the closed subset defined by the ideal $\mathfrak{a} \subseteq \mathcal{O}(X)$. Then the closure of the image $\varphi(A)$ is defined by ${\varphi^*}^{-1}(\mathfrak{a}) \subseteq \mathcal{O}(Y)$:*
$$\overline{\varphi(A)} = \mathcal{V}_Y({\varphi^*}^{-1}(\mathfrak{a})).$$

PROOF. For $x \in X$ we have
$$x \in \varphi^{-1}(B) \iff \varphi(x) \in B \iff f(\varphi(x)) = 0 \text{ for all } f \in S,$$
and this is equivalent to $\varphi^*(f)(x) = 0$ for all $f \in S$, hence to $x \in \mathcal{V}_X(\varphi^*(S))$, proving the first claim.

For the second claim, let $f \in \mathcal{O}(Y)$. Then
$$f|_{\overline{\varphi(A)}} = 0 \iff f|_{\varphi(A)} = 0 \iff \varphi^*(f)|_A = 0 \iff \varphi^*(f) \in I(A) = \sqrt{\mathfrak{a}}$$
The latter is equivalent to the condition that a power of f belongs to ${\varphi^*}^{-1}(\mathfrak{a})$. Thus the zero set of ${\varphi^*}^{-1}(\mathfrak{a})$ equals the closed set $\overline{\varphi(A)}$. □

EXERCISE 2.2.2. If $\varphi_1, \varphi_2\colon X \to Y$ are two morphisms, then the "kernel of coincidence"
$$\ker(\varphi_1, \varphi_2) := \{x \in X \mid \varphi_1(x) = \varphi_2(x)\} \subseteq X$$
is closed in X

EXERCISE 2.2.3. Let $\varphi\colon X \to Y$ be a morphism of affine varieties.

(1) If X is irreducible, then $\overline{\varphi(X)}$ is irreducible.
(2) Every irreducible component of X is mapped into an irreducible component of Y.
(3) If $U \subseteq Y$ is a special open set, then so is $\varphi^{-1}(U)$.

EXERCISE 2.2.4. Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism, $\varphi = (f_1, f_2, \dots, f_m)$ where $f_i \in \mathbb{C}[x_1, x_2, \dots, x_n]$, and let $Y := \overline{\varphi(\mathbb{C}^n)}$ be the closure of the image of φ. Then
$$I(Y) = (y_1 - f_1, y_2 - f_2, \dots, y_m - f_m) \cap \mathbb{C}[y_1, y_2, \dots, y_m]$$
where both sides are considered as subsets of $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$. So $I(Y)$ is obtained from the ideal $(y_1 - f_1, \dots, y_m - f_m)$ by *eliminating the variables* $x_1, \dots, x_n$.
(Hint: Use the graph Γ_φ defined in Exercise 2.1.8 and show that the ideal $I(\Gamma_\varphi)$ is generated by $\{y_j - f_j \mid j = 1, \dots, m\}$.)

EXERCISE 2.2.5. Let $\varphi\colon X \xrightarrow{\sim} X$ be an automorphism and $Y \subseteq X$ a closed subset such that $\varphi(Y) \subseteq Y$. Then $\varphi(Y) = Y$ and $\varphi|_Y\colon Y \to Y$ is an automorphism, too.
(Hint: Look at the descending chain $Y \supseteq Y_1 := \varphi(Y) \supseteq Y_2 := \varphi(Y_1) \supseteq \cdots$. If $Y_n = Y_{n+1}$, then $\varphi(Y_{n-1}) = Y_n = \varphi(Y_n)$ and so $Y_{n-1} = Y_n$.)

DEFINITION 2.2.6. A morphism $\varphi\colon X \to Y$ is called a *closed immersion* if $\varphi(X) \subseteq Y$ is closed and the induced map $X \to \varphi(X)$ is an isomorphism.

LEMMA 2.2.7. *A morphism $\varphi\colon X \to Y$ is a closed immersion if and only if the comorphism $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective.*

Proof. If φ is a closed immersion, then $\mathcal{O}(X) \simeq \mathcal{O}(\varphi(X))$ and the regular functions on $\varphi(X)$ are restrictions from regular functions on Y, hence φ^* is surjective.

Now assume that φ^* is surjective, and put $\mathfrak{a} := \ker \varphi^*$. This is a radical ideal and so $\mathfrak{a} = I(A)$ where $A := \mathcal{V}_X(\mathfrak{a})$. By definition, φ^* has the decomposition $\mathcal{O}(Y) \twoheadrightarrow \mathcal{O}(A) \xrightarrow{\sim} \mathcal{O}(X)$, i.e. φ induces an isomorphism $X \xrightarrow{\sim} A \subseteq Y$. $\square$

Exercise 2.2.8. Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be morphisms, and assume that the composition $\psi \circ \varphi$ is a closed immersion. Then φ is a closed immersion.

A special case of preimages are the *fibers* of a morphism $\varphi\colon X \to Y$. Let $y \in Y$. Then

$$\varphi^{-1}(y) := \{x \in X \mid \varphi(x) = y\}$$

is called the *fiber of* $y \in Y$. By the proposition above, the fiber of y is a closed subvariety of X defined by $\varphi^*(\mathfrak{m}_y)$:

$$\varphi^{-1}(y) = \mathcal{V}_X(\varphi^*(\mathfrak{m}_y)).$$

Of course, the fiber of a point $y \in Y$ can be empty. In algebraic terms this means that $\varphi^*(\mathfrak{m}_y)$ generates the unit ideal $(1) = \mathcal{O}(X)$.

Exercise 2.2.9. Describe the fibers of the morphism $\varphi\colon \mathrm{M}_2 \to \mathrm{M}_2$, $A \mapsto A^2$. (Hint: Use the fact that $\varphi(gAg^{-1}) = g\varphi(A)g^{-1}$ for $g \in \mathrm{GL}_2$.)

Definition 2.2.10. Let $\varphi\colon X \to Y$ be a morphism of affine varieties and consider the fiber $F := \varphi^{-1}(y)$ of a point $y \in \varphi(X) \subseteq Y$. Then the fiber F is called *reduced* if $\varphi^*(\mathfrak{m}_y)$ generates a perfect ideal in $\mathcal{O}(X)$, i.e. if

$$\sqrt{\mathcal{O}(X) \cdot \varphi^*(\mathfrak{m}_y)} = \mathcal{O}(X) \cdot \varphi^*(\mathfrak{m}_y).$$

The fiber F is called *reduced in the point* $x \in F$ if this holds in the local ring $\mathcal{O}_{X,x}$, i.e.

$$\sqrt{\mathcal{O}_{X,x} \cdot \varphi^*(\mathfrak{m}_y)} = \mathcal{O}_{X,x} \cdot \varphi^*(\mathfrak{m}_y).$$

Example 2.2.11. Look again at the morphism $\varphi\colon \mathbb{C} \to C := \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$, $t \mapsto (t^2, t^3)$. Then φ^* is the injection $\mathcal{O}(C) \xrightarrow{\sim} \mathbb{C}[t^2, t^3] \hookrightarrow \mathbb{C}[t]$ and so

$$\mathbb{C}[t] \cdot \varphi^*(\mathfrak{m}_{(0,0)}) = (t^2, t^3) \subsetneq \sqrt{(t^2, t^3)} = (t).$$

Thus the zero fiber $\varphi^{-1}(0)$ is not reduced. On the other hand, all other fibers are reduced. In fact, φ induces an isomorphism of $\mathbb{C}^*$ with the special open set $C \setminus \{(0,0)\} (= C_{\bar{x}} = C_{\bar{y}})$, where the inverse map is given by $(a, b) \mapsto \frac{b}{a}$.

The following lemma shows that reducedness is a local property.

Lemma 2.2.12. *Let* $\varphi\colon X \to Y$ *be a morphism and* $F := \varphi^{-1}(y)$ *the fiber of* $y \in Y$.

(1) *If* F *is reduced in* $x \in F$, *then* F *is reduced in a neighborhood of* x.
(2) *If* F *is reduced in every* $x \in F$, *then* F *is reduced.*

Proof. We will use here some standard facts related to "localization", see [**Eis95**, I.2.1]. Set $R := \mathcal{O}(X)/\varphi^*(\mathfrak{m}_y)\mathcal{O}(X)$, and let $\mathfrak{r} := \sqrt{(0)} \subseteq R$ denote the nilradical.

(1) Since $R_{\mathfrak{m}_x}$ is reduced, the ideal $\mathfrak{r}$ is in the kernel of the map $R \to R_{\mathfrak{m}_x}$. It follows that there is an element $s \notin \mathfrak{m}_x$ such that $\mathfrak{r}$ belongs to the kernel of $R \to R_s$, i.e. R_s is reduced. This means that the fiber F is reduced in every point of F_s.

(2) If F is reduced in every point, it follows from (1) that there are finitely many $s_i \in R$ such that R_{s_i} is reduced for all i and that $(s_1, \ldots, s_m) = R$. This implies that $s_i^N \cdot \mathfrak{r} = (0)$ for all i and some $N > 0$, hence $\mathfrak{r} = (0)$, because $1 \in (s_1, \ldots, s_m)$. $\square$

Exercise 2.2.13. Show that all fibers of the morphism $\psi\colon \mathbb{C} \to D := \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{C}^2$, $t \mapsto (t^2-1, t(t^2-1))$, are reduced and that ψ induces an isomorphism $\mathbb{C} \setminus \{1,-1\} \xrightarrow{\sim} D \setminus \{(0,0)\}$.

Exercise 2.2.14. Consider the morphism $\varphi\colon \mathrm{SL}_2 \to \mathbb{C}^3$, $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) := (ab, ad, cd)$.

(1) The image of φ is a closed hypersurface $H \subseteq \mathbb{C}^3$ defined by $xz - y(y-1) = 0$.
(2) The fibers of φ are the left cosets of the subgroup $T := \{\begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \mid t \in \mathbb{C}^*\}$.
(3) All fibers are reduced.

(Hint: Show that the left multiplication with some $g \in \mathrm{SL}_2$ induces an automorphism λ_g of H and isomorphisms $\varphi^{-1}(y) \xrightarrow{\sim} \varphi^{-1}(\lambda_g(y))$ for all $y \in H$. This implies that it suffices to study just one fiber, e.g. $\varphi^{-1}(\varphi(E))$.)

Exercise 2.2.15. Consider the morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $\varphi(x,y) := (x, xy)$.
(1) $\varphi(\mathbb{C}^2) = \mathbb{C}^2 \setminus \{(0,y) \mid y \neq 0\}$ which is not locally closed.
(2) What happens with the lines parallel to the x-axis or parallel to the y-axis?
(3) $\varphi^{-1}(0) = y$-axis. Is this fiber reduced?
(4) φ induces an isomorphism $\mathbb{C}^2 \setminus y$-axis $\xrightarrow{\sim} \mathbb{C}^2 \setminus y$-axis.

2.3. Dominant morphisms and degree. Let $\varphi\colon X \to Y$ be a morphism of affine varieties, x a point of X and $y := \varphi(x)$ its image in Y. Then $\varphi^*(\mathfrak{m}_y) \subseteq \mathfrak{m}_x$, and so φ^* induces a local homomorphism

$$\varphi_x^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}.$$

(A homomorphism between local rings is called *local* if it maps the maximal ideal into the maximal ideal.)

The next proposition tells us that, in a neighborhood of a point $x \in X$, a morphism φ is uniquely determined by the local homomorphism φ_x^*.

Proposition 2.3.1. (1) *Let $\varphi, \psi\colon X \to Y$ be two morphisms and $x \in X$ a point such that $\varphi(x) = \psi(x)$ and $\varphi_x^* = \psi_x^*$. Then φ and ψ coincide on every irreducible component of X which contains x.*
(2) *If $x \in X$, $y \in Y$ and if $\rho\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a local homomorphism, then there is a special open sets $X' \subseteq X$ containing x and a morphism $\varphi\colon X' \to Y$ such that $\varphi_x^* = \rho$.*
(3) *If $x \in X$, $y \in Y$ and $\rho\colon \mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x}$ an isomorphism, then there exist special open sets $X' \subseteq X$ and $Y' \subseteq Y$ containing x and y, respectively, and an isomorphism $\varphi\colon X' \xrightarrow{\sim} Y'$ such that $\varphi_x^* = \rho$.*

Proof. (1) Let R be a finitely generated reduced $\mathbb{C}$-algebra and $\mathfrak{m} \subseteq R$ a maximal ideal. The canonical map $\mu\colon R \to R_{\mathfrak{m}}$ is injective if and only if $\mathfrak{m}$ contains all minimal prime ideals of R. (In fact, $\ker\mu = \{r \in R \mid sr = 0 \text{ for some } s \in R \setminus \mathfrak{m}\}$.)

Denote by $\bar{X} \subseteq X$ the union of irreducible components passing through x and by $\bar{Y} \subseteq Y$ the union of irreducible components passing through $\varphi(x)$. Then $\varphi(\bar{X}) \subseteq \bar{Y}$, because the image of an irreducible component of X is contained in an irreducible component of Y (see Exercise 2.2.3). Thus we obtain a morphism $\bar{\varphi}\colon \bar{X} \to \bar{Y}$ with the following commutative diagram of $\mathbb{C}$-algebras and homomorphisms which shows that $\bar{\varphi}$ is completely determined by φ_x^*:

$$\begin{CD}
\mathcal{O}(Y) @>>> \mathcal{O}(\bar{Y}) @>\subseteq>> \mathcal{O}_{\bar{Y},\varphi(x)} = \mathcal{O}_{Y,\varphi(x)} \\
@VV\varphi^*V @VV\bar{\varphi}^*V @VV\varphi_x^*V \\
\mathcal{O}(X) @>>> \mathcal{O}(\bar{X}) @>\subseteq>> \mathcal{O}_{\bar{X},x} = \mathcal{O}_{X,x}
\end{CD}$$

(2) We can assume that all irreducible components of X pass through x and all irreducible components of Y pass through y. Then

$$\mathcal{O}(Y) \subseteq \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \supseteq \mathcal{O}(X).$$

Let $h_1, \ldots, h_m \in \mathcal{O}(Y)$ be a set of generators and put $g_j := \rho(h_j)$. Then we can find an element $t \in \mathcal{O}(X) \setminus \mathfrak{m}_x$ such that $g_j \in \mathcal{O}(X)_t$ for all j, i.e. $\rho(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)_t$. Hence there is a morphism $\varphi\colon X_t \to Y$ such that $\varphi^* = \rho|_{\mathcal{O}(X)_t}$, and so $\varphi_x^* = \rho$.

(3) By (2) we can assume that there is a morphism $\varphi\colon X \to Y$ such that $\varphi_x^* = \rho$, i.e. $\rho(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)$. Let $f_1, \ldots, f_n \in \mathcal{O}(X)$ be generators. Then $f_i = \frac{\rho(h_i)}{\rho(s)}$ where $h_i \in \mathcal{O}(Y)$ and $s \in \mathcal{O}(Y) \setminus \mathfrak{m}_y$. This implies that $\rho(\mathcal{O}(Y)_s) = \mathcal{O}(X)_t$ where $t = \rho(s)$. Thus ρ induces an isomorphism $\mathcal{O}(Y)_s \xrightarrow{\sim} \mathcal{O}(X)_t$, and the claim follows. □

Definition 2.3.2. Let X, Y be irreducible affine varieties. A morphism $\varphi\colon X \to Y$ is called *dominant* if the image is dense in Y, i.e. $\overline{\varphi(X)} = Y$. This is equivalent to the condition that $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective (see Proposition 2.2.1(2)).

It follows that every dominant morphism $\varphi\colon X \to Y$ induces a *finitely generated field extension* $\varphi^*\colon \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$. If this is a finite field extension of degree $d := [\mathbb{C}(X) : \mathbb{C}(Y)]$ we will say that φ is a *morphism of finite degree* d. If $d = 1$, i.e. if φ^* induces an isomorphism $\mathbb{C}(Y) \xrightarrow{\sim} \mathbb{C}(X)$, then φ is called a *birational morphism*.

Exercise 2.3.3. Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant morphism. Then φ has finite degree d, and there is a non-empty open set $U \subseteq \mathbb{C}$ such that $\#\varphi^{-1}(x) = d$ for all $x \in U$.

There is a similar result as the second part of Proposition 2.3.1 saying that affine varieties with isomorphic function fields are locally isomorphic.

Proposition 2.3.4. *Let X and Y be irreducible affine varieties and assume that we have an isomorphism $\rho\colon \mathbb{C}(Y) \xrightarrow{\sim} \mathbb{C}(X)$. Then there exist special open sets $X' \subseteq X$ and $Y' \subseteq Y$ and an isomorphism $\psi\colon X' \xrightarrow{\sim} Y'$ such that $\rho = \psi^*$.*

Proof. Since $\mathcal{O}(Y) \subseteq \mathbb{C}(Y)$ is finitely generated, there is an $f \in \mathcal{O}(X)$ such that $\rho(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)_f$. Replacing X by X_f we can therefore assume that $\rho(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)$. By the same argument we can find an $h \in \mathcal{O}(Y)$ such that $\rho^{-1}(\mathcal{O}(X)) \subseteq \mathcal{O}(Y)_h$. Thus $\rho(\mathcal{O}(Y)_h) \subseteq \mathcal{O}(X)_{\rho(h)}$ and $\rho^{-1}(\mathcal{O}(X)_{\rho(h)}) \subseteq \mathcal{O}(Y)_h$. Hence $\rho(\mathcal{O}(Y)_h) = \mathcal{O}(X)_{\rho(h)}$, and we get an isomorphism $\psi\colon X_{\rho(h)} \xrightarrow{\sim} Y_h$ with $\psi^* = \rho$. □

2.4. Rational varieties and Lüroth's Theorem. An irreducible affine variety X is called *rational* if its field of rational functions $\mathbb{C}(X)$ is a purely transcendental extension of $\mathbb{C}$ (section 1.7). By Proposition 2.3.4 this means that X contains a special open set U which is isomorphic to a special open set of $\mathbb{C}^n$.

Proposition 2.4.1. *Let $\varphi\colon X \to Y$ be a dominant morphism where X is rational and* $\dim Y = 1$. *Then Y is a rational curve.*

Proof. We can assume that X is a special open set of $\mathbb{C}^n$. Then there is a line L in $\mathbb{C}^n$ such that $\varphi(L \cap X) \subseteq Y$ is dense. This implies that $\mathbb{C}(C) \subseteq \mathbb{C}(L \cap X) \xrightarrow{\sim} \mathbb{C}(x)$, and the claim follows from the Lüroth's Theorem below. □

Theorem 2.4.2 (Lüroth's Theorem). *Let $K \subseteq \mathbb{C}(x)$ be a subfield which contains $\mathbb{C}$. Then there is an $h \in K$ such that $K = \mathbb{C}(h)$.*

Proof. We can assume that $K \neq \mathbb{C}$. Any $f(t) \in \mathbb{C}(x)[t]$ can be written in the form $f(t) = \frac{p(x,t)}{q(x)}$ where $p(x,t) \in \mathbb{C}[x,t]$, $q(x) \in \mathbb{C}[x]$, and p, q are relatively prime. Define the *degree* of f by $\deg(f) := \max\{\deg_x p, \deg_x q\}$. It is easy to see that $\deg(f) = \deg(f_1) + \deg(f_2)$ in case $f = f_1 f_2$ and both factors f_i are monic as polynomials in t.

Let $h \in K \setminus \mathbb{C}$ be an element of minimal degree d, $h = \frac{r(x)}{s(x)}$ where $r, s \in \mathbb{C}[x]$. We can assume that r, s are monic and that $\deg_x s < \deg_x r = d$. Set $f = f(t) := r(t) - hs(t) \in K[t] \subseteq \mathbb{C}(x)[t]$. Then $\deg_t f = d$ and $f(x) = 0$. We claim that f is irreducible in $K[t]$. This implies that f is the minimal polynomial of x over K, but also the minimal polynomial of x over $\mathbb{C}(h)$, hence $K = \mathbb{C}(h)$.

It remains to see that f is irreducible as a polynomial in $K[t]$. If $f(t) = f_1(t)f_2(t)$, then $\deg(f) = \deg(f_1) + \deg(f_2)$ since f is monic. If $\deg(f_1) = 0$, then $f_1(t) \in \mathbb{C}[t]$, and thus $f_1(t)$ divides $r(t)$ and $s(t)$, because h is purely transcendental over $\mathbb{C}$. Therefore, we can assume that $0 < \deg(f_1) < d$. But then one of the coefficients of $f_1(t)$ belongs to $K \setminus \mathbb{C}$ and has height $< d$, contradicting the minimality of d. $\square$

2.5. Products. If f is a function on X and h a function on Y, then we denote by $f \cdot h$ the $\mathbb{C}$-valued function on the defined by $(f \cdot h)(x, y) := f(x) \cdot h(y)$.

PROPOSITION 2.5.1. *The product $X \times Y$ of two affine varieties together with the algebra*

$$\mathcal{O}(X \times Y) := \mathbb{C}[f \cdot h \mid f \in \mathcal{O}(X), h \in \mathcal{O}(Y)]$$

of $\mathbb{C}$-valued functions is an affine variety. Moreover, the canonical homomorphism $\mathcal{O}(X) \otimes \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$, $f \otimes h \mapsto f \cdot h$, is an isomorphism.

PROOF. Let $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ be closed subvarieties. Then $X \times Y \subseteq \mathbb{C}^{n+m}$ is closed, namely equal to the zero set $\mathcal{V}(I(X) \cup I(Y))$. So it remains to show that $\mathcal{O}(X \times Y) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]/I(X \times Y)$ is generated by the products $f \cdot h$ and that $f \cdot h \in \mathcal{O}(X \times Y)$ for $f \in \mathcal{O}(X)$ and $h \in \mathcal{O}(Y)$. But this is clear since $\bar{x}_i = x_i|_{X \times Y} = x_i|_X \cdot 1$ and $\bar{y}_j = y_j|_{X \times Y} = 1 \cdot y_j|_Y$, and $f|_X \cdot h|_Y = (fh)|_{X \times Y}$ for $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $h \in \mathbb{C}[y_1, \ldots, y_m]$.

For the last claim, we only have to show that the map $\mathcal{O}(X) \otimes \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$, $f \otimes h \mapsto f \cdot h$, is injective. For this, let $(f_i \mid i \in I)$ be a basis of $\mathcal{O}(Y)$. Then every element $s \in \mathcal{O}(X) \otimes \mathcal{O}(Y)$ can be uniquely written as $s = \sum_{\text{finite}} s_i \otimes f_i$. If s is in the kernel of the map, then $\sum s_i(x) f_i(y) = 0$ for all $(x, y) \in X \times Y$ and so, for every fixed $x \in X$, $\sum s_i(x) f_i$ is the zero function on Y. This implies that $s_i(x) = 0$ for all $x \in X$ and so $s_i = 0$ for all i. Thus $s = 0$ proving the claim. $\square$

EXAMPLE 2.5.2.

(1) By definition, we have $\mathbb{C}^m \times \mathbb{C}^n = \mathbb{C}^{m+n}$.

(2) The two projections $\mathrm{pr}_X \colon X \times Y \to X$, $(x, y) \mapsto x$, and $\mathrm{pr}_Y \colon X \times Y \to Y$, $(x, y) \mapsto y$, are morphisms with comorphisms $\mathrm{pr}_X^*(f) = f \cdot 1$ and $\mathrm{pr}_Y^*(h) = 1 \cdot h$.

(3) If $\varphi \colon X \to X'$ and $\psi \colon Y \to Y'$ are morphisms, then so is

$$\varphi \times \psi \colon X \times Y \to X' \times Y', \quad (x, y) \mapsto (\varphi(x), \psi(y)).$$

(4) **Diagonal:** $\Delta \colon X \to X \times X$, $x \mapsto (x, x)$ is a closed immersion where $\Delta(X) \subseteq X \times X$ is the closed subset defined by $\{f \cdot 1 - 1 \cdot f \mid f \in \mathcal{O}(X)\}$.

(5) **Graph:** Let $\varphi \colon X \to Y$ be a morphism. Then

$$\Gamma(\varphi) := \{(x, \varphi(x)) \mid x \in X\} \subseteq X \times Y$$

is a closed subset. Moreover, the projection pr_X induces an isomorphism $p \colon \Gamma(\varphi) \xrightarrow{\sim} X$ and $\varphi = \mathrm{pr}_Y \circ p^{-1}$.

(6) **Matrix multiplication:** The composition of linear maps

$$\mu \colon \operatorname{Hom}(U, V) \times \operatorname{Hom}(V, W) \to \operatorname{Hom}(U, W), \quad (A, B) \mapsto B \circ A$$

is a morphism. Choosing coordinates we find $\mu^*(z_{ij}) = \sum_k y_{ik} x_{kj}$.

EXERCISE 2.5.3. Show that the ideal of the diagonal $\Delta(X) \subseteq X \times X$ is generated by the function $f \cdot 1 - 1 \cdot f$, $f \in \mathcal{O}(X)$ (see Example 2.5.2(4)).

Lemma 2.5.4. *The projection* $\mathrm{pr}_X : X \times Y \to X$ *is an open morphism, i.e. the image of an open set under* pr_X *is open.*

Proof. It suffices to show that the image of a special open set $U := (X\times Y)_g$ is open. Writing $g = \sum f_i \cdot h_i$ with linearly independent h_i one gets $\mathrm{pr}_X(U) = \bigcup_i X_{f_i}$ and the claim follows. □

Proposition 2.5.5. *If* X, Y *are irreducible affine varieties, then* $X \times Y$ *is irreducible.*

Proof. Assume that $X \times Y = A \cup B$ with closed subsets A, B. Define

$$X_A := \{x \in X \mid \{x\} \times Y \subseteq A\} \quad \text{and} \quad X_B := \{x \in X \mid \{x\} \times Y \subseteq B\}$$

Since Y is irreducible we see that $X = X_A \cup X_B$. Now we claim that X_A and X_B are both closed in X and so one of them equals X, say $X_A = X$. But then $A = X \times Y$ and we are done. To prove the claim we remark that $X \setminus X_A = \mathrm{pr}_X(X \times Y \setminus A)$ which is open by Lemma 2.5.4 above. □

Corollary 2.5.6. *If* $X = \bigcup_i X_i$ *and* $Y = \bigcup_j Y_j$ *are the irreducible decompositions of* X *and* Y*, then* $X \times Y = \bigcup_{i,j} X_i \times Y_j$ *is the irreducible decomposition of the product.*

Remark 2.5.7. In terms of algebras, Proposition 2.5.5 above says that a tensor product $A \otimes B$ of two finitely generated domains is a domain.

2.6. Fiber products. Let X, Y, S be affine varieties and let $\varphi : X \to S$, $\psi : Y \to S$ two morphisms. Then

$$X \times_S Y := \{(x, y) \in X \times Y \mid \varphi(x) = \psi(y)\} \subseteq X \times Y$$

is a closed subset. In fact, it is the inverse image $(\varphi \times \psi)^{-1}(\Delta(S))$ of the diagonal $\Delta(S) \subseteq S\times S$ which is a closed subset (Example 2.5.2(4)). We have the commutative diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\psi"] \\
X \arrow[r, "\varphi"] & S
\end{tikzcd}$$

where the morphisms p and q are induced by the projections $X \times Y \to X$ and $X \times Y \to Y$. The affine variety is called the *fiber product* of X, Y over S. It has the following universal property which defines it up to unique isomorphisms.

Proposition 2.6.1. *If* $\alpha : Z \to X$ *and* $\beta : Z \to Y$ *are two morphisms such that* $\varphi \circ \alpha = \psi \circ \beta$*, then there is a unique morphism* $(\alpha, \beta) : Z \to X \times_S Y$ *such that* $p \circ (\alpha, \beta) = \alpha$ *and* $q \circ (\alpha, \beta) = \beta$*:*

$$\begin{tikzcd}
Z \arrow[dr, "(\alpha,\beta)"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "\psi"] \\
& X \arrow[r, "\varphi"] & S
\end{tikzcd}$$

Proof. Clearly, the morphism $z \mapsto (\alpha(z), \beta(z)) \in X \times Y$ has its image in $X \times_Z Y$ and satisfies the conditions. It is also obvious that it is unique. □

Example 2.6.2. (1) If $\varphi : X \hookrightarrow S$ is a closed immersion, then q is a closed immersion with image $\psi^{-1}(X)$.

(2) If $s \in S$ and $X = \{s\} \hookrightarrow S$, then $\{s\} \times_S Y = \psi^{-1}(s)$.
(3) If $f \in \mathcal{O}(S)$ and $\varphi\colon X = S_f \hookrightarrow S$, then $S_f \times_S Y = Y_{\psi^*(f)} \subseteq Y$.

Example 2.6.3. We look again at the curve $D := \mathcal{V}(y^2 - x^2 - x^3)$ and the morphism $\psi\colon \mathbb{C} \to D$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$ from Example 2.1.3 (see also Exercise 2.2.13). Then $\mathbb{C} \times_D \mathbb{C} = \Delta \cup \{(1, -1), (-1, 1)\} \subseteq \mathbb{C}^2$ where Δ is the diagonal.

Exercise 2.6.4. Show that $\mathcal{O}(X \times_S Y) \simeq (\mathcal{O}(X) \otimes_{\mathcal{O}(S)} \mathcal{O}(Y))_{\text{red}}$ where $R_{\text{red}} := R/\sqrt{(0)}$.

Example 2.6.5. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a morphism defined by a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ of degree d. Then all fibers $f^{-1}(\lambda)$ for $\lambda \neq 0$ are isomorphic and smooth. They are irreducible if and only if f is not a power of another polynomial.

Proof. The first part is clear, because $\sum_i \frac{\partial f}{\partial x_i} x_i = d \cdot f$. It is also obvious that $f - 1$ is reducible, if f is a power of another polynomial. So assume that $f - 1$ is reducible, and consider the polynomial $F(x_1, \ldots, x_n, z) := f(x_1, \ldots, x_n) - z^d$. Then the zero set $\mathcal{V}(F)$ is the fiber product

$$\begin{tikzcd}
\mathcal{V}(F) = \mathbb{C} \times_{\mathbb{C}} \mathbb{C}^n \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{C}^n \arrow[d, "f"] \\
\mathbb{C} \arrow[r, "?^d"] & \mathbb{C}
\end{tikzcd}$$

and $\mathcal{V}(F) \setminus p^{-1}(\mathbb{C}^*) \simeq \mathbb{C}^* \times f^{-1}(1)$, because f is homogeneous of degree d. This shows that $\mathcal{V}(F)$ and hence F is reducible. Considering F as a polynomial $F = f - z^n \in K[z]$ where $f \in K := \mathbb{C}[x_1, \ldots, x_n]$, we can use a standard result from Galois theory to deduce that f is a power (Exercise 2.6.6). $\square$

Exercise 2.6.6. Let K be a field of characteristic zero which contains the roots of unity. Let $d \in \mathbb{N}$ and assume that $a \in K \setminus \bigcup_{p|d} K^p$. Then the polynomial $z^d - a \in K[z]$ is irreducible.
(Hint: If $b^d = a$, then $z^d - a = \prod_j (z - \zeta^j b)$ where $\zeta \in K$ is a primitive d-th root of unity. It follows that $K[b]/K$ is a Galois extension, and that the Galois group G embeds into the group $\mu_d \subseteq K$ of d-th roots of unity by $\sigma \mapsto \frac{\sigma(b)}{b}$. Thus G is cyclic, and if the order is $m|d$, then the power of b^m is fixed by G.)

3. Dimension

3.1. Definitions and basic results. If k is a field and A a k-algebra, then a set $a_1, a_2, \ldots, a_n \in A$ of elements from A are called *algebraically independent over* k if they do not satisfy a non-trivial polynomial equation $F(a_1, a_2, \ldots, a_n) = 0$ where $F \in k[x_1, \ldots, x_n]$. Equivalently, the canonical homomorphism of k-algebras $k[x_1, \ldots, x_n] \to A$ defined by $x_i \mapsto a_i$ is injective.

In order to define the dimension of a variety we will need the concept of *transcendence degree* $\operatorname{tdeg}_k K$ of a field extension K/k. It is defined to be the maximal number of algebraically independent elements in K. Such a set is called a *transcendence basis*, and all such bases have the same number of elements. We refer to [**Art91**, Chap. 13, Sect. 8] for the basic properties of transcendental extensions.

Definition 3.1.1. Let X be an irreducible affine variety and $\mathbb{C}(X)$ its field of rational functions. Then the *dimension* of X is defined by

$$\dim X := \operatorname{tdeg}_{\mathbb{C}} \mathbb{C}(X).$$

If X is reducible and $X = \bigcup X_i$ the irreducible decomposition (see 1.6), then

$$\dim X := \max_i \dim X_i.$$

Finally, we define the *local dimension* of X in a point $x \in X = \bigcup X_i$ to be

$$\dim_x X := \max_{X_i \ni x} \dim X_i.$$

EXAMPLE 3.1.2.

(1) We have $\dim \mathbb{C}^n = n$. More generally, if V is a complex vector space of dimension n, then $\dim V = n$.
(In fact, $x_1, \dots, x_n$ is a transcendence basis of the field $\mathbb{C}(x_1, \dots, x_n)$.)

(2) If $U \subseteq X$ is a special open subset which is dense in X, then $\dim U = \dim X$.
(This is obvious if X is irreducible. If $X_i \subseteq X$ is an irreducible component, then $U_i := U \cap X_i$ is a special open set and $U = \bigcup_i U_i$ is the decomposition into irreducible components.)

(3) Every maximal set of algebraically independent elements of $\mathcal{O}(X)$ consists of $\dim X$ elements.
(For an irreducible X this is clear, and one easily reduces to this case.)

EXERCISE 3.1.3. If $\varphi\colon X \xrightarrow{\sim} Y$ is an isomorphism, then $\dim_x X = \dim_{\varphi(x)} Y$ for all $x \in X$.

EXERCISE 3.1.4. Let $G \subseteq \mathrm{GL}_n$ be a closed subgroup. Then $\dim_g G = \dim G$ for all $g \in G$.
(Hint: Use the fact that left multiplication with g is an isomorphisms $G \xrightarrow{\sim} G$.)

LEMMA 3.1.5. *For affine varieties X, Y we have* $\dim(X \times Y) = \dim X + \dim Y$.

PROOF. It suffices to consider the case where X, Y are irreducible, see Corollary 2.5.6. Then $\mathcal{O}(X) \otimes \mathcal{O}(Y)$ is a domain as well as $\mathbb{C}(X) \otimes \mathbb{C}(Y)$. Now $\mathbb{C}(X)$ is finite over a subfield $\mathbb{C}(x_1, \dots, x_n)$ where $n = \dim X$, and $\mathbb{C}(Y)$ is finite over a subfield $\mathbb{C}(y_1, \dots, y_m)$ where $m = \dim Y$. Hence $\mathbb{C}(X) \otimes \mathbb{C}(Y)$ is finitely generated over $\mathbb{C}(x_1, \dots, x_n) \otimes \mathbb{C}(y_1, \dots, y_m)$. Since $\mathbb{C}(X \times Y)$ is the field of fractions of $\mathbb{C}(X) \otimes \mathbb{C}(Y)$, it follows that it is finite over $\mathbb{C}(x_1, \dots, x_n, y_1, \dots, y_m)$ which is the field of fractions of $\mathbb{C}(x_1, \dots, x_n) \otimes \mathbb{C}(y_1, \dots, y_m)$. $\square$

EXERCISE 3.1.6. Let X be an affine variety. Assume that $\mathcal{O}(X)$ is generated by r elements. Then $\dim X \leq r$, and if $\dim X = r$, then $X \simeq \mathbb{C}^r$.

EXERCISE 3.1.7. The function $x \mapsto \dim_x X$ is upper semi-continuous on X. (This means that for all $\alpha \in \mathbb{R}$ the set $\{x \in X \mid \dim_x X < \alpha\}$ is open in X.)

LEMMA 3.1.8. *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a nonconstant polynomial and $X := \mathcal{V}(f) \subseteq \mathbb{C}^n$ its zero set. Then* $\dim X = n - 1$.

PROOF. We can assume that f is irreducible and that the variable x_n occurs in f. Denote by $\bar{x}_i \in \mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]/(f)$ the restrictions of the coordinate functions x_i. Then $\mathbb{C}(X) = \mathbb{C}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$. Since $f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = 0$ we see that $\bar{x}_n \in \mathbb{C}(X)$ is algebraic over the subfield $\mathbb{C}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1})$. Therefore, $\operatorname{tdeg} \mathbb{C}(X) = \operatorname{tdeg} \mathbb{C}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}) \leq n-1$. On the other hand, the composition

$$\mathbb{C}[x_1, \dots, x_{n-1}] \hookrightarrow \mathbb{C}[x_1, \dots, x_n] \xrightarrow{\text{res}} \mathcal{O}(X)$$

is injective, since the kernel is the intersection $(f) \cap \mathbb{C}[x_1, \dots, x_{n-1}]$ which is zero. Thus, $\operatorname{tdeg} \mathbb{C}(X) \geq n - 1$, and the claim follows. $\square$

The first part of the proof above, namely that $\dim \mathcal{V}(f) < n = \dim \mathbb{C}^n$ has the following generalization.

Lemma 3.1.9. *If X is irreducible and $Y \subsetneqq X$ a proper closed subset, then* $\dim Y < \dim X$.

Proof. We can assume that Y is irreducible. If $h_1, \dots, h_m \in \mathcal{O}(Y)$ are algebraically independent where $m = \dim Y$, and $h_i = \tilde{h}_i|_Y$ for $\tilde{h}_1, \dots, \tilde{h}_m \in \mathcal{O}(X)$, then $\tilde{h}_1, \dots, \tilde{h}_m$ are algebraically independent, too, and so $\dim X \geq \dim Y$. If $\dim Y = \dim X$, then every $f \in \mathcal{O}(X)$ is algebraic over $\mathbb{C}(\tilde{h}_1, \dots, \tilde{h}_m)$. Choose $f \in \mathcal{O}(X)$ in the kernel of the restriction map, i.e. $f|_Y = 0$. Then f satisfies an equation of the form

$$f^k + p_1 f^{k-1} + \cdots + p_{k-1} f + p_k = 0$$

where $p_j \in \mathbb{C}(\tilde{h}_1, \dots, \tilde{h}_m)$ and k is minimal. Multiplying this equation with a suitable $q \in \mathbb{C}[\tilde{h}_1, \dots, \tilde{h}_m]$ we can assume that $p_j \in \mathbb{C}[\tilde{h}_1, \dots, \tilde{h}_m]$. But this implies that $p_k|_Y = 0$. Thus $p_k = 0$ and we end up with a contradiction. □

Example 3.1.10. We have $\dim X = 0$ if and only if X is finite, and this is equivalent to $\dim_{\mathbb{C}} \mathcal{O}(X) < \infty$.
(This is clear: If X is irreducible of dimension 0, then $\mathbb{C}(X)$ is algebraic over $\mathbb{C}$ and so $\mathbb{C} = \mathcal{O}(X) = \mathbb{C}(X)$, and the claim follows.)

Exercise 3.1.11. Let A be a finitely generated algebra. Then the following statements are equivalent.

(i) A is finite dimensional.
(ii) $A_{\text{red}} := A/\sqrt{(0)}$ is finite dimensional.
(iii) The number of maximal ideals in A is finite.

Exercise 3.1.12. Let $U \subseteq X$ be a dense open set. Then $\dim X \setminus U < \dim X$.

Proposition 3.1.13. *Let X be an irreducible affine variety of dimension n. Then there is a special open set $U \subseteq X$ which is isomorphic to a special open set of a hypersurface $\mathcal{V}(h) \subseteq \mathbb{C}^{n+1}$.*

Proof. The existence of a primitive element implies that the field of rational functions $\mathbb{C}(X)$ has the form

$$\mathbb{C}(X) = \mathbb{C}(x_1, \dots, x_n)[f]$$

where f satisfies a minimal equation: $f^m + p_1 f^{m-1} + \cdots + p_m = 0$, $p_j \in \mathbb{C}(x_1, \dots, x_n)$, see [**Art91**, Chap. 14, Theorem 4.1]. Multiplying with a suitable polynomial from $\mathbb{C}[x_1, \dots, x_n]$ we can assume that all p_j belong to $\mathbb{C}[x_1, \dots, x_n]$. Then the polynomial $h := y^m + p_1 y^{n-1} + \cdots + p_m \in \mathbb{C}[x_1, \dots, x_n, y]$ is irreducible and defines a hypersurface $H := \mathcal{V}(h) \subseteq \mathbb{C}^{n+1}$ whose field of rational functions $\mathbb{C}(H)$ is isomorphic to $\mathbb{C}(X)$, by construction. Now the claim follows from Proposition 2.3.4. □

3.2. Finite morphisms. Finite morphisms will play an important role in the following. In particular, they will help us to "compare" an arbitrary affine variety X with an affine space $\mathbb{C}^n$ of the same dimension by using the famous Normalization Lemma of Noether.

Definition 3.2.1. Let $A \subseteq B$ be two rings. We say that B *is finite over* A if B is a finite A-module, i.e. there are $b_1, \dots, b_s \in B$ such that $B = \sum_j A b_j$.

A morphism $\varphi\colon X \to Y$ between two affine varieties is called *finite* if $\mathcal{O}(X)$ is finite over $\varphi^*(\mathcal{O}(Y))$.

If $A \subseteq B \subseteq C$ are rings such that B is finite over A and C is finite over B, then C is finite over A. In particular, if $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are finite morphisms, then the composition $\psi \circ \varphi\colon X \to Z$ is finite, too. Another useful remark is the following: If $\varphi\colon X \to Y$ is finite and $X' \subseteq X$, $Y' \subseteq Y$ closed subsets such that $\varphi(X') \subseteq Y'$, then the induced morphism $\varphi'\colon X' \to Y'$ is also finite.

EXAMPLE 3.2.2. Typical examples of finite morphisms are the ones given in Example 2.1.3, namely $\varphi\colon \mathbb{C} \to C = \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$ and $\psi\colon \mathbb{C} \to D = \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{C}^2$. In both cases, the morphisms are the so-called *normalizations*, a concept which we will discuss later.

On the other hand, the inclusion of a special open set $X_f \hookrightarrow X$ is not finite if f is neither invertible nor zero.

EXERCISE 3.2.3. Every nonconstant morphism $\varphi\colon \mathbb{C} \to \mathbb{C}$ is finite, and the same holds for the nonconstant morphisms $\psi\colon \mathbb{C}^* \to \mathbb{C}^*$.

The basic geometric property of a finite morphism is given in the next proposition.

PROPOSITION 3.2.4. *Let $\varphi\colon X \to Y$ be a finite morphism. Then φ is closed and has finite fibers.*

PROOF. If $y \in Y$, then $\varphi^{-1}(y) = \mathcal{V}_X(\varphi^*(\mathfrak{m}_y))$ (see 2.2). If $\varphi^{-1}(y) \neq \emptyset$, then the induced morphism $\varphi^{-1}(y) \to \{y\}$ is finite, too, and so $\mathcal{O}(\varphi^{-1}(y))$ is a finite dimensional $\mathbb{C}$-algebra. Thus, the fiber $\varphi^{-1}(y)$ is finite (Example 3.1.10) proving the second claim.

For the first claim it suffices to show that $\overline{\varphi(X)} = \varphi(X)$. Hence we can assume that $\overline{\varphi(X)} = Y$, i.e. that $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. If $\varphi^{-1}(y) = \emptyset$, then $\mathcal{O}(X)\mathfrak{m}_y = \mathcal{O}(X)$ where we identify $\mathfrak{m}_y$ with its image $\varphi^*(\mathfrak{m}_y)$. The Lemma of NAKAYAMA (see Lemma 3.2.5 below) now implies that $(1 + a)\mathcal{O}(X) = 0$ for some $a \in \mathfrak{m}_y$ which is a contradiction since $1 + a \neq 0$. ☐

LEMMA 3.2.5 (Lemma of NAKAYAMA). *Let R be a ring, $\mathfrak{a} \subseteq R$ an ideal and M a finitely generated R-module. If $\mathfrak{a}M = M$, then there is an element $a \in \mathfrak{a}$ such that $(1 + a)M = 0$. In particular, if M is torsionfree and $\mathfrak{a} \neq R$, then $M = 0$.*

PROOF. Let $M = \sum_{j=1}^k Rm_j$. Then $m_i = \sum_j a_{ij}m_j$ for all i where $a_{ij} \in \mathfrak{a}$. If A denotes the $k \times k$-matrix $(a_{ij})_{i,j}$ and m the column vector $(m_1, \ldots, m_k)^t$ this means that $m = A \cdot m$. Thus $(E - A)m = 0$, and so $\det(E - A)m_j = 0$ for all j. But

$$\det(E - A) = \det \begin{bmatrix} 1 - a_{11} & -a_{12} & \cdots \\ -a_{21} & 1 - a_{22} & \cdots \\ \vdots & & \ddots \end{bmatrix} = 1 + a \quad \text{where } a \in \mathfrak{a}.$$

and the claim follows. ☐

EXERCISE 3.2.6. Define $\varphi\colon \mathbb{C}^* \to \mathbb{C}$ by $t \mapsto t + \frac{1}{t}$. Show that his morphism is closed, has finite fibers, but is not finite. Thus the converse statement of the Proposition 3.2.4 above is not true.

EXERCISE 3.2.7. Let X be an affine variety and $x \in X$. Assume that $f_1, \ldots, f_r \in \mathfrak{m}_x$ generate the ideal $\mathfrak{m}_x$ modulo $\mathfrak{m}_x^2$, i.e., $\mathfrak{m}_x = (f_1, \ldots, f_r) + \mathfrak{m}_x^2$. Then $\{x\}$ is an irreducible component of $\mathcal{V}_X(f_1, \ldots, f_r)$.
(Hint: If $C \subseteq \mathcal{V}_X(f_1, \ldots, f_r)$ is an irreducible component containing x and $\mathfrak{m} \subseteq \mathcal{O}(C)$ the maximal ideal of x, then $\mathfrak{m}^2 = \mathfrak{m}$. Hence $\mathfrak{m} = 0$ by the Lemma of NAKAYAMA above.)

EXERCISE 3.2.8. Let $\varphi\colon X \to Y$ be a finite surjective morphism. Then $\dim X = \dim Y$.

EXERCISE 3.2.9. Let X be an affine variety and $X = \bigcup_i X_i$ the irreducible decomposition. A morphism $\varphi\colon X \to Y$ is finite if and only if the restrictions $\varphi|_{X_i}\colon X_i \to Y$ are finite for all i.

The following easy lemma will be very useful in sequel.

LEMMA 3.2.10. *Let $A \subseteq B$ be rings and $b \in B$. Assume that b satisfies an equation of the form*

$$b^m + a_1 b^{m-1} + a_2 b^{m-2} + \cdots + a_m = 0 \tag{6}$$

where $a_1, a_2, \ldots, a_m \in A$. Then the subring $A[b] \subseteq B$ is finite over A.

PROOF. It follows from the equation satisfied by b that for $N \geq m$ we have

$$b^N = -a_1 b^{N-1} - a_2 b^{N-2} - \cdots - a_m b^{N-m},$$

and so, by induction, that $A[b] = \sum_{i=0}^{m-1} Ab^i$. □

DEFINITION 3.2.11. An element $b \in B$ satisfying an equation of the form (6) is called *integral over A*.

The next result is usually called the "Normalization Lemma". It is due to EMMY NOETHER, but was first formulated, in a special case, by DAVID HILBERT.

THEOREM 3.2.12 (Normalization Lemma). *Let K be an infinite field and A a finitely generated K-algebra. Then there are algebraically independent elements $a_1, \ldots, a_n \in A$ such that A is finite over $K[a_1, \ldots, a_n]$*

PROOF. We proceed by induction on the number m of generators of A as a K-algebra. If $m = 0$, then $A = K$ and there is nothing to prove. If $A = K[b_1, \ldots, b_m]$ and if $b_1, \ldots, b_m$ are algebraically independent, we are done, too. So let's assume that $F(b_1, \ldots, b_m) = 0$ where $F \in K[x_1, \ldots, x_m]$ is a nonzero polynomial. We can also assume that x_m occurs in F. Write

$$F = \sum_{r_1, r_2, \ldots, r_m} \alpha_{r_1 r_2 \ldots r_m} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}$$

and put $r := \max\{r_1 + r_2 + \cdots + r_m \mid \alpha_{r_1 r_2 \ldots r_m} \neq 0\}$. Substituting $x_j = x_j' + \gamma_j x_m$ for $j = 1, \ldots, m-1$, we find

$$F = \Big(\sum_{r_1 + r_2 + \cdots + r_m = r} \alpha_{r_1 \ldots r_m} \gamma_1^{r_1} \cdots \gamma_{m-1}^{r_{m-1}} \Big) x_m^r + H(x_1', \ldots, x_{m-1}', x_m) \tag{7}$$

where x_m occurs in H with an exponent $< r$. Since K is infinite we can find $\gamma_1, \ldots, \gamma_{m-1} \in K$ such that $\sum_{r_1 + \cdots + r_m = r} \alpha_{r_1 \ldots r_m} \gamma_1^{r_1} \cdots \gamma_{m-1}^{r_{m-1}} \neq 0$. Setting $b_j' := b_j - \gamma_j b_m$ for $j = 1, \ldots, m-1$, we get $A = K[b_1', b_2', \ldots, b_{m-1}', b_m]$. Now equation (7) implies that b_m satisfies an equation of the form (6), hence A is finite over $K[b_1', \ldots, b_{m-1}']$ by Lemma 3.2.10, and the claim follows by induction. □

REMARK 3.2.13. The proof above shows the following. If $A = K[b_1, \ldots, b_m]$, then there is a number $n \leq m$ and n linear combinations $a_i := \sum_j \gamma_{ij} b_j \in A$ such that $a_1, \ldots, a_n$ are algebraically independent over K and that A is finite over $K[a_1, \ldots, a_n]$.

A first consequence is the following result which is usually called NOETHER'S *normalization*.

PROPOSITION 3.2.14. *Let X is an affine variety of dimension n. Then there is a finite surjective morphism $\varphi \colon X \to \mathbb{C}^n$.*

PROOF. It follows from the Normalization Lemma (Theorem 3.2.12) that there exist $f_1, \ldots, f_n \in \mathcal{O}(X)$ such that $\mathcal{O}(X)$ is finite over the subring $\mathbb{C}[f_1, \ldots, f_n]$. Hence $\dim X = n$ (Example 3.1.2(3)), and the morphism $\varphi = (f_1, \ldots, f_n) \colon X \to \mathbb{C}^n$ is finite and surjective (Proposition 3.2.4). □

This result can be improved, using Remark 3.2.13 above.

Proposition 3.2.15. *Let $X \subseteq \mathbb{C}^m$ be a closed subvariety of dimension $n \le m$. Then there is a linear projection $\lambda\colon \mathbb{C}^m \to \mathbb{C}^n$ such that $\lambda|_X\colon X \to \mathbb{C}^n$ is finite and surjective.*

In fact, more is true: *There is an open dense set $U \subseteq \operatorname{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ such that the proposition above holds for any $\lambda \in U$.* We will not give a proof here since it does not follow immediately from our previous results. A special case is given in Exercise 3.2.18 below.

Example 3.2.16. Let $f_1, f_2, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ be nonconstant homogeneous polynomials, and put $A := \mathbb{C}[f_1, f_2, \dots, f_m]$. Then the following statements are equivalent:

(i) $\mathbb{C}[x_1, \dots, x_n]/(f_1, f_2, \dots, f_m)$ is a finite dimensional algebra.
(ii) There is a $k \in \mathbb{N}$ such that $(x_1, x_2, \dots, x_n)^k \subseteq (f_1, f_2, \dots, f_m)$.
(iii) $\mathbb{C}[x_1, \dots, x_n]$ is finite over A.

Proof. Let $\mathfrak{m} := (x_1, \dots, x_n) \subseteq \mathbb{C}[x_1, \dots, x_n]$ be the homogeneous maximal ideal.

(i)⇒(ii): Since $R := \mathbb{C}[x_1, \dots, x_n]/(f_1, f_2, \dots, f_m)$ is graded and finite dimensional we have $\overline{\mathfrak{m}}^k = 0$ for some k where $\overline{\mathfrak{m}} \subseteq R$ is the image of $\mathfrak{m}$. Hence $\mathfrak{m}^k \subseteq (f_1, \dots, f_m)$.

(ii)⇒(iii): Set $V := \bigoplus_{i=0}^{k-1} \mathbb{C}[x_1, \dots, x_n]_i \subseteq \mathbb{C}[x_1, \dots, x_n]$. We will show, by induction, that $\mathfrak{m}^\ell \subseteq AV$ for all ℓ, hence $AV = \mathbb{C}[x_1, \dots, x_n]$. Clearly, $\mathfrak{m}^\ell \subseteq AV$ for $\ell < k$. If $\ell \ge k$ and $f \in \mathfrak{m}^\ell$, then $f = \sum_{i=1}^m h_i f_i$ where we can assume that all h_i are homogeneous. Therefore, $\deg h_i < \ell$, hence $h_i \in AV$ by induction, and so $f \in AV$.

(iii)⇒(i): If $\mathbb{C}[x_1, \dots, x_n]$ is finite over A, then $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)$ is finite over $A/(f_1, \dots, f_m) = \mathbb{C}$, hence the claim. □

Exercise 3.2.17. Assume that the morphism $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ is given by nonconstant homogeneous polynomials $f_1, \cdots, f_m$. If $\varphi^{-1}(0)$ is finite, then $\varphi^{-1}(0) = \{0\}$ and φ is a finite morphism.
(Hint: Use the example above together with Exercise 3.1.11.)

Exercise 3.2.18. Let $X \subseteq \mathbb{C}^n$ be a closed cone and $\lambda\colon \mathbb{C}^n \to \mathbb{C}^m$ a linear map. If $X \cap \ker \lambda = \{0\}$, then $\lambda|_X\colon X \to \mathbb{C}^m$ is finite. Moreover, the set of linear maps $\lambda\colon \mathbb{C}^n \to \mathbb{C}^m$ such that $\lambda|_X$ is finite is open in $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m) = \mathrm{M}_{m,n}(\mathbb{C})$.

Noether's normalization often allows to reduce problems about general affine varieties X to the case $X = \mathbb{C}^n$. One useful application is the following, and more will follow in the next sections.

Proposition 3.2.19. *An irreducible affine variety X cannot be covered by a countable set of proper closed subsets.*

Proof. This is clear for $X = \mathbb{C}$. Now let $X = \bigcup_{i \in I} X_i$ where I is countable and all $X_i \subseteq X$ are closed. If $X = \mathbb{C}^n$, then, by induction, every linear subspace of dimension $n-1$ is contained in one of the X_i. Since there are uncountable many such subspaces, there are infinitely many of them contained in the same X_i. Thus $X_i = \mathbb{C}^n$, because the union of infinitely many linear subspace of codimension 1 is Zariski-dense in $\mathbb{C}^n$. In fact, a polynomial vanishing on such a union is divisible by infinitely many linear functions.

In general, choose a finite surjective morphism $\varphi\colon X \to \mathbb{C}^n$ (Proposition 3.2.14). Then $\mathbb{C}^n = \bigcup_{i \in I} \varphi(X_i)$, and so $\varphi(X_{i_0}) = \mathbb{C}^n$ for some i_0, because all $\varphi(X_i)$ are closed (Proposition 3.2.4). But then $\dim X_{i_0} = n = \dim X$ and so $X_{i_0} = X$. □

3.3. Krull's principal ideal theorem. We have seen in Lemma 3.1.8 that the dimension of a hypersurface $\mathcal{V}(f) \subseteq \mathbb{C}^n$ is equal to $n-1$, i.e. $\operatorname{codim}_{\mathbb{C}^n} \mathcal{V}(f) = 1$ where the *codimension* of a closed subvariety $Y \subseteq X$ is defined by $\operatorname{codim}_X Y := \dim X - \dim Y$. We want to generalize this to arbitrary affine varieties X. First we prove a converse of Lemma 3.2.10.

LEMMA 3.3.1. *Let $A \subseteq B$ be rings. Assume that A is Noetherian and that B is finite over A. Then every $b \in B$ is integral over A, i.e., b satisfies an equation of the form*

$$b^m + a_1 b^{m-1} + a_2 b^{m-2} + \cdots + a_m = 0$$

where $a_1, a_2, \ldots, a_m \in A$.

PROOF. Since A is Noetherian the subalgebra $A[b] \subseteq B$ is finite over A. Therefore, the sequence $A \subseteq A + Ab \subseteq A + Ab + Ab^2 \subseteq \cdots \subseteq A + Ab + \cdots + Ab^k \subseteq \cdots$ becomes stationary. Hence, there is a $m \geq 1$ such that $b^m \in A + Ab + \cdots + Ab^{m-1}$. □

EXERCISE 3.3.2. Let $r \in \mathbb{C}(x_1, \ldots, x_n)$ satisfy an equation of the form

$$r^m + p_1 r^{m-1} + \cdots + p_m = 0 \text{ where } p_j \in \mathbb{C}[x_1, \ldots, x_n].$$

Then $r \in \mathbb{C}[x_1, \ldots, x_n]$. In particular, if $A \subseteq \mathbb{C}(a_1, \ldots, a_n)$ is a subalgebra which is finite over $\mathbb{C}[a_1, \ldots, a_n]$, then $A = \mathbb{C}[a_1, \ldots, a_n]$.

LEMMA 3.3.3. *Let A be a $\mathbb{C}$-domain and K its field of fractions. Let $a_1, \ldots, a_n \in A$ be algebraically independent such that A is finite over $\mathbb{C}[a_1, \ldots, a_n]$. Denote by $N \colon K \to \mathbb{C}(a_1, \ldots, a_n)$ the norm. Then*

(1) *$N(A) \subseteq \mathbb{C}[a_1, \ldots, a_n]$;*
(2) *For all $a \in A$ we have $\sqrt{Aa \cap \mathbb{C}[a_1, \ldots, a_n]} = \sqrt{\mathbb{C}[a_1, \ldots, a_n]N(a)}$.*

PROOF. For $a \in A$ denote by $a^{(1)} := a, a^{(2)}, \ldots, a^{(r)} \in \bar{K}$ the conjugates of a over $\mathbb{C}(a_1, \ldots, a_n)$ where $\bar{K}$ is the algebraic closure of K. Since a is integral over $\mathbb{C}[a_1, \ldots, a_n]$, the same holds for all $a^{(j)}$. This implies, by Lemma 3.2.10, that the subalgebra $\tilde{A} := \mathbb{C}[a_1, \ldots, a_n][a^{(1)}, \ldots, a^{(r)}] \subseteq \bar{K}$ is finite over $\mathbb{C}[a_1, \ldots, a_n]$. Therefore, $N(a) = a^{(1)} a^{(2)} \cdots a^{(r)}$ belongs to $\tilde{A} \cap \mathbb{C}(a_1, \ldots, a_n)$ which is equal to $\mathbb{C}[a_1, \ldots, a_n]$ by Exercise 5.8 above. This prove the first claim.

Now we have

$$\prod_j (t - a^{(j)}) = t^r + h_1 t^{r-1} + \cdots + h_{r-1} t + h_r$$

where $h_j \in \tilde{A} \cap \mathbb{C}(a_1, \ldots, a_n) = \mathbb{C}[a_1, \ldots, a_n]$ and $h_r = (-1)^r N(a)$. It follows that $N(a) = ab$ where $b = (-1)^{r-1}(a^{r-1} + h_1 a^{r-2} + \cdots + h_{r-1}) \in A$ and so $N(a) \in Aa$. Thus, $\mathbb{C}[a_1, \ldots, a_n]N(a) \subseteq Aa \cap \mathbb{C}[a_1, \ldots, a_n]$.

In order to see that $Aa \cap \mathbb{C}[a_1, \ldots, a_n] \subseteq \sqrt{\mathbb{C}[a_1, \ldots, a_n]N(a)}$ we choose an element $sa \in Aa \cap \mathbb{C}[a_1, \ldots, a_n]$. Then $N(sa) = (sa)^r$, and since $N(sa) = N(s)N(a) \in \mathbb{C}[a_1, \ldots, a_n]N(a)$ we finally get $sa \in \sqrt{\mathbb{C}[a_1, \ldots, a_n]N(a)}$. □

THEOREM 3.3.4 (KRULL'S Principal Ideal Theorem). *Let X be an irreducible affine variety and $f \in \mathcal{O}(X)$, $f \neq 0$. Assume that $\mathcal{V}_X(f)$ is non-empty. Then every irreducible component of $\mathcal{V}_X(f)$ has codimension 1 in X. In particular,* $\dim \mathcal{V}_X(f) = \dim X - 1$.

PROOF. Let $\mathcal{V}_X(f) = C_1 \cup C_2 \cup \cdots \cup C_r$ be the irreducible decomposition. Choose an $h \in \mathcal{O}(X)$ vanishing on $C_2 \cup C_3 \cup \cdots \cup C_r$ which does not vanish on C_1. Then $\mathcal{V}_{X_h}(f) = C_1 \cap X_h$ is irreducible. Thus, it suffices to consider the case where $\mathcal{V}_X(f) \subseteq X$ is irreducible. By the Normalization Lemma (Theorem 3.2.12) there is a finite surjective morphism $\varphi \colon X \to \mathbb{C}^n$, $n = \dim X$. By Lemma 3.3.3(2) we get $\varphi(\mathcal{V}_X(f)) = \mathcal{V}(N(f))$, and so $\dim \mathcal{V}_X(f) = \dim \mathcal{V}(N(f)) = n - 1$ (see Lemma 3.1.8). □

It is easy to see that this result also holds for *equidimensional* varieties (i.e. varieties X where all irreducible components have the same dimension) in case f is a nonzero divisor. For a general X and a nonzero divisor $f \in \mathcal{O}(X)$, we can only say that every irreducible component of $\mathcal{V}_X(f)$ has dimension $\leq \dim X - 1$.

A first consequence is the following result.

PROPOSITION 3.3.5. *Let X be an irreducible variety and $f_1, f_2, \dots, f_r \in \mathcal{O}(X)$. If the zero set $\mathcal{V}_X(f_1, \dots, f_r)$ is non-empty, then every irreducible component C of $\mathcal{V}_X(f_1, \dots, f_r)$ has dimension $\dim C \geq \dim X - r$.*

PROOF. We proceed by induction on $\dim X$. Define $Y := \mathcal{V}_X(f_1)$, and let $Y = Y_1 \cup \dots \cup Y_s$ be the decomposition into irreducible components. Then

$$\mathcal{V}_X(f_1, \cdots, f_r) = \bigcup_j \mathcal{V}_{Y_j}(f_2, \dots, f_r)$$

Since $\dim Y_j = \dim X - 1$ for all j we see, by induction, that every irreducible component of $\mathcal{V}_{Y_j}(f_2, \dots, f_r)$ has dimension $\geq (\dim X - 1) - (r-1) = \dim X - r$, and the claim follows. $\square$

EXERCISE 3.3.6. Let X be an affine variety and $f \in \mathcal{O}(X)$ a nonzero divisor. For any $x \in \mathcal{V}_X(f)$ we have $\dim_x \mathcal{V}_X(f) = \dim_x X - 1$.
(Hint: If f is a nonzero divisor, then f is nonzero on every irreducible component X_i of X and so $\mathcal{V}_{X_i}(f)$ is either empty or every irreducible component has codimension 1. Now the claim follows easily.)

Another consequence of KRULL'S Principal Ideal Theorem is the following which gives an alternative definition of the dimension of a variety.

PROPOSITION 3.3.7. *Let X be an irreducible variety and $Y \subsetneqq X$ a closed irreducible subset. Then there is a strictly decreasing chain of length $n := \dim X$,*

$$X_n = X \supsetneqq X_{n-1} \supsetneqq \cdots \supsetneqq X_d = Y \supsetneqq \cdots \supsetneqq X_1 \supsetneqq X_0$$

of irreducible closed subsets X_j. In particular, $\dim X$ equals the length of a maximal chain of irreducible closed subsets.

PROOF. By induction, we only have to show that Y is contained in an irreducible hypersurface $H \subseteq X$. Let $f \in I(Y)$ be a nonzero function. Then $X \supseteq \mathcal{V}_X(f) \supseteq Y$ and so Y is contained in an irreducible component of $\mathcal{V}_X(f)$ which all have codimension 1 by Theorem 3.3.4. $\square$

REMARK 3.3.8. This result allows to define the *dimension* $\dim A$ *of a* $\mathbb{C}$*-algebra* A as the maximal length of a chain of prime ideal $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_m \subseteq A$. If A is finitely generated, then $\dim A$ is finite, and every maximal chain has length $\dim A$. Moreover, $\dim A = \dim A_{\text{red}}$ where $A_{\text{red}} := A/\sqrt{(0)}$, and so $\dim A = \dim X$ where X is an affine variety with coordinate ring isomorphic to A_{red}.

We also see that for a variety X and a point $x \in X$ we have $\dim_x X = \dim \mathcal{O}_{X,x}$.

COROLLARY 3.3.9. *Let A be a finitely generated $\mathbb{C}$-algebra and let $a \in A$ be a nonzero divisor. Then $\dim A/Aa \leq \dim A - 1$, and equality holds if A_{red} is a domain.*

PROOF. Put $\bar{A} := A/(a)$ and denote by $a' \in A_{\text{red}}$ the image of a. Then a' is a nonzero divisor in A_{red} and so $\dim A_{\text{red}}/\sqrt{(a')} \leq \dim A_{\text{red}} - 1$ by Theorem 3.3.4. Since $\bar{A}_{\text{red}} \simeq A_{\text{red}}/\sqrt{(a')}$ we finally get $\dim \bar{A} = \dim \bar{A}_{\text{red}} \leq \dim A_{\text{red}} - 1 = \dim A - 1$ $\square$

3.4. Decomposition Theorem and dimension formula. Let $\varphi\colon X \to Y$ be a dominant morphism where X, Y are both irreducible. We want to show that the dimension of a non-empty fiber $\varphi^{-1}(y)$ is always $\geq \dim X - \dim Y$ and that we have equality on a dense open set of Y. A crucial step is the following *Decomposition Theorem* for a morphism.

THEOREM 3.4.1. *Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. There is a non-empty special open set $U \subseteq Y$ and a factorization of φ of the form*

$$\begin{tikzcd} \varphi^{-1}(U) \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & U \times \mathbb{C}^r \arrow[d, "\mathrm{pr}_U"] \\ & U \end{tikzcd}$$

where ρ is a finite surjective morphism and $r := \dim X - \dim Y$. In particular, the fibers $\varphi^{-1}(y) = \rho^{-1}(\{y\} \times \mathbb{C}^r)$ have the same dimension for all $y \in U$, namely $\dim X - \dim Y$.

REMARK 3.4.2. We will see later in Proposition 3.4.7 that the fibers $\varphi^{-1}(y)$ for $y \in U$ are *equidimensional*, i.e., all irreducible components have the same dimension, namely $\dim X - \dim Y$.

PROOF. Since φ is dominant we will regard $\mathcal{O}(Y)$ as a subalgebra of $\mathcal{O}(X)$. Let $K = \mathbb{C}(Y)$ be the quotient field of $\mathcal{O}(Y)$ and put $A := K \cdot \mathcal{O}(X) \subseteq \mathbb{C}(X)$, the K-algebra generated by K and $\mathcal{O}(X)$. Then A is finitely generated over K and so we can find algebraically independent elements $h_1, \ldots, h_r \in A$ such that A is finite over $K[h_1, \ldots, h_r]$ (Theorem 3.2.12). It follows that $r = \dim X - \dim Y$.

We claim that there is an $f \in \mathcal{O}(Y)$ such that $h_i = \frac{a_i}{f}$ with $a_i \in \mathcal{O}(X)$ for all i and that $\mathcal{O}(X_f) = \mathcal{O}(X)_f$ is finite over $\mathcal{O}(Y_f)[h_1, \ldots, h_r]$. The first statement is clear, and we can therefore assume that $h_1, \ldots, h_r \in \mathcal{O}(X)$.

For the second statement, let $b_1, \ldots, b_s$ be generators of A over $K[h_1, \ldots, h_r]$. Multiplying with a suitable element of $\mathcal{O}(Y) \subseteq K$ we can first assume that $b_j \in \mathcal{O}(X)$ and then, by adding more elements if necessary, that $b_1, \ldots, b_s$ generate $\mathcal{O}(X)$ as a $\mathbb{C}$-algebra. Now $b_i b_j = \sum_k c_k^{(ij)} b_k$ where $c_k^{(ij)} \in K[h_1, \ldots, h_r]$. Thus we can find an $f \in \mathcal{O}(Y)$ such that $f \cdot c_k^{(ij)} \in \mathcal{O}(Y)[h_1, \ldots, h_r]$. It follows that

$$\sum_j \mathcal{O}(Y_f)[h_1, \ldots, h_r]\, b_j \subseteq \mathcal{O}(X)_f = \mathcal{O}(X_f)$$

is a subalgebra containing $\mathcal{O}(X)$ and $\frac{1}{f}$, hence is equal to $\mathcal{O}(X_f)$, and the claim follows.

Setting $U := Y_f$ we get $\varphi^{-1}(U) = X_f$ and obtain a morphism

$$\rho = \varphi \times (h_1, \ldots, h_r)\colon X_f \to Y_f \times \mathbb{C}^r,\ x \mapsto (\varphi(x), h_1(x), \ldots, h_r(x))$$

which satisfies the requirements of the proposition.

The last statement is clear (see Exercise 3.2.8). □

EXAMPLE 3.4.3. Let $f \in \mathbb{C}[x, y]$ be a nonconstant polynomial. Then there is a finite morphism $\rho\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $f = \mathrm{pr}_1 \circ \rho$:

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\rho"] \arrow[dr, "f"'] & \mathbb{C}^2 \arrow[d, "\mathrm{pr}_1"] \\ & \mathbb{C} \end{tikzcd}$$

Proof. We can assume that the variable y occurs in f. Consider the isomorphism $\Phi\colon \mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2$ given by $(x,y) \mapsto (x, y + x^n)$ and choose n large enough so that $\tilde{f} = \Phi^*(f) = f(x, y + x^n)$ has leading term ax^N where $a \in \mathbb{C}^*$. Then $\mathbb{C}[x,y]$ is finite over $\mathbb{C}[\tilde{f}, y]$, hence defines a finite surjective morphism $\tilde{\rho}\colon \mathbb{C}^2 \to \mathbb{C}^2$, $(x,y) \mapsto (\tilde{f}(x,y), y)$, and we get the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow[\simeq]{\Phi} & \mathbb{C}^2 \\ {\scriptstyle\tilde{\rho}}\downarrow & & {\scriptstyle f}\downarrow \\ \mathbb{C}^2 & \xrightarrow{\mathrm{pr}_1} & \mathbb{C} \end{array}$$

Now the claim follow with $\rho := \tilde{\rho} \circ \Phi^{-1}$. $\square$

Example 3.4.4. In this example we work out the decomposition of Theorem 3.4.1 for the morphism $\varphi\colon \mathrm{M}_2(\mathbb{C}) \to \mathrm{M}_2(\mathbb{C})$, $A \mapsto A^2$, i.e., we want to find an $f \in \mathcal{O}(\mathrm{M}_2)$ such that the induced morphism $\varphi^{-1}(\mathrm{M}_2(\mathbb{C})_f) \to \mathrm{M}_2(\mathbb{C})_f$ is finite and surjective.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so that $\mathcal{O}(\mathrm{M}_2) = \mathbb{C}[a,b,c,d]$ and

$$R := \varphi^*(\mathcal{O}(\mathrm{M}_2)) = \mathbb{C}[a^2 + bc, d^2 + bc, b(a+d), c(a+d)] \subseteq \mathbb{C}[a,b,c,d].$$

We have $\mathrm{tr}(A)^2 - \mathrm{tr}(A^2) = 2\det(A)$, hence $\mathrm{tr}(A)$ satisfies the integral equation

$$x^4 - 2\,\mathrm{tr}(A^2)x^2 = 4\det(A^2) - \mathrm{tr}(A^2)^2, \tag{8}$$

over R, showing that $R[\mathrm{tr}(A)]$ is finite over R and contains $\det(A)$. Since R contains the elements $\mathrm{tr}(A)b$, $\mathrm{tr}(A)c$ and $a^2 - b^2 = \mathrm{tr}(A)(a-b)$ it follows that

$$R[\mathrm{tr}(A)]_{\mathrm{tr}(A)} = \mathbb{C}[a,b,c,d]_{\mathrm{tr}(A)}.$$

Moreover, equation (8) has the two solutions $\pm\,\mathrm{tr}(A)$, and that the other two solutions satisfy the equation $x^2 - \mathrm{tr}(A^2) = -2\det(A)$. It follows that the norm of $\mathrm{tr}(A)$ which is $N(\mathrm{tr}(A)) = \mathrm{tr}(A^2)^2 - 4\det(A^2)$, has in $R[\mathrm{tr}(A)]$ the decomposition

$$N(\mathrm{tr}(A)) = \mathrm{tr}(A)^2(2\det(A) - \mathrm{tr}(A^2)),$$

hence $R[\mathrm{tr}(A)]_{N(\mathrm{tr}(A))} \supseteq R[\mathrm{tr}(A)]_{\mathrm{tr}(A)}$. This implies that the induced morphism $\varphi^{-1}(\mathrm{M}_2(\mathbb{C})_{N(\mathrm{tr}(A))}) \to \mathrm{M}_2(\mathbb{C})_{N(\mathrm{tr}(A))}$ is finite and surjective of degree 4. Note that $N(\mathrm{tr}(A)) \neq 0$ is equivalent to the condition that A^2 has distinct eigenvalues.

Exercise 3.4.5. Work out the decomposition of Theorem 3.4.1 for the morphisms $\varphi\colon \mathrm{SL}_2 \to \mathbb{C}^3$, $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) := (ab, ad, cd)$ (see Exercise 2.2.14). What is the degree of the finite morphism ρ?

Corollary 3.4.6. *If $\varphi\colon X \to Y$ is a morphism, then there is a set $U \subseteq \varphi(X)$ which is open and dense in $\overline{\varphi(X)}$.*

Proof. If X is irreducible, this is an immediate consequence of Theorem 3.4.1. In general, let $X = \bigcup_{i \in I} X_i$ be the decomposition into irreducible components. Then, for a suitable subset $J \subseteq I$, we can assume that $\overline{\varphi(X)} = \bigcup_{j \in J} \overline{\varphi(X_j)}$ is the decomposition into irreducible components. For each $j \in J$ there is a proper closed subset $A_j \subsetneqq \overline{\varphi(X_j)}$ such that $\overline{\varphi(X_j)} \setminus A_j \subseteq \varphi(X_j)$. Hence $\overline{\varphi(X)} \setminus \bigcup_j A_j$ is an open dense subset of $\overline{\varphi(X)}$ contained in the image $\varphi(X)$. $\square$

Proposition 3.4.7 (Dimension formula for morphisms). *Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. If $y \in \varphi(X)$ and C is an irreducible component of the fiber $\varphi^{-1}(y)$, then*

$$\dim C \geq \dim X - \dim Y,$$

with equality for all y from a dense open set $U \subseteq Y$.

PROOF. Set $m := \dim Y$ and let $\psi\colon Y \to \mathbb{C}^m$ be a finite surjective morphism (Theorem 3.2.12). If we denote by $\tilde{\varphi}\colon X \to \mathbb{C}^m$ the composition $\psi \circ \varphi$, then every fiber of $\tilde{\varphi}$ is a finite union of fibers of φ. Hence it suffices to prove the claim for the morphism $\tilde{\varphi} = (f_1, \ldots, f_m)\colon X \to \mathbb{C}^m$. If $a = (a_1, \ldots, a_m) \in \tilde{\varphi}(X)$, then $\tilde{\varphi}^{-1}(a) = \mathcal{V}_X(f_1 - a_1, f_2 - a_2, \ldots, f_m - a_m)$, and the claim follows from Proposition 3.3.5, a consequence of Krull's Principal Ideal Theorem. The last part is Theorem 3.4.1 above. $\square$

One might believe that the two propositions above imply that for any morphism $\varphi\colon X \to Y$ the function $y \mapsto \dim \varphi^{-1}(y)$ is upper-semicontinuous. This is not true as one can show by examples (see Exercise 3.4.8). However, a famous theorem of CHEVALLEY says that the function $x \mapsto \dim_x \varphi^{-1}(\varphi(x))$ is upper-semicontinuous on X. The proof is quite involved, and we will not present it here.

EXERCISE 3.4.8. Consider the morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $(x, y) \mapsto (x, xy)$. Show that the image $\varphi(\mathbb{C}^2)$ is not locally closed in $\mathbb{C}^2$ and that the map $a \mapsto \dim \varphi^{-1}(a)$ is not upper-semicontinuous.

Another application of the above is the following density result. We call a morphism $\varphi\colon X \to Y$ *strongly dominant* if for every irreducible component $C \subseteq X$ the closure $\overline{\varphi(C)}$ is an irreducible component of Y. In case X and Y are both irreducible, this is equivalent to dominant. Note that for a morphism $\varphi\colon X \to Y$ with dense image it is not true in general that the inverse image of a dense open set is dense. But this holds for a strongly dominant morphisms where we have the following much stronger result.

PROPOSITION 3.4.9. *Let $\varphi\colon X \to Y$ be a strongly dominant morphism. If $D \subseteq Y$ is a dense subset, then $\varphi^{-1}(D)$ is dense in X.*

PROOF. We can assume that X, Y are both irreducible and that all fibers have the same dimension $d := \dim X - \dim Y$. Consider the closed subset $X' := \overline{\varphi^{-1}(D)} \subseteq X$ and denote by $C_1, \ldots, C_k$ the irreducible components of X'. Define, for $i = 1, \ldots, k$,

$$D_i := \{y \in D \mid \dim C_i \cap \varphi^{-1}(y) = d\}.$$

Clearly, $D = \bigcup_i D_i$, and so there is an index i_0 such that $Y = \overline{D_{i_0}}$. This implies that the induced morphism $\varphi_{i_0}\colon C_{i_0} \to Y$ is dominant and that $\dim \varphi_{i_0}^{-1}(y) = d$ for all y of the dense set $D_{i_0} \subseteq Y$. Therefore, $\dim C_{i_0} = \dim Y + d = \dim X$ (see the following Exercise 3.4.10), hence $X = C_{i_0} \subseteq \overline{\varphi^{-1}(D)}$. $\square$

EXERCISE 3.4.10. Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. If $D \subseteq Y$ is a dense subset such that $\dim \varphi^{-1}(y) = d$ for all $y \in D$, then $\dim X = \dim Y + d$.

3.5. Constructible sets. Recall that a subset $A \subseteq X$ of a variety X is called *locally closed* if A is the intersection of an open and a closed subset, or, equivalently, if A is open in its closure $\bar{A}$. We have seen in Exercise 3.4.8 that images of morphisms need not to be locally closed. However, we will show that images of morphisms are always "constructible" in the following sense.

DEFINITION 3.5.1. A subset C of an affine variety X is called *constructible* if it is a finite union of locally closed subsets.

EXERCISE 3.5.2.
(1) Finite unions, finite intersections and complements of constructible sets are again constructible.
(2) If C is a constructible, then C contains a set U which is open and dense in $\bar{C}$.

Proposition 3.5.3. *If $\varphi\colon X \to Y$ is a morphism, then the image of a constructible subset is constructible.*

Proof. Since every open set is the union of finitely many special open sets it suffices to show, in view of the exercise above, that the image of a morphism is constructible. By Corollary 3.4.6 there is a dense open set $U \subseteq \overline{\varphi(X)}$ contained in the image $\varphi(X)$. Then the complement $Y' := \overline{\varphi(X)} \setminus U$ is closed and $\dim Y' < \dim Y$ (Exercise 3.1.12). By induction on $\dim \overline{\varphi(X)}$, we can assume that the claim holds for the morphism $\varphi'\colon X' := \varphi^{-1}(Y') \to Y'$ induced by φ. But then $\varphi(X) = U \cup \varphi'(X')$ and we are done. $\square$

Exercise 3.5.4. Let X be an irreducible affine variety and $C \subseteq X$ a dense constructible subset. Then C can written in the form

$$C = C_0 \cup \bigcup_{j=1}^{m} C_j$$

where $C_0 \subseteq X$ is open and dense, C_j is locally closed, $\overline{C_j}$ is irreducible of codimension ≥ 1, and $\overline{C_j} \cap C_0 = \emptyset$.

3.6. Degree of a morphism. Recall that a dominant morphism $\varphi\colon X \to Y$ between irreducible varieties is called of *finite degree* d if $\dim X = \dim Y$ and $d = [\mathbb{C}(X) : \mathbb{C}(Y)]$ (see 2.3). This has the following geometric interpretation.

Proposition 3.6.1. *Let X, Y be irreducible affine varieties and $\varphi\colon X \to Y$ a dominant morphism of finite degree d. Then there is a dense open set $U \subseteq Y$ such that $\#\varphi^{-1}(y) = d$ for all $y \in U$.*

Proof. We have $\mathbb{C}(X) = \mathbb{C}(Y)[r]$ where r satisfies the minimal equation

$$r^d + a_1 r^{d-1} + \cdots + a_d = 0.$$

Replacing Y and X by suitable special open sets Y_f and X_f ($f \in \mathcal{O}(Y) \subseteq \mathcal{O}(X)$) we can assume that

(1) $r \in \mathcal{O}(X)$;
(2) $a_1, \ldots, a_d \in \mathcal{O}(Y)$;
(3) $\mathcal{O}(X)$ is finite over $\mathcal{O}(Y)$ (Theorem 3.4.1);
(4) $\mathcal{O}(X) = \mathcal{O}(Y)[r]$.

In fact, (1) and (2) are clear and so $A := \mathcal{O}(Y)[r] = \bigoplus_{i=0}^{d-1} \mathcal{O}(Y) r^i \subseteq \mathcal{O}(X)$. For $S := \mathcal{O}(Y) \setminus \{0\}$ we get $A_S = \mathbb{C}(Y)[r] = \mathbb{C}(X) = \mathcal{O}(X)_S$, we can find an $s \in S$ such that $A_s = \mathcal{O}(X)_s$, hence (3) and (4). In particular

$$\mathcal{O}(X) = \bigoplus_{j=0}^{d-1} \mathcal{O}(Y) r^j \xleftarrow{\sim} \mathcal{O}(Y)[t]/(t^d + a_1 t^{d-1} + \cdots + a_d)$$

and so, for every $y \in Y$, we get

$$\mathcal{O}(X)/\mathcal{O}(X)\mathfrak{m}_y \simeq \mathbb{C}[t]/(t^d + a_1(y) t^{d-1} + \cdots + a_d(y))$$

This means that the number of elements in the fiber $\varphi^{-1}(y)$ is equal to the number of different solutions of the equation

$$t^d + a_1(y) t^{d-1} + \cdots + a_d(y) = 0. \tag{9}$$

Let D_d be the discriminant of an equation of degree d (see Example 1.1.3) and define $f(y) := D(a_1(y), \ldots, a_d(y))$. Then $f \in \mathcal{O}(Y)$, and $f(y) \neq 0$ if and only if equation (9) has d different solutions, or, equivalently, the fiber $\varphi^{-1}(y)$ has d points. Thus, the special open set $U := Y_f \subseteq Y$ has the required property. $\square$

REMARK 3.6.2. One can show that the open set U constructed in the proof has the property that the morphism $\varphi^{-1}(U) \to U$ is an *unramified covering* with respect to the $\mathbb{C}$-topology.

EXERCISE 3.6.3. What is the degree of the morphism $\mathrm{M}_n \to \mathrm{M}_n$ given by $A \mapsto A^k$?

EXERCISE 3.6.4. Let $\varphi\colon X \to Y$ be a dominant morphism where X and Y are irreducible. If there is an open dense set $U \subseteq X$ such that $\varphi|_U$ is injective, then φ is birational.

EXERCISE 3.6.5. Let $\varphi\colon X \to Y$ be a *quasi-finite* morphism, i.e. all fibers are finite. Then $\dim \overline{\varphi(X)} = \dim X$.

3.7. Möbius transformations. Let $f \in \mathbb{C}(z) \setminus \mathbb{C}$, $f = \frac{p}{q}$ where $p, q \in \mathbb{C}[z]$ are prime. Define $\deg f := \max\{\deg p, \deg q\}$.

LEMMA 3.7.1. $[\mathbb{C}(z) : \mathbb{C}(f)] = \deg f$.

PROOF. The rational function f defines a dominant morphism $f\colon \mathbb{C}\setminus\mathcal{V}(q) \to \mathbb{C}$, corresponding to the embedding $\mathbb{C}(z) \hookrightarrow \mathbb{C}(z)$ given by $z \mapsto f$. For $\alpha \in \mathbb{C}$ we find

$$f - \alpha = \frac{p}{q} - \alpha = \frac{p - \alpha q}{q}.$$

For a general $\alpha \in \mathbb{C}$ the numerator $p - \alpha q$ has degree $\deg f$ and has no multiple roots. Thus, by Proposition 3.6.1, the map f has degree $\deg f$. □

For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$ the corresponding MÖBIUS transformation $\mu_A\colon \mathbb{C}(z) \xrightarrow{\sim} \mathbb{C}(z)$ is defined by

$$\mu_A(z) = \frac{az + b}{cz + d}.$$

Lemma 3.7.1 above shows that μ_A is an isomorphism, and a easy calculation gives $\mu_A \circ \mu_B = \mu_{AB}$ for all $A, B \in \mathrm{GL}_2(\mathbb{C})$. It is also clear that $\mu_A = \mu_B$ if and only if $B = \lambda A$ for some $\lambda \in \mathbb{C}^*$. Finally, again by Lemma 3.7.1, every automorphism of the field $\mathbb{C}(z)$ is a MÖBIUS transformation. Thus we have proved the following result.

PROPOSITION 3.7.2. *The map $A \mapsto \mu_A$ is a surjective group homomorphism*

$$\mu\colon\ \mathrm{GL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{Aut}(\mathbb{C}(z))$$

with kernel $\mathbb{C}^* E_2$.

4. Tangent Spaces, Differentials, and Vector Fields

4.1. Zariski tangent space. A *tangent vector* δ in a point x_0 of an affine variety X is "rule" to differentiate regular functions, i.e., it is a $\mathbb{C}$-linear map $\delta\colon \mathcal{O}(X) \to \mathbb{C}$ satisfying

$$\delta(f \cdot g) = f(x_0)\,\delta(g) + g(x_0)\,\delta(f) \text{ for all } f, g \in \mathcal{O}(X). \tag{10}$$

Such a map is called a *derivation of $\mathcal{O}(X)$ in x_0*. For $n \geq 0$ we have $\delta(f^n) = nf^{n-1}(x_0) \cdot \delta(f)$, and so, for any polynomial $F = F(y_1, \ldots, y_m)$, we get

$$\delta(F(f_1, \ldots, f_m)) = \sum_{j=1}^{m} \frac{\partial F}{\partial y_j}(f_1(x_0), \ldots, f_m(x_0)) \cdot \delta(f_j).$$

This implies that a derivation in x_0 is completely determined by its values on a generating set of the algebra $\mathcal{O}(X)$. Moreover, a linear combination of derivations in x_0 is again a derivation in x_0. As a consequence, the derivations in x_0 form a finite dimensional subspace of $\mathrm{Hom}(\mathcal{O}(X), \mathbb{C})$.

DEFINITION 4.1.1. The ZARISKI *tangent space* $T_{x_0}X$ of a variety X in a point x_0 is defined to be the set of all tangent vectors in x_0:

$$T_{x_0}X := \operatorname{Der}_{x_0}(\mathcal{O}(X)) := \{\delta\colon \mathcal{O}(X) \to \mathbb{C} \mid \delta \text{ a } \mathbb{C}\text{-linear derivation in } x_0\}.$$

We have already seen above that $T_{x_0}X$ is a finite dimensional linear subspace of $\operatorname{Hom}(\mathcal{O}(X), \mathbb{C})$.

EXERCISE 4.1.2. Let $\delta \in T_xX$ be a tangent vector in x. Then

(1) $\delta(c) = 0$ for every constant $c \in \mathcal{O}(X)$.
(2) If $f \in \mathcal{O}(X)$ is invertible, then $\delta(f^{-1}) = -\dfrac{\delta f}{f(x)^2}$.

EXAMPLE 4.1.3. If $X = \mathbb{C}^n$ and $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, then

$$T_a\mathbb{C}^n = \bigoplus_i \mathbb{C}\, \frac{\partial}{\partial x_i}\Big|_a$$

where $\frac{\partial}{\partial x_i}\Big|_a(f) := \frac{\partial f}{\partial x_i}(a)$. Thus we have a canonical isomorphism $T_a\mathbb{C}^n \simeq \mathbb{C}^n$ by identifying $\delta \in \operatorname{Der}_a(\mathbb{C}[x_1, \ldots, x_n])$ with $(\delta x_1, \ldots, \delta x_n) \in \mathbb{C}^n$.

More generally, if V is a finite dimensional vector space and $x_0 \in V$ we define, for every $v \in V$, the tangent vector $\partial_{v,x_0}\colon \mathcal{O}(V) \to \mathbb{C}$ in x_0 by

$$\partial_{v,x_0}(f) := \frac{f(x_0 + tv) - f(x_0)}{t}\Big|_{t=0},$$

and thus obtain a canonical isomorphism $V \xrightarrow{\sim} T_{x_0}V$, for every $x_0 \in V$. We will mostly identify $T_{x_0}V$ with V.

Let $\delta \in T_xX$ be a tangent vector. Since $\mathcal{O}(X) = \mathbb{C} \oplus \mathfrak{m}_x$ we see that δ is determined by its restriction to $\mathfrak{m}_x$. Moreover, formula (10) above shows that δ vanishes on $\mathfrak{m}_x^2$. Hence, δ induces a linear map $\bar{\delta}\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$.

LEMMA 4.1.4. *Given an affine variety X and a point $x \in X$ there is a canonical isomorphism*

$$T_xX \xrightarrow{\sim} \operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C}).$$

given by $\delta \mapsto \bar{\delta} := \delta|_{\mathfrak{m}_x}$.

PROOF. We have already seen that $\delta \mapsto \bar{\delta}$ is injective. On the other hand, let $C \subseteq \mathfrak{m}_x$ be a complement of $\mathfrak{m}_x^2$ so that $\mathcal{O}(X) = \mathbb{C} \oplus C \oplus \mathfrak{m}_x^2$. If $\lambda\colon C \to \mathbb{C}$ is linear, then one easily sees that the extension of λ to a linear map δ on $\mathcal{O}(X)$ by putting $\delta|_{\mathbb{C}\oplus\mathfrak{m}_x^2} = 0$ is a derivation in x. □

EXERCISE 4.1.5. The canonical homomorphism $\mathcal{O}(X) \to \mathcal{O}_{X,x}$ induces an isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\sim} \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ is the maximal ideal.

If $U = X_f \subseteq X$ is a special open set and $x \in U$, then $T_xU = T_xX$ in a canonical way. In fact, a derivation δ' of $\mathcal{O}(U)$ induces a derivation δ of $\mathcal{O}(X)$ by restriction: $\delta(h) := \delta'(h|_U)$, and every derivation δ of $\mathcal{O}(X)$ "extends" to a derivation δ' of $\mathcal{O}(U) = \mathcal{O}(X)_f$ by setting $\delta'(\frac{h}{f^m}) = -\frac{f(x)\cdot\delta h - mg(x)\cdot\delta f}{f(x)^{m+1}}$ (see Exercise 4.1.2; one has to check that every derivation vanishes on the kernel of the map $\mathcal{O}(X) \to \mathcal{O}(X_f)$). The same result follows from Exercise 4.1.5 using Lemma 4.1.4.

EXERCISE 4.1.6. If $Y \subseteq X$ is a closed subvariety and $x \in Y$, then $\dim T_xY \leq \dim T_xX$. (Hint: The surjection $\mathcal{O}(X) \to \mathcal{O}(Y)$ induces a surjection $\mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \to \mathfrak{m}_{x,Y}/\mathfrak{m}_{x,Y}^2$.)

PROPOSITION 4.1.7. $\dim T_xX \geq \dim_x X$.

PROOF. If $C \subseteq X$ is an irreducible component passing through x we have $\dim T_x C \leq \dim T_x X$ (Exercise 4.1.6). Thus we can assume that X is irreducible. Choose $f_1, \dots, f_r \in \mathfrak{m}_x$ such that the residue classes modulo $\mathfrak{m}_x^2$ form a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$, hence $r = \dim T_x X$, by Lemma 4.1.4. Since the zero set $\mathcal{V}_X(f_1, \dots, f_r)$ has $\{x\}$ as an irreducible component (see Exercise 3.2.7) it follows from Proposition 3.3.5 that

$$0 = \dim\{x\} \geq \dim X - r = \dim X - \dim T_x X.$$

Hence the claim. □

DEFINITION 4.1.8. The variety X is called *nonsingular* or *smooth* in $x \in X$ if $\dim T_x X = \dim_x X$. Otherwise it is *singular* in x. The variety X is called *nonsingular* or *smooth* if it is nonsingular in every point. We denote by X_{sing} the set of singular points of X.

PROPOSITION 4.1.9. *For $x \in X$ and $y \in Y$ there is a canonical isomorphism*

$$T_{(x,y)}(X \times Y) \xrightarrow{\sim} T_x X \oplus T_y Y.$$

PROOF. Every derivation δ of $\mathcal{O}(X \times Y)$ in (x, y) induces, by restriction, derivations δ_X of $\mathcal{O}(X)$ in x and δ_Y of $\mathcal{O}(Y)$ in y. This defines a linear map $T_{(x,y)} X \times Y \to T_x X \oplus T_y Y$ which is injective, because $\delta(f \cdot h) = h(y) \cdot \delta_X f + f(x) \cdot \delta_Y h$ for $f \in \mathcal{O}(X)$ and $h \in \mathcal{O}(Y)$.

In order to see that the map is surjective we first claim that given two derivations $\delta_1 \in T_x X$ and $\delta_2 \in T_y Y$ there is a unique linear map $\delta \colon \mathcal{O}(X \times Y) \to \mathbb{C}$ such that $\delta(f \cdot h) = h(y) \cdot \delta_1 f + f(x) \cdot \delta_2 h$. This follows from Proposition 2.5.1 and the universal property of the tensor product. Now it is easy to see that this map δ is a derivation in (x, y) and that $\delta_X = \delta_1$ and $\delta_Y = \delta_2$. □

4.2. Tangent spaces of subvarieties. Let $X \subseteq V$ be closed subvariety of the vector space V and $x_0 \in X$. If $\delta \in T_{x_0} V = V$ is a tangent vector which vanishes on $I(X) = \ker(\text{res} \colon \mathcal{O}(V) \to \mathcal{O}(X))$, then the induced map $\bar{\delta} \colon \mathcal{O}(X) \to \mathbb{C}$ is a derivation in x_0, and vice versa. Thus we have the following result.

PROPOSITION 4.2.1. *If $X \subseteq V$ is a closed subvariety and $x_0 \in X$, then*

$$T_{x_0} X = \{v \in V \mid \partial_v(f) = 0 \text{ for all } f \in I(X)\} \subseteq V = T_{x_0} V.$$

More explicitly, let $V = \mathbb{C}^n$ and assume that the ideal $I(X)$ is generated by $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$. Then, for $x_0 \in X$, we get

$$T_x X = \{a = (a_1, \dots, a_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x) a_j = 0 \text{ for } i = 1, \dots, s\}.$$

In particular,

$$\dim T_x X = n - \operatorname{rk} \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{(i,j)}.$$

The $s \times n$-matrix

$$\operatorname{Jac}(f_1, \dots, f_s) := \left(\frac{\partial f_i}{\partial x_j} \right)_{(i,j)}$$

with entries in $\mathbb{C}[x_1, \dots, x_n]$ is called the *Jacobian matrix* of $f_1, \dots, f_s$. We get

$$T_x(X) = \ker \operatorname{Jac}(f_1, \dots, f_m)_x.$$

The proposition above gives the following criterion for smoothness.

PROPOSITION 4.2.2 (JACOBI-Criterion). *Let $X \subseteq \mathbb{C}^n$ be a closed subvariety where $I(X) = (f_1, \dots, f_s)$. Then $x \in X$ is non-singular if and only if*

$$\operatorname{rk}(\operatorname{Jac}(f_1, \dots, f_s)_x) \geq n - \dim_x X.$$

EXAMPLE 4.2.3. Consider the plane curve $C = \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$. Then $I(C) = (y^2 - x^3)$ and so the tangent space in an arbitrary point $x_0 = (a, b) \in C$ is given by $T_{(a,b)}C = \{(u, v) \in \mathbb{C}^2 \mid -3a^2u + 2bv = 0\}$. Since $(a, b) = (t^2, t^3)$ for some $t \in \mathbb{C}$ we get

$$T_{(t^2,t^3)}C = \begin{cases} \mathbb{C}^2 & \text{for } t = 0, \\ \mathbb{C}\begin{bmatrix} 2 \\ 3t \end{bmatrix} & \text{for } t \neq 0. \end{cases}$$

In particular, C is singular in $(0, 0)$ and smooth elsewhere.

EXAMPLE 4.2.4. Let $H := \mathcal{V}(f) \subseteq \mathbb{C}^n$ be a hypersurface where $f \in \mathbb{C}[x_1, \dots, x_n]$ is square-free. Then $H_{sing} = \{a \in H \mid \frac{\partial f}{\partial x_i}(a) = 0 \text{ for all } i\} = \mathcal{V}(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. It follows that $\dim H_{sing} < \dim H = n - 1$. In fact, no irreducible component C of H belongs to H_{sing}, because no prime divisor p of f divides all $\frac{\partial f}{\partial x_i}$.

EXERCISE 4.2.5. Calculate the tangent spaces of the plane curves $C_1 := \mathcal{V}(y - x^2)$ and $C_2 = \mathcal{V}(y^2 - x^2 - x^3)$ in arbitrary points (a, b).

4.3. *R*-valued points and epsilonization. Let $X \subseteq \mathbb{C}^n$ be a closed subvariety. For any $\mathbb{C}$-algebra R we define the *R-valued points of X* by

$$X(R) := \{a = (a_1, \dots, a_n) \in R^n \mid f(a) = 0 \text{ for all } f \in I(X)\}.$$

This definition does not depend on the embedding $X \subseteq \mathbb{C}^n$, because we have a canonical bijection $\operatorname{Alg}_{\mathbb{C}}(\mathcal{O}(X), R) \xrightarrow{\sim} X(R)$ given by $\rho \mapsto (\rho(\bar{x}_1), \dots, \rho(\bar{x}_n))$.

Now consider the $\mathbb{C}$-algebra $\mathbb{C}[\varepsilon] := \mathbb{C}[t]/(t^2)$ where $\varepsilon := t + (t^2)$ which is called the *algebra of dual numbers*. By definition, we have $\mathbb{C}[\varepsilon] = \mathbb{C} \oplus \mathbb{C}\varepsilon$ and $\varepsilon^2 = 0$. If X is an affine variety and $\rho\colon \mathcal{O}(X) \to \mathbb{C}[\varepsilon]$ an algebra homomorphism, then an easy calculation shows that ρ is of the form $\rho = \operatorname{ev}_x \oplus \delta_x \varepsilon$ for some $x \in X$ where ev_x is the evaluation map $f \mapsto f(x)$ and δ_x is a derivation in x, i.e., $\rho(f) = f(x) + \delta_x(f)\,\varepsilon$. Conversely, if δ_x is a derivation in x, then $\rho := \operatorname{ev}_x \oplus \delta_x \varepsilon$ is an algebra homomorphism. Hence

$$X(\mathbb{C}[\varepsilon]) = \{(x, \delta) \mid x \in X \text{ and } \delta \in T_x X\}. \tag{11}$$

This formula is very useful for calculating tangent spaces as we will see below. This method is sometimes called *epsilonization*.

If $X = V$ is a vector space, then the homomorphisms $\rho\colon \mathcal{O}(V) \to \mathbb{C}[\varepsilon]$ are in one-to-one correspondence with the elements of $V \oplus V\varepsilon$. In fact, there are canonical bijections

$$V(\mathbb{C}[\varepsilon]) \xrightarrow{\sim} \operatorname{Alg}_{\mathbb{C}}(\mathcal{O}(V), \mathbb{C}[\varepsilon]) \xrightarrow{\sim} V \oplus V\varepsilon.$$

The inverse map to $\operatorname{Alg}_{\mathbb{C}}(\mathcal{O}(V), \mathbb{C}[\varepsilon]) \xrightarrow{\sim} V \oplus V\varepsilon$ associates to $x + v\varepsilon \in V \oplus V\varepsilon$ the algebra homomorphism $\rho\colon f \mapsto f(x + v\varepsilon)$, and since

$$f(x + v\varepsilon) = f(x) + \partial_{v,x} f\,\varepsilon$$

it follows again from the above that $T_x V$ can be canonically identified with V.

EXAMPLE 4.3.1. (a) The tangent space of GL_n at E is the space of all $n \times n$-matrices and the tangent space of SL_n at $E \in \mathrm{SL}_n$ is the subspace of traceless matrices:

$$T_E \mathrm{SL}_n =_n := \{X \in \mathrm{M}_n \mid \operatorname{tr} X = 0\} \subseteq T_E \mathrm{GL}_n = \mathfrak{gl}_n := \mathrm{M}_n\,.$$

In fact, $I(\mathrm{SL}_n) = (\det -1)$, and an easy calculation shows that $\det(E + X\varepsilon) = 1 + (\operatorname{tr} X)\varepsilon$ which implies, by Proposition 4.2.1, that $X \in \mathrm{M}_n$ belongs to $T_E \mathrm{SL}_n$ if and only if $\operatorname{tr} X = 0$.

(b) Next we look at the orthogonal group $\mathrm{O}_n := \{A \in \mathrm{M}_n \mid AA^t = E\}$. As a closed subset O_n is defined by $\binom{n+1}{2}$ quadratic equations and so $\dim \mathrm{O}_n \geq n^2 - \binom{n+1}{2} = \binom{n}{2}$. On the other hand, we have

$$(E + X\varepsilon)(E + X\varepsilon)^t = E + (X + X^t)\varepsilon$$

which shows that $T_E \mathrm{O}_n \subseteq \{X \in \mathrm{M}_n \mid X \text{ skew symmetric}\}$. Since this space has dimension $\binom{n}{2}$ and since $\dim_E \mathrm{O}_n = \dim \mathrm{O}_n$ (Exercise 3.1.4) it follows from Proposition 4.1.7 that

$$T_E \mathrm{O}_n = T_E \mathrm{SO}_n = \mathfrak{so}_n := \{X \in \mathrm{M}_n \mid X \text{ skew symmetric}\}.$$

Exercise 4.3.2. If $X, Y \subseteq \mathbb{C}^n$ are closed subvarieties and $z \in X \cap Y$, then $T_z(X \cap Y) \subseteq T_z X \cap T_z Y \subseteq \mathbb{C}^n$. Give an example where $T_z(X \cap Y) \subsetneqq T_z X \cap T_z Y$.

4.4. Nonsingular varieties. We want to show that every variety X contains an open dense set of smooth points. Later in Corollary 4.10.6 we will even see that the smooth points form a open set.

Example 4.4.1. Let $H := \mathcal{V}(f) \subseteq \mathbb{C}^n$ be a hypersurface where $f \in \mathbb{C}[x_1, \dots, x_n]$ is square-free and nonconstant, and so $I(H) = (f)$. Then the tangent space in a point $x_0 \in H$ is given by

$$T_{x_0} H := \{a = (a_1, \dots, a_n) \mid \sum_i a_i \frac{\partial f}{\partial x_i}(x_0) = 0\},$$

and so

$$H_{sing} = \mathcal{V}(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}) \subseteq H.$$

It follows that H_{sing} is a proper closed subset whose complement is dense. (This is clear for irreducible hypersurfaces since a nonzero derivative $\frac{\partial f}{\partial x_i}$ cannot be a multiple of f and so $\mathcal{V}(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is a proper closed subset of $\mathcal{V}(f)$. This implies that every irreducible component of H contains a non-empty open set of nonsingular points which does not meet the other components, and the claim follows.)

It is also interesting to remark that a common point of two or more irreducible components of H is always singular. We will see that this true in general (Corollary 4.10.6).

Proposition 4.4.2. *Let X be an irreducible affine variety. Then the set X_{sing} of singular points of X is a proper closed subset of X whose complement is dense.*

Proof. We can assume that X is an irreducible closed subvariety of $\mathbb{C}^n$ of dimension d. If $I(X) = (f_1, \dots, f_s)$, then, by Proposition 4.2.1,

$$X_{sing} = \{x \in X \mid \operatorname{rk} \left(\frac{\partial f_j}{\partial x_i}(x)\right)_{(i,j)} < n - d\}$$

which is the closed subset defined by the vanishing of all $(n-d) \times (n-d)$ minors of the Jacobian matrix $\operatorname{Jac}(f_1, \dots, f_s)$. In order to see that X_{sing} has a dense complement, we use the fact, that every irreducible variety contains a special open set which is isomorphic to a special open set of an irreducible hypersurface H (see Proposition 3.1.13). Since H contains a dense open set of nonsingular points (see Example 4.4.1 above) the claim follows. $\square$

We will see later in Corollary 4.10.6 that the proposition above holds for every variety. At this moment we only know that there is always a dense open set $U \subseteq X$ consisting of nonsingular points.

EXERCISE 4.4.3. If X is an affine variety such that all irreducible components have the same dimension. Then X_{sing} is closed and has a dense complement.

EXERCISE 4.4.4. The hypersurface $H = \mathcal{V}(xz - y(y-1)) \subseteq \mathbb{C}^3$ from Exercise 2.2.14 is nonsingular.

EXERCISE 4.4.5. Let $q \in \mathbb{C}[x_1, \ldots, x_n]$ be a quadratic form and $Q := \mathcal{V}(q) \subseteq \mathbb{C}^n$. Then 0 is a singular point of Q. It is the only singular point if and only if q is nondegenerate.

EXERCISE 4.4.6. Determine the singular points of the plane curves

$$E_p := \mathcal{V}(y^2 - p(x))$$

where $p(x)$ is an arbitrary polynomial, and deduce a necessary and sufficient condition for E_p to be smooth.

EXERCISE 4.4.7. Let $X \subseteq \mathbb{C}^n$ be a closed cone (see Exercise 1.2.9). Then X_{sing} is a cone, too. Moreover, $0 \in X$ is a nonsingular point if and only if X is subspace.

EXERCISE 4.4.8. Let X be an affine variety such that the group of automorphisms acts transitively on X. Then X is smooth.

4.5. Tangent bundle and vector fields. Let X be an affine variety. Denote by $TX := \bigcup_{x \in X} T_x X$ the disjoint union of the tangent spaces and by $p \colon TX \to X$ the natural projection, $\delta \in T_x X \mapsto x$. We call TX the *tangent bundle* of X. We will see later that TX has a natural structure of an affine variety and that p is a morphism.

A *section* $\xi \colon X \to TX$ of p, i.e. $p \circ \xi = \mathrm{Id}_X$, is a collection $(\xi_x)_{x \in X}$ of tangent vectors $\xi_x \in T_x X$. It is usually called a *vector field* and can be considered as an operator on regular functions $f \in \mathcal{O}(X)$:

$$(\xi f)(x) := \xi_x f \text{ for } x \in X.$$

DEFINITION 4.5.1. An *algebraic vector field on X* is a section $\xi \colon X \to TX$ with the property that $\xi f \in \mathcal{O}(X)$ for all $f \in \mathcal{O}(X)$. The space of algebraic vector fields is denoted by $\mathrm{Vec}(X)$.
In the following, we will mostly talk about "vector fields" and omit the term "algebraic" whenever it is clear from the context.

Thus a vector field ξ can be considered as a linear map $\xi \colon \mathcal{O}(X) \to \mathcal{O}(X)$, and so $\mathrm{Vec}(X)$ is a subspace of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}(X))$. More generally, the vector fields form a *module over* $\mathcal{O}(X)$ where the product $f\xi$ for $f \in \mathcal{O}(X)$ is defined in the obvious way: $(f\xi)_x := f(x)\xi_x$.

EXAMPLE 4.5.2. Let $X = V$ be a $\mathbb{C}$-vector space and fix a vector $v \in V$. Then $\partial_v \in \mathrm{Vec}(V)$ is defined by $x \mapsto \partial_{v,x}$. It follows that

$$\partial_v f := \left. \frac{f(x+tv) - f(x)}{t} \right|_{t=0} \in \mathcal{O}(X)$$

which implies that this vector field is indeed algebraic. We claim that every algebraic vector field on V is of this form. In fact, if $V = \mathbb{C}^n$, then

$$\mathrm{Vec}(\mathbb{C}^n) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n] \frac{\partial}{\partial x_i}$$

which means that every algebraic vector field ξ on $\mathbb{C}^n$ is of the form $\xi = \sum_i h_i \frac{\partial}{\partial x_i}$ where $h_i \in \mathbb{C}[x_1, \ldots, x_n] = \mathcal{O}(\mathbb{C}^n)$. (This follows from the two facts that every vector field ξ on $\mathbb{C}^n$ is of this form with arbitrary functions h_i and that $\xi(x_i) = h_i$.)

Another observation is that for every vector field ξ on X the corresponding linear map $\xi\colon \mathcal{O}(X) \to \mathcal{O}(X)$ is a derivation, i.e. ξ is a *linear differential operator*:

$$\xi(fh) = f\,\xi h + h\,\xi f \text{ for all } f, h \in \mathcal{O}(X).$$

Proposition 4.5.3. *The map sending a vector field to the corresponding linear differential operator defines a bijection* $\mathrm{Vec}(X) \xrightarrow{\sim} \mathrm{Der}(\mathcal{O}(X), \mathcal{O}(X)) \subseteq \mathrm{End}(\mathcal{O}(X))$.

Proof. It remains to show that every derivation $\xi\colon \mathcal{O}(X) \to \mathcal{O}(X)$ is given by an algebraic vector field. For this, define $\xi_x := \mathrm{ev}_x \circ \xi$. Then the vector field $(\xi_x)_{x \in X}$ is algebraic and the corresponding linear map is ξ. □

Example 4.5.2 above shows that for $X = V$ we have a canonical bijection $TV \simeq V \times V$, using the identifications $T_v V = V \simeq \{x\} \times V$. Then $p\colon TV \to V$ becomes the projection pr_V, and algebraic vector fields are section of pr_V, i.e. morphisms $\xi\colon V \to V \times V$ of the form $\xi(x) = (x, \xi_x)$. We will mostly identify TV with $V \times V$.

Proposition 4.5.4. *Let $X \subseteq V$ be a closed subset.*

(1) *If $\xi \in \mathrm{Vec}(V)$, then $\xi|_X$ defines a vector field on X (i.e. $\xi_x \in T_x X$ for all $x \in X$) if and only if $(\xi f)|_X = 0$ for all $f \in I(X)$. Moreover, it suffices to test a system of generators of the ideal $I(X)$.*

(2) *There is a canonical bijection $TX \xrightarrow{\sim} \{(x, \delta) \mid \delta \in T_x X \subseteq V\}$ where the latter is a closed subset of $X \times V$. Thus TX has the structure of an affine variety. Using coordinates, we get*

$$TX \xrightarrow{\sim} \{(x, a_1, \ldots, a_n) \mid \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(x) = 0 \text{ for all } f \in I(X)\} \subseteq X \times \mathbb{C}^n$$

(3) *A vector field ξ on X is algebraic if and only if $\xi\colon X \to TX$ is a morphism.*

Proof. (1) We have $\xi_x \in T_x X$ for all $x \in X$ if and only if $\xi_x f = 0$ for all x and all $f \in I(X)$ which is equivalent to $(\xi f)|_X = 0$ for all $f \in I(X)$.

(2) We can assume that $V = \mathbb{C}^n$ and $\mathcal{O}(V) = \mathbb{C}[x_1, \ldots, x_n]$. If $I(X) = (f_1, \ldots f_m)$, then, by (1),

$$\begin{aligned} T' &:= \{(x, \delta_x) \in X \times V \mid \delta \in T_x X\} \\ &= \{(x, a_1, \ldots, a_n) \mid \sum_{i=1}^{n} a_i \frac{\partial f_j}{\partial x_i}(x) = 0 \text{ for } j = 1, \ldots, m\} \subseteq X \times \mathbb{C}^n \end{aligned}$$

which shows that this is a closed subspace of $X \times \mathbb{C}^n$. Now (2) follows easily.

(3) Using the identification of TX with the closed subvariety T' above, an arbitrary section $\xi\colon X \to TX$ has the form $\xi_x = \sum h_i(x) \frac{\partial}{\partial x_i}$ with arbitrary functions h_i on X. Set $\bar{x}_i := x_i|_X$. Then the vector field ξ is algebraic if and only if $h_i = \xi \bar{x}_i$ is regular on X which is equivalent to the condition that $\xi\colon X \to TX$ is a morphism. □

Remark 4.5.5. We will see later in Proposition 4.6.7 that the structure of TX as an affine variety does not depend on the embedding $X \subseteq V$.

Example 4.5.6. Consider the curve $H := \mathcal{V}(xy - 1) \subseteq \mathbb{C}^2$. Then $I(H) = (xy - 1)$. For a vector field $\xi = a(x, y)\partial_x + b(x, y)\partial_y$ on $\mathbb{C}^2$ we get

$$\xi(xy - 1) = a(x, y)y + b(x, y)x.$$

Thus $\xi(xy - 1)|_H = 0$ if and only if $ay + bx = 0$ on H. It follows that $x\partial_x - y\partial_y$ defines a vector field ξ_0 on H and that $\mathrm{Vec}(H) = \mathcal{O}(C)\xi_0$. (In fact, setting $h := ay|_H = -bx|_H$ we get $a|_H = h \cdot x|_H$ and $b|_H = -h \cdot y|_H$.)

The tangent bundle $TH \subseteq H \times \mathbb{C}^2$ has the following description (see Proposition 4.5.4(1)):

$$TH = \{(t, t^{-1}, \alpha, \beta) \mid \alpha t^{-1} + \beta t = 0\} = \{(t, t^{-1}, -\beta t^2, \beta \mid t \in \mathbb{C}^*, \beta \in \mathbb{C}\} \xrightarrow{\sim} H \times \mathbb{C}.$$

EXAMPLE 4.5.7. Consider NEIL's parabola $C := \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$ (see Example 1.3.11). Then a vector field $a\partial_x + b\partial_y$ defines a vector field on C if and only if

$$-3ax^2 + 2by = 0 \text{ on } C.$$

To find the solutions we use the isomorphism $\mathcal{O}(C) \xrightarrow{\sim} \mathbb{C}[t^2, t^3]$, $x \mapsto t^2, y \mapsto t^3$ (see Example 2.2.11). Thus we have to solve the equation $3\bar{a}t = 2\bar{b}$ in $\mathbb{C}[t^2, t^3]$. This is easy: Every solution is a linear combination (with coefficients in $\mathbb{C}[t^2, t^3]$) of the two solutions $(2t^2, 3t^3)$ and $(2t^3, 3t^4)$. This shows that

$$\xi_0 := (2x\partial_x + 3y\partial_y)|_D \quad \text{and} \quad \xi_1 := (2y\partial_x + 3x^2\partial_y)|_D$$

are vector fields on C and that $\operatorname{Vec}(C) = \mathcal{O}(C)\xi_0 + \mathcal{O}(C)\xi_1$. Moreover, $\bar{x}^2\xi_0 = \bar{y}\xi_1$.

Our calculation also shows that every vector field on C vanishes in the singular point 0 of the curve. For the tangent bundle we get

$$TC = \{(t^2, t^3, \alpha, \beta) \mid -3\alpha t^4 + 2\beta t^3 = 0\} \subseteq C \times \mathbb{C}^2$$

which has two irreducible components, namely

$$TC = \{(t^2, t^3, 2\alpha, 3\alpha t) \mid t, \alpha \in \mathbb{C}\} \cup \{(0,0)\} \times \mathbb{C}^2$$

EXERCISE 4.5.8. Determine the vector fields on the curve $D := \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{C}^2$. Do they all vanish in the singular point of D?

EXERCISE 4.5.9. Determine the vector fields on the curves $D_1 := \{(t, t^2, t^3) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$ and $D_2 := \{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$.
(Hint: For D_2 one can use that $\mathcal{O}(D_2) \simeq \mathbb{C}[t^3, t^4, t^5] = \mathbb{C} \oplus \bigoplus_{i \geq 3} \mathbb{C}t^i$.)

If the variety X is smooth, then all fibers of $p\colon TX \to X$ are vector spaces of the same dimension. We will show now that in this case TX is a *vector bundle* of rank $r := \dim X$ over X. This means that for every point $x \in X$ there is a special open neighborhood U of x in X and an isomorphism $p^{-1}(U) \xrightarrow{\sim} \psi_U\colon U \times \mathbb{C}^r$ over U which is linear in the fibers, i.e. $\psi\colon T_uU = p^{-1}(u) \xrightarrow{\sim} \{u\} \times \mathbb{C}^r = \mathbb{C}^r$ is a linear map.

PROPOSITION 4.5.10. *If X is smooth and irreducible, then $TX \to X$ is a vector bundle of rank $r = \dim X$*

PROOF. We can assume that $X \subseteq \mathbb{C}^n$ is a closed subset where $I(X) = (f_1, \ldots, f_m)$. Denote by $J = \operatorname{Jac}(f_1, \ldots, f_m)$ the Jacobian matrix, with entries in $\mathbb{C}[x_1, \ldots, x_n]$. Then $\ker J(x) = T_x(X) \subseteq \mathbb{C}^n$ (Proposition **??**), and, by assumption, $\operatorname{rk}(J(x)) = n - r$ for all $x \in X$. Fix $x_0 \in X$ and choose $n - r$ columns of $J(x_0)$ which are linearly independent. Then this holds for all x in an special open neighborhood U of x_0. Let $1 \leq i_1 < \cdots < i_r \leq n$ be the indices of the remaining columns and denote by $q\colon \mathbb{C}^n \to \mathbb{C}^r$ the corresponding linear projection. Then q induces an isomorphism $\ker J(x) \xrightarrow{\sim} \mathbb{C}^r$ for all $x \in U$. □

In general, the fibers of $TX \to X$ have different dimensions, and the minimal dimension is reached on the smooth points of X which form an open set of X. The next result generalizes this. It is a special case of a famous theorem of CHEVALLEY saying that for every morphism $Z \to X$ the function $z \mapsto \dim_z \varphi^{-1}(\varphi(z))$ is upper-semicontinuous, see section A.3.4.

PROPOSITION 4.5.11. *The function $x \mapsto \dim T_xX$ is upper-semicontinuous.*

PROOF. □

The next result is well-known in differential geometry; for the definition of a *Lie algebra* we refer to section II.4.1.

PROPOSITION 4.5.12. *The vector fields* $\mathrm{Vec}(X)$ *on* X *form a Lie algebra with Lie bracket*

$$[\xi, \eta] := \xi \circ \eta - \eta \circ \xi.$$

PROOF. By Proposition 4.5.3 it suffices to show that for any two derivations ξ, η of $\mathcal{O}(X)$ the commutator $\xi \circ \eta - \eta \circ \xi$ is again a derivation. But this is a general fact and holds for any associative algebra, see the following Exercise 4.5.14. □

EXERCISE 4.5.13. Let A be an arbitrary associative $\mathbb{C}$-algebra. Then A is a Lie algebra with Lie bracket $[a, b] := ab - ba$, i.e., the bracket $[\ ,\ ]$ satisfies the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \text{ for all } a, b, c \in A.$$

EXERCISE 4.5.14. Let R be an associative $\mathbb{C}$-algebra. If $\xi, \eta\colon R \to R$ are both $\mathbb{C}$-derivations, then so is the commutator $\xi \circ \eta - \eta \circ \xi$. This means that the derivations $\mathrm{Der}(R)$ form a Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(R)$.

EXERCISE 4.5.15. Let $X \subseteq \mathbb{C}^n$ be a closed and irreducible. Then $\dim TX \geq 2 \dim X$. If X is smooth, then TX is irreducible and smooth of dimension $\dim TX = 2 \dim X$. (Hint: If $I(X) = (f_1, \dots f_m)$, then $TX \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is defined by the equations

$$f_j = 0 \text{ and } \sum_{i=1}^{n} y_i \frac{\partial f_j}{\partial x_i}(x) = 0 \text{ for } j = 1, \dots, m.$$

The Jacobian matrix of this system of $2m$ equations in $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$ has the following block form

$$\begin{bmatrix} \mathrm{Jac}(f_1, \dots, f_m) & 0 \\ * & \mathrm{Jac}(f_1, \dots, f_m) \end{bmatrix}$$

and thus has rank $\geq 2 \cdot \mathrm{rk}\, \mathrm{Jac}(f_1, \dots, f_m) = 2(n - \dim X)$.)

4.6. Differential of a morphism. Let $\varphi\colon X \to Y$ be a morphism of affine varieties, and let $x \in X$.

DEFINITION 4.6.1. The *differential of* φ *in* x is the linear map

$$d\varphi_x\colon T_x X \to T_{\varphi(x)} Y$$

defined by $\delta \mapsto d\varphi_x(\delta) := \delta \circ \varphi^*$.

If $Z \subseteq X$ is a closed subvariety and $z \in Z$, then we get for the induced morphism $\varphi|_Z\colon Z \to Y$ that $d(\varphi|_Z)_z = d\varphi_z|_{T_z Z}$. Another obvious remark is that the differential of a constant morphism is the zero map.

REMARK 4.6.2. Set $y := \varphi(x)$. The comorphism $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ defines a homomorphism $\mathfrak{m}_y \to \mathfrak{m}_x$ and thus a linear map $\bar{\varphi}^*\colon \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$. It is easy to see that the differential $d\varphi_x$ corresponds to the dual map of $\bar{\varphi}^*$ under the isomorphisms $T_x X \simeq \mathrm{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C})$ and $T_y Y \simeq \mathrm{Hom}(\mathfrak{m}_y/\mathfrak{m}_y^2, \mathbb{C})$ (see Lemma 4.1.4).

EXAMPLE 4.6.3. Using the identification $T_{(x,y)}(X \times Y) = T_x X \oplus T_y Y$ (see Proposition 4.1.9) one easily sees that the differential $d(\mathrm{pr}_X)_x\colon T_{(x,y)}(X \times Y) \to T_x X$ coincides with the linear projection $\mathrm{pr}_{T_x X}$.

PROPOSITION 4.6.4. *Consider a morphism* $\varphi = (f_1, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$, $f_j \in \mathcal{O}(\mathbb{C}^n) = \mathbb{C}[x_1, \dots, x_n]$. *Then the differential*

$$d\varphi_x\colon T_x\mathbb{C}^n = \mathbb{C}^n \to T_{\varphi(x)}\mathbb{C}^m = \mathbb{C}^m$$

of φ *in* $x \in \mathbb{C}^n$ *is given by the Jacobian matrix*

$$\mathrm{Jac}(f_1, \dots, f_m)_x = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{(i,j)}.$$

PROOF. The identification of the tangent space $T_x\mathbb{C}^n = \mathrm{Der}_x(\mathcal{O}(\mathbb{C}^n))$ with $\mathbb{C}^n$ is given by $\delta \mapsto (\delta x_1, \dots, \delta x_n)$ (see Example 4.1.3). This implies that

$$d\varphi_x(\delta) = ((\delta \circ \varphi^*)(y_1), \dots, (\delta \circ \varphi^*)(y_m)) = (\delta f_1, \dots, \delta f_m).$$

Now the claim follows since

$$\delta f_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) \cdot \delta x_i.$$

$\square$

PROPOSITION 4.6.5. *Let $\varphi\colon X \to Y$ be a morphism, and let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be closed subvarieties such that $\varphi(X_0) \subseteq Y_0$. Denote by $\varphi_0\colon X_0 \to Y_0$ the induced morphism. Then, for all $x \in X_0$, we have $d\varphi_x(T_xX_0) \subseteq T_{\varphi(x)}Y_0$, and $d\varphi_0 = d\varphi|_{TX_0}\colon TX_0 \to TY_0$.*

PROOF. We know that $\delta \in T_xX$ belongs to T_xX_0 if and only if $\delta(f) = 0$ for all $f \in I_X(X_0)$ (Proposition 4.2.1), and similarly for Y. Since $\varphi(X_0) \subseteq Y_0$ we have $\varphi^*(I_Y(Y_0)) \subseteq I_X(X_0)$. Thus, for $\delta \in T_xX_0$ we obtain

$$d\varphi_x(\delta)(h) = \delta(\varphi^*(h)) = 0 \quad \text{for all } h \in I_Y(Y_0),$$

and the claim follows. $\square$

EXERCISE 4.6.6. Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be morphisms of affine varieties and let $x \in X$. Then

$$d(\psi \circ \varphi)_x = d\psi_y \circ d\varphi_x$$

where $y := \varphi(x) \in Y$.

For any morphism $\varphi\colon X \to Y$ the differentials $d\varphi_x$ define a map $d\varphi\colon TX \to TY$ of the tangent bundles in the obvious way. Embedding X and Y into vector spaces, the tangent bundle inherits the structure of an affine variety (Proposition 4.5.4).

PROPOSITION 4.6.7. *The differential $d\varphi\colon TX \to TY$, $(x, \delta) \mapsto (\varphi(x), d\varphi_x(\delta)$, is a morphism of varieties. In particular, the structure of TX as an affine variety is independent of the embedding of X into a vector space.*

PROOF. Consider first the case $X = \mathbb{C}^n$, $Y = \mathbb{C}^m$ and $\varphi = (f_1, \dots, f_n)$. Then $d\varphi\colon T\mathbb{C}^n = \mathbb{C}^n \times \mathbb{C}^n \to T\mathbb{C}^m = \mathbb{C}^m \times \mathbb{C}^m$ is given by

$$d\varphi(x, a_1, \dots, a_n) = (f_1(x), \dots, f_m(x), \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(x)a_i, \dots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(x)a_i)$$

(Proposition 4.6.4) which is clearly a morphism.

Now choose embeddings $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$, and extend the morphism φ to a morphism $\Phi\colon \mathbb{C}^n \to \mathbb{C}^m$ (Lemma 2.1.6):

$$\begin{array}{ccc} X & \xrightarrow{\subseteq} & \mathbb{C}^n \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ Y & \xrightarrow{\subseteq} & \mathbb{C}^m \end{array}$$

The claim follows from Proposition 4.6.5 above. $\square$

4.7. Epsilonization. In order to calculate explicitly differentials of morphisms we will again use the *epsilonization* (4.3). Recall that for $\delta \in T_xX$ the map $\rho := \mathrm{ev}_x \oplus \delta\varepsilon \colon \mathcal{O}(X) \to \mathbb{C}[\varepsilon]$ is a homomorphism of algebras and vice versa. If $\varphi\colon X \to Y$ is a morphism and $x \in X$, $y := \varphi(x) \in Y$, then we obtain, by definition, the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\mathrm{ev}_x \oplus \delta\varepsilon"] & \mathbb{C}[\varepsilon] \\
\mathcal{O}(Y) \arrow[u, "\varphi^*"] \arrow[ur, "\mathrm{ev}_y \oplus d\varphi_x(\delta)\varepsilon"'] &
\end{tikzcd}$$

If $X := V$ and $Y := W$ are vector spaces, then a homomorphism $\rho\colon \mathcal{O}(V) \to \mathbb{C}[\varepsilon]$ corresponds to an element $x \oplus v\varepsilon \in V \oplus V\varepsilon$ where $\rho(f) = f(x + v\varepsilon)$, and so $\rho \circ \varphi^*$ corresponds to the element $\varphi(x + v\varepsilon) \in W \oplus W\varepsilon$. Thus we obtain the following result which is very useful for calculating differentials of morphisms.

Lemma 4.7.1. *Let $\varphi\colon V \to W$ be a morphism between vector spaces, and let $x \in V$ and $v \in T_xV = V$. Then we have*

$$\varphi(x + \varepsilon v) = \varphi(x) + d\varphi_x(v)\,\varepsilon$$

where both sides are considered as elements of $W \oplus W\varepsilon$.

Example 4.7.2. The differential of the morphism $?^m\colon \mathrm{M}_n \to \mathrm{M}_n$, $A \mapsto A^m$, in E is $m \cdot \mathrm{Id}$. In fact, $(E + X\varepsilon)^m = E + mX\varepsilon$.

The differential of $\varphi\colon \mathrm{M}_2 \to \mathrm{M}_2$, $A \mapsto A^2$, in an arbitrary matrix B is given by $d\varphi_B(X) = BX + XB$, because $(B + X\varepsilon)^2 = B^2 + (BX + XB)\varepsilon$.

The differential of the matrix multiplication $\mu\colon \mathrm{M}_n \times \mathrm{M}_n \to \mathrm{M}_n$ in (E, E) is the addition: $(E + X\varepsilon)(E + Y\varepsilon) = E + (X + Y)\varepsilon$.

Exercise 4.7.3. Consider the multiplication $\mu\colon \mathrm{M}_2 \times \mathrm{M}_2 \to M_2$ and show:

(1) $d\mu_{(A,B)}$ is surjective, if A or B is invertible.
(2) If $\operatorname{rk} A = \operatorname{rk} B = 1$, then $d\mu_{(A,B)}$ has rank 3.
(3) We have $\operatorname{rk} d\mu_{(A,0)} = \operatorname{rk} d\mu_{(0,A)} = 2 \operatorname{rk} A$.

Exercise 4.7.4. Calculate the differential of the morphism $\varphi\colon \mathrm{End}(V) \times V \to V$ given by $(\rho, v) \mapsto \rho(v)$, and determine the pairs (ρ, v) where $d\varphi_{(\rho,v)}$ is surjective.

4.8. Tangent spaces of fibers. Let $\varphi\colon X \to Y$ be a morphism, $x \in X$ and $F := \varphi^{-1}(\varphi(x))$ the fiber through x. Since $\varphi|_F$ is the constant map, its differential in any point is zero and so $T_xF \subseteq \ker d\varphi_x$. This proves the first part of the following result.

Proposition 4.8.1. *Let $\varphi\colon X \to Y$ be a morphism, $x \in X$ and $F := \varphi^{-1}(\varphi(x))$ the fiber through x.*

(1) $T_xF \subseteq \ker d\varphi_x$.
(2) *If the fiber F is reduced in x, then $T_xF = \ker d\varphi_x$.*
(3) *If X is smooth in x and $\operatorname{rk} d\varphi_x = \dim_x X - \dim_x F$, then F is reduced and smooth in x.*

Proof. (2) Put $y := \varphi(x) \in Y$. By definition the fiber is reduced in x if and only if the ideal in the local ring $\mathcal{O}_{X,x}$ generated by $\varphi^*(\mathfrak{m}_y)$ is perfect which means that $\mathcal{O}_{F,x} = \mathcal{O}_{X,x}/\varphi^*(\mathfrak{m}_y)\mathcal{O}_{X,x}$ (see Definition 2.2.10).

Now let $\delta \in T_xX$ be a derivation of $\mathcal{O}(X)$ in x. If $\delta \in \ker d\varphi_x$, then $\delta \circ \varphi^* = 0$. Hence δ, regarded as a derivation of $\mathcal{O}_{X,x}$, vanishes on $\varphi^*(\mathfrak{m}_y)\mathcal{O}_{X,x}$ and thus induces a derivation of $\mathcal{O}_{F,x}$ in x, i.e., $\delta \in T_xF$.

(3) Set $R := \mathcal{O}(X)/\varphi^*(\mathfrak{m}_y)\mathcal{O}(X) \supseteq \mathfrak{m} := \mathfrak{m}_x/\varphi^*(\mathfrak{m}_y)\mathcal{O}(X)$. Clearly, $R_{\text{red}} = \mathcal{O}(F)$, and the composition $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2 \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is the zero map. Since X is smooth in x we get $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim_x X$, and since the first map is dual to $d\varphi_x$ it has rank $\dim_x X - \dim_x F$. It follows that $\dim \mathfrak{m}/\mathfrak{m}^2 \le \dim_x F = \dim R_{\mathfrak{m}}$. Now it follows from Proposition 4.10.5 that $R_{\mathfrak{m}}$ is a domain, hence $R_{\mathfrak{m}} = \mathcal{O}_{F,x}$, and that F is smooth in x, because $\dim T_x F = \dim \mathfrak{m}/\mathfrak{m}^2 \le \dim_x F$. $\square$

EXAMPLE 4.8.2. Let $X \subseteq \mathbb{C}^n$ be a closed subset and $I(X) = (f_1, \dots, f_m)$. Consider the morphism $\varphi = (f_1, \dots, f_m) \colon \mathbb{C}^n \to \mathbb{C}^m$. Then $X = \varphi^{-1}(0)$, and this fiber is reduced in every point. Thus, for every $x \in X$,

$$T_x X = \ker d\varphi_x = \ker \operatorname{Jac}(f_1, \dots, f_m)_x$$

as we have already seen in Proposition 4.2.1. The following result is a partial inverse.

PROPOSITION 4.8.3. *Let $Z = \mathcal{V}(f_1, \dots, f_m) \subseteq \mathbb{C}^n$ be a closed subset. Assume that* $\operatorname{rk} \operatorname{Jac}(f_1, \dots, f_m)_z = n - \dim_z Z$ *for all $z \in Z$. Then Z is smooth and $I(Z) = (f_1, \dots, f_m)$.*

PROOF. Consider the morphism $\varphi = (f_1, \dots, f_m) \colon \mathbb{C}^n \to \mathbb{C}^m$. Then $Z = \varphi^{-1}(0)$, and $d\varphi_z = \operatorname{Jac}(f_1, \dots, f_m)_z \colon \mathbb{C}^n \to \mathbb{C}^m$. Thus $T_z Z \subseteq \ker \operatorname{Jac}(f_1, \dots, f_m)_z$, and we have equality, because $\dim_z Z \le \dim T_z Z \le \dim \ker \operatorname{Jac}(f_1, \dots, f_m)_z = \dim_z Z$. Now Proposition 4.8.1(3) shows that the fiber $\varphi^{-1}(0)$ is reduced and smooth in every point z, hence the claim. $\square$

EXERCISE 4.8.4. For every point $(x, y) \in X \times Y$ we have $T_x X = \ker d(\mathrm{pr}_Y)_{(x,y)}$ and $T_y X = \ker d(\mathrm{pr}_X)_{(x,y)}$ where $\mathrm{pr}_X, \mathrm{pr}_Y$ are the canonical projections (see Proposition 4.1.9).

EXERCISE 4.8.5. For the closed subset $N \subseteq \mathrm{M}_2$ of nilpotent 2×2-matrices we have $I(N) = (\mathrm{tr}, \det)$.

PROPOSITION 4.8.6. *Let $\varphi \colon X \to Y$ be a dominant morphism of irreducible varieties, and let $x \in X$ and $y := \varphi(x) \in Y$. Assume that the fiber $F := \varphi^{-1}(y)$ is reduced and smooth in x and that* $\dim_x F = \dim X - \dim Y$.

(1) *If Y is smooth in y, then X is smooth in x.*
(2) *If X is smooth in x and Y normal in y, then Y is smooth in y.*

PROOF. By Proposition 4.8.1(2) we have an exact sequence of vector spaces

$$0 \longrightarrow T_x F \xrightarrow{\subseteq} T_x X \xrightarrow{d\varphi_x} T_y Y$$

where $\dim T_x F = \dim_x F = \dim X - \dim Y$.

(1) If Y is smooth in y, then $\dim T_y Y = \dim Y$, hence $\dim T_x X \le \dim T_x F + \dim T_y Y = \dim_x F + \dim Y = \dim X$, and so X is smooth in x and $d\varphi_x$ is surjective.

(2) This is more complicated; the statement can be found in [**GKM**[+]**13**, Lemma 2.22]. We will give a proof later in section 4.11. $\square$

The normality assumption in the statement (2) is necessary, as shown by Exercise 2.2.13.

4.9. Morphisms of maximal rank. The main result of this section is the following theorem.

THEOREM 4.9.1. *Let $\varphi \colon X \to Y$ be a dominant morphism between two irreducible varieties X and Y. Then there is a dense open set $U \subseteq X$ such that $d\varphi_x \colon T_x X \to T_{\varphi(x)} Y$ is surjective for all $x \in U$.*

We first work out an important example which will be used in the proof of the proposition above.

EXAMPLE 4.9.2. Let Y be an irreducible affine variety and $X \subseteq Y \times \mathbb{C}$ an irreducible hypersurface. Assume that $I(X) = (f)$ where $f = \sum_{i=0}^{n} f_i t^i \in \mathcal{O}(Y)[t] = \mathcal{O}(Y \times \mathbb{C})$ and $f_n = 1$. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "\subseteq"] \arrow[dr, "p"'] & Y \times \mathbb{C} \arrow[d, "\mathrm{pr}_Y"] \\ & Y \end{tikzcd}$$

Then the differential $dp_{(y,a)} \colon T_{(y,a)}X \to T_y Y$ is surjective if $\dfrac{\partial f}{\partial t}(y,a) \neq 0$, and this holds on a dense open set of X.

PROOF. We have $T_{(y,a)}X \subseteq T_{(y,a)}Y \times \mathbb{C} = T_y Y \oplus \mathbb{C}$, and this subspace is given by $T_{(y,a)}X = \{(\delta, \lambda) \mid (\delta, \lambda)f = 0\}$, because $I(X) = (f)$. Now we have

$$(\delta, \lambda)f = \sum_{i=0}^{n} (\delta f_i \cdot a^i + f_i(y) \cdot i \cdot a^{i-1} \cdot \lambda) = \sum_{i=0}^{n} \delta f_i \cdot a^i + \frac{\partial f}{\partial t}(y,a) \cdot \lambda$$

Since $dp_{(y,a)}(\delta, \lambda) = \delta$ we see that $dp_{(y,a)}$ is surjective if $\frac{\partial f}{\partial t}(y,a) \neq 0$ which proves the first claim. But $\frac{\partial f}{\partial t}$ cannot be a multiple of f and thus does not vanish on X, proving the second claim. $\square$

The next lemma shows that the situation described in the example above always holds on an open set for every morphism of finite degree.

LEMMA 4.9.3. *Let X, Y be irreducible affine varieties and $\varphi \colon X \to Y$ a morphism of finite degree. Then there is a special open set $U \subseteq Y$ and a closed embedding $\gamma \colon \varphi^{-1}(U) \hookrightarrow U \times \mathbb{C}$ with the following properties:*

(i) *$I(\gamma(U)) = (f)$ where $f = \sum_{i=0}^{n} f_i t^i \in \mathcal{O}(U)[t]$;*
(ii) *$\mathrm{pr}_U \circ \gamma = \varphi|_{\varphi^{-1}(U)}$.*

$$\begin{tikzcd} \varphi^{-1}(U) \arrow[r, "\simeq"] \arrow[rr, bend left, "\gamma"] \arrow[dr, "\varphi"'] & \mathcal{V}_{U \times \mathbb{C}}(f) \arrow[r, "\subseteq"] \arrow[d, "p"] & U \times \mathbb{C} \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

PROOF. We have to show that there is a nonzero $s \in \mathcal{O}(Y)$ such that $\mathcal{O}(X)_s \simeq \mathcal{O}(Y)_s[t]/(f)$ with a polynomial $f \in \mathcal{O}(Y)_s[t]$. Then the claim follows by setting $U := Y_s$.

By assumption, the field $\mathbb{C}(X)$ is a finite extension of $\mathbb{C}(Y)$ of degree n, say,

$$\mathbb{C}(X) = \mathbb{C}(Y)[h] \simeq \mathbb{C}(Y)[t]/(f)$$

where $f = \sum_{i=0}^{n} f_i t^i$, $f_i \in \mathbb{C}(Y)$ and $f_n = 1$. There is a nonzero element $s \in \mathcal{O}(Y)$ such that

(a) $f_i \in \mathcal{O}(Y)_s$ for all i,
(b) $h \in \mathcal{O}(X)_s$ and
(c) $\mathcal{O}(X)_s = \mathcal{O}(Y)_s[h] = \bigoplus_{i=0}^{n-1} \mathcal{O}(Y)_s h^i$.

In fact, (a) and (b) are clear. For (c) we first remark that $\mathcal{O}(Y)_s[h] = \bigoplus_{i=0}^{n-1} \mathcal{O}(Y)_s h^i \subseteq \mathcal{O}(X)_s$, because of (a) and (b). If $h_1, \ldots, h_m$ is a set of generators of $\mathcal{O}(X)$ we can find a nonzero $s \in \mathcal{O}(Y)$ such that $h_i \in \mathcal{O}(Y)_s[h]$, proving (c).

Setting $U := Y_s$ we get $\varphi^{-1}(U) = X_s$ and $\mathcal{O}(X_s) = \mathcal{O}(Y_s)[t]/(f)$, by (c), and the claim follows. $\square$

Proof of Theorem 4.9.1. By the Decomposition Theorem (Theorem 3.4.1) we can assume that φ is the composition of a finite surjective morphism and a projection of the form $Y \times \mathbb{C}^r \to Y$. Since the differential of the second morphism is surjective in any point we are reduced to the case of a finite morphism. Now the claim follows from Lemma 4.9.3 above and the Example 4.9.2. $\square$

Lemma 4.9.4. *Let $\varphi\colon X \to Y$ be a morphism, $x \in X$ and $y := \varphi(x) \in Y$. Assume that X is smooth in x and $d\varphi_x$ is surjective.*

(1) *Y is smooth in y.*
(2) *The fiber $\varphi^{-1}(y)$ is reduced and smooth in x, and $\dim_x F = \dim_x X - \dim_y Y$.*

Proof. By assumption,

$$\dim T_x F \le \dim \ker d\varphi_x = \dim T_x X - \dim T_y Y \le \dim X - \dim Y \le \dim_x F$$

which implies that we have equality everywhere. In particular, F is smooth in x and Y is smooth in y.

If we denote by $\bar{\mathfrak{m}} \subseteq \mathcal{O}(X)/\mathfrak{m}_y\mathcal{O}(X)$ the maximal ideal corresponding to $x \in F$ one easily sees that $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$ is the cokernel of the natural map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ induced by φ^*. The duality between $\mathfrak{m}_x/\mathfrak{m}_x^2$ and $T_x X$ (see Lemma 4.1.4 and Remark 4.6.2) implies that $\dim \ker d\varphi_x = \dim_{\mathbb{C}} \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$. Since $\dim \ker d\varphi_x = \dim_x F = \dim \mathcal{O}(X)_x/\mathfrak{m}_y\mathcal{O}(X)_x$ it follows that $\mathcal{O}(X)_x/\mathfrak{m}_y\mathcal{O}(X)_x$ is a domain (Proposition 4.10.5), and so F is reduced in x. $\square$

Corollary 4.9.5. *For every morphism $\varphi\colon X \to Y$ there is a dense special open set $U \subseteq X$ such that all fibers of the morphism $\varphi|_U \colon U \to Y$ are reduced and smooth.*

Proof. One easily reduces to the case where X is irreducible. Then there is a special open set $U \subseteq X$ which is smooth (Corollary 4.10.6) and such that $d\varphi_x$ is surjective for all $x \in U$ (Theorem 4.9.1). Now the claim follows from the previous Lemma 4.9.4. $\square$

Corollary 4.9.6 (Lemma of Sard). *Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ be a dominant morphism and set $S := \{x \in \mathbb{C}^n \mid d\varphi_x \text{ is not surjective}\}$. Then S is closed and $\overline{\varphi(S)}$ is a proper closed subset of $\mathbb{C}^m$. In particular, there is a dense open set $U \subseteq \mathbb{C}^m$ such that all fibers $\varphi^{-1}(y)$ for $y \in U$ are reduced and smooth of dimension $n - m$.*

Proof. If $\varphi = (f_1, \ldots, f_m)$, then $S = \{x \in \mathbb{C}^n \mid \operatorname{rk} \operatorname{Jac}(f_1, \ldots, f_m)(x) < m\}$ and so S is closed in $\mathbb{C}^n$. Moreover, the differential of $\varphi|_S \colon S \to \mathbb{C}^m$ at any point of S is not surjective. Therefore, by Theorem 4.9.1, the closure of the image $\varphi(S)$ has dimension strictly less than m. $\square$

Exercise 4.9.7. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a nonconstant polynomial. Then $\mathcal{V}(f - \lambda)$ is a smooth hypersurface for almost all $\lambda \in \mathbb{C}$.

Corollary 4.9.8. *If $\varphi\colon X \to Y$ is a morphism such that $d\varphi_x = 0$ for all $x \in X$, then the image $\varphi(X)$ is finite. In particular, if X is connected, then φ is constant.*

Proof. If $X' \subseteq X$ is an irreducible component and $Y' := \overline{\varphi(X')}$, then the induced morphism $\varphi' \colon X' \to Y'$ has the same property, namely $d\varphi'_x = 0$ for all $x \in X'$. It follows now from Theorem 4.9.1 that $\dim Y' = 0$. Hence φ is constant on X'. $\square$

Example 4.9.9. Let V be a vector space and $W \subseteq V$ a subspace. If $X \subseteq V$ is a closed irreducible subvariety such that $T_x X \subseteq W$ for all $x \in X$, then $X \subseteq x + W$ for any $x \in X$.

(This follows from the previous corollary applied to the morphism $\varphi\colon X \to V/W$ induced by the linear projection $V \to V/W$.)

4.10. Associated graded algebras. Let R be $\mathbb{C}$-algebra and $\mathfrak{a} \subseteq R$ an ideal. The *associated graded algebra* is defined in the following way. Consider the $\mathbb{C}$-vector space

$$\operatorname{gr}_{\mathfrak{a}} R := \bigoplus_{i\geq 0} \mathfrak{a}^i/\mathfrak{a}^{i+1} = R/\mathfrak{a} \oplus \mathfrak{a}/\mathfrak{a}^2 \oplus \mathfrak{a}^2/\mathfrak{a}^3 \oplus \cdots$$

and define the multiplication of (homogeneous) elements by

$$(f + \mathfrak{a}^{i+1}) \cdot (h + \mathfrak{a}^{j+1}) := fh + \mathfrak{a}^{i+j+1}$$

for $f \in \mathfrak{a}^i, h \in \mathfrak{a}^j$. It is easy to see that this defines a multiplication on $\operatorname{gr}_{\mathfrak{a}} R$. By definition, $R/\mathfrak{a}$ is a subalgebra of $\operatorname{gr}_{\mathfrak{a}} R$, and $\operatorname{gr}_{\mathfrak{a}} R$ is generated by $\mathfrak{a}/\mathfrak{a}^2$ as a $R/\mathfrak{a}$-algebra. In particular, if R is finitely generated as a $\mathbb{C}$-algebra, then so is $\operatorname{gr}_{\mathfrak{a}} R$.

We want to use this construction to give the following characterization of nonsingular points.

Theorem 4.10.1. *Let X be an affine variety. A point $x \in X$ is nonsingular if and only if the associated graded algebra $\operatorname{gr}_{\mathfrak{m}_x} \mathcal{O}(X)$ is a polynomial ring. In particular, the local ring $\mathcal{O}_{X,x}$ of a nonsingular point x is a domain and so x belongs to a unique irreducible component of X.*

Before we can give the proof we have to explain some technical results from commutative algebra. Let R be a $\mathbb{C}$-algebra and $\mathfrak{m} \subseteq R$ a maximal ideal. Consider the subalgebra $\tilde{R}$ of $R[t, t^{-1}]$ generated as an R-algebra by t and $\mathfrak{m}t^{-1}$:

$$\tilde{R} := R[t, \mathfrak{m}t^{-1}] = \cdots \oplus \mathfrak{m}^2 t^{-2} \oplus \mathfrak{m}t^{-1} \oplus R \oplus Rt \oplus Rt^2 \oplus \cdots \subseteq R[t, t^{-1}].$$

In the following lemma we collect some basic properties of this construction.

Lemma 4.10.2.
1. *If R is finitely generated, then so is $\tilde{R}$.*
2. *There is a canonical isomorphism $\tilde{R}/\tilde{R}t \xrightarrow{\sim} \operatorname{gr}_{\mathfrak{m}} R$.*
3. *If $\mathfrak{a} \subseteq \mathfrak{m}$ is an ideal and $\tilde{\mathfrak{a}} := \mathfrak{a}[t, t^{-1}] \cap \tilde{R}$, then $\tilde{R}/\tilde{\mathfrak{a}} \xrightarrow{\sim} \widetilde{R/\mathfrak{a}}$.*
4. *If $\mathfrak{n} \subseteq R$ is the nilradical, then $\tilde{\mathfrak{n}} := \mathfrak{n}[t, t^{-1}] \cap \tilde{R}$ is the nilradical of $\tilde{R}$, and $\tilde{R}/\tilde{\mathfrak{n}} \xrightarrow{\sim} \widetilde{R/\mathfrak{n}}$.*
5. *Assume that R is a finitely generated domain. Then $\tilde{R}$ is a domain, and we have*
$$\dim \tilde{R} = \dim R + 1 \quad \textit{and} \quad \dim \tilde{R}/\tilde{R}t = \dim R.$$
6. *Assume that R finitely generated and that the minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are all contained in $\mathfrak{m}$. Then the $\tilde{\mathfrak{p}}_1, \ldots, \tilde{\mathfrak{p}}_r$ are the minimal primes of $\tilde{R}$.*

Proof. (1) If $R = \mathbb{C}[h_1, \cdots, h_m]$ and $\mathfrak{m} = (f_1, \ldots, f_n)$, then

$$\tilde{R} = \mathbb{C}[h_1, \ldots, h_m, t, f_1 t^{-1}, \ldots, f_n t^{-1}],$$

and so $\tilde{R}$ is finitely generated.

(2) By definition, we have

$$\tilde{R}t = \cdots \oplus \mathfrak{m}^3 t^{-2} \oplus \mathfrak{m}^2 t^{-1} \oplus \mathfrak{m} \oplus Rt \oplus Rt^2 \oplus \cdots .$$

Hence

$$\tilde{R}/\tilde{R}t = \cdots \oplus (\mathfrak{m}^2/\mathfrak{m}^3)t^{-2} \oplus (\mathfrak{m}/\mathfrak{m}^2)t^{-1} \oplus R/\mathfrak{m}$$

and the claim follows.

(3) The canonical map $\pi\colon R[t, t^{-1}] \to (R/\mathfrak{a})[t, t^{-1}]$ induces, by our construction, a surjective homomorphism $\tilde{\pi}\colon \tilde{R} \to \widetilde{R/\mathfrak{a}}$ with kernel $\ker \pi \cap \tilde{R} = \mathfrak{a}[t, t^{-1}] \cap \tilde{R}$.

(4) Put $R_{\text{red}} := R/\mathfrak{n}$. Then $R_{\text{red}}[t,t^{-1}]$ is reduced, i.e. without nilpotent elements $\neq 0$, and so is $\widetilde{R_{\text{red}}}$. Since the kernel of the map $R[t,t^{-1}] \to R_{\text{red}}[t,t^{-1}]$ is equal to $\mathfrak{n}[t,t^{-1}]$ and consists of nilpotent elements the claim follows from (3).

(5) The first part is clear since $R[t,t^{-1}]$ is a domain. Since $\tilde{R}_t = R[t,t^{-1}]$ we get $\dim \tilde{R} = \dim R[t,t^{-1}] = \dim R[t] = \dim R + 1$. Moreover, by the Principal Ideal Theorem (Theorem 3.3.4) we have $\dim \tilde{R}/\tilde{R}t = \dim \tilde{R} - 1$.

(6) It follows from (3) and (5) that the ideals $\tilde{\mathfrak{p}}_i$ are prime. Since $\bigcap_i \mathfrak{p}_i = \mathfrak{n}$ we obtain from (2)

$$\bigcap_i \tilde{\mathfrak{p}}_i = \bigcap_i \mathfrak{p}_i[t,t^{-1}] \cap \tilde{R} = \mathfrak{n}[t,t^{-1}] \cap \tilde{R} = \tilde{\mathfrak{n}}.$$

Since $\tilde{\mathfrak{p}}_i \cap R[t] = \mathfrak{p}_i[t]$ there are no inclusions $\tilde{\mathfrak{p}}_i \subseteq \tilde{\mathfrak{p}}_j$ for $i \neq j$, and the claim follows. (We use here the well-know fact that the minimal primes in a finitely generated $\mathbb{C}$-algebra are characterized by the condition $\bigcap \mathfrak{p}_i = \mathfrak{n}$, cf. Remark 1.6.7.) □

In the next lemma we give some properties of the associated graded algebra $\operatorname{gr}_{\mathfrak{m}} R$ where $\mathfrak{m}$ is a maximal ideal of R.

Lemma 4.10.3. *Let R be a $\mathbb{C}$-algebra and $\mathfrak{m} \subseteq R$ a maximal ideal.*

(1) *Assume that $\bigcap_j \mathfrak{m}^j = (0)$. If $\operatorname{gr}_{\mathfrak{m}} R$ is a domain, then so is R.*
(2) *Denote by $\mathfrak{m}R_{\mathfrak{m}} \subseteq R_{\mathfrak{m}}$ the maximal ideal of the localization $R_{\mathfrak{m}}$. There is a natural isomorphism $\operatorname{gr}_{\mathfrak{m}} R \xrightarrow{\sim} \operatorname{gr}_{\mathfrak{m}R_{\mathfrak{m}}} R_{\mathfrak{m}}$ of graded $\mathbb{C}$-algebras.*

Proof. (1) If $ab = 0$ for nonzero elements $a, b \in R$, we can find $i, j \geq 0$ such that $a \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ and $b \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}$. Thus $\bar{a} := a + \mathfrak{m}^{i+1}$ and $\bar{b} := b + \mathfrak{m}^{j+1}$ are nonzero elements in $\operatorname{gr}_{\mathfrak{m}} A$, and $\bar{a}\bar{b} = ab + \mathfrak{m}^{i+j+1} = 0$. This contradiction proves the claim.

(2) Set $\mathfrak{M} := \mathfrak{m}R_{\mathfrak{m}} \subseteq R_{\mathfrak{m}}$. The image of $S := R \setminus \mathfrak{m}$ in $R/\mathfrak{m}^k$ consists of invertible elements and so $R/\mathfrak{m}^k \to R_{\mathfrak{m}}/\mathfrak{M}^k$ is surjective. It is also injective, because $R_{\mathfrak{m}}/\mathfrak{M}^k$ can be identified with the localization of $R/\mathfrak{m}^k$ at S. Thus $R/\mathfrak{m}^k \xrightarrow{\sim} R_{\mathfrak{m}}/\mathfrak{M}^k$ and so $\mathfrak{m}^i/\mathfrak{m}^{i+1} \xrightarrow{\sim} \mathfrak{M}^i/\mathfrak{M}^{i+1}$ for all $i \geq 0$. □

Finally, we need the following result due to Krull. It implies that in a local Noetherian $\mathbb{C}$-algebra R with maximal ideal $\mathfrak{m}$ we have $\bigcap_{j \geq 0} \mathfrak{m}^j = (0)$.

Lemma 4.10.4 (Krull). *Let R be a Noetherian $\mathbb{C}$-algebra, $\mathfrak{a} \subseteq R$ an ideal and $\mathfrak{b} := \bigcap_{j \geq 0} \mathfrak{a}^j$. Then $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$. In particular, there is an $a \in \mathfrak{a}$ such that $(1 + a)\mathfrak{b} = 0$.*

Proof. The second claim follows from the first and the Lemma of Nakayama (Lemma 3.2.5). Let $\mathfrak{a} = (a_1, \ldots, a_s)$ and put

$$I := \langle f \mid f \in R[x_1, \ldots, x_s] \text{ homogeneous and } f(a_1, \ldots, a_s) \in \mathfrak{b} \rangle \subseteq R[x_1, \ldots, x_s].$$

It is easy to see that I is an ideal of $R[x_1, \ldots, x_s]$ and so $I = (f_1, \ldots, f_k)$ where the f_j are homogeneous. Choose an $n \in \mathbb{N}$, $n > \deg f_j$ for all j. By definition, $\mathfrak{b} \subseteq \mathfrak{a}^n$ and so, for every $b \in \mathfrak{b}$, there is a homogeneous polynomial $f \in R[x_1, \cdots, x_s]$ of degree n such that $f(a_1, \ldots, a_s) = b$. It follows that $f = \sum_j h_j f_j$ where the h_j are homogeneous of degree > 0, and so $b = f(a_1, \ldots, a_s) = \sum_j h_j(a_1, \ldots, a_s) f_j(a_1, \ldots, a_s) \in \mathfrak{a}\mathfrak{b}$. □

The next proposition is a reformulation of our main Theorem 4.10.1. For later use we will prove it in this slightly more general form.

Proposition 4.10.5. *Let R be a finitely generated $\mathbb{C}$-algebra and let $\mathfrak{m} \subseteq R$ be a maximal ideal. Then $\dim \operatorname{gr}_{\mathfrak{m}} R = \dim R_{\mathfrak{m}}$. Moreover, $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim R_{\mathfrak{m}}$ if and only if $\operatorname{gr}_{\mathfrak{m}} R$ is a polynomial ring. If this holds, then $R_{\mathfrak{m}}$ is a domain.*

Proof. Inverting an element from $R \setminus \mathfrak{m}$ does not change $\operatorname{gr}_{\mathfrak{m}} R$ (Lemma 4.10.3(2)). Therefore we can assume that all minimal primes of R are contained in $\mathfrak{m}$. In particular, we have $\dim R_{\mathfrak{m}} = \dim R = \max_i \dim R/\mathfrak{p}_i$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals. Moreover, every element from $R \setminus \mathfrak{m}$ is a nonzero divisor.

Now consider the $\mathbb{C}$-algebra $\tilde{R} = R[t, \mathfrak{m}t^{-1}] \subseteq R[t, t^{-1}]$ introduced above. It follows from Lemma 4.10.2 that $\tilde{R}$ has the following two properties:

(i) $\tilde{R}/\tilde{R}t \xrightarrow{\sim} \operatorname{gr}_{\mathfrak{m}} R$, by (2).

(ii) $\dim \tilde{R}/\tilde{R}t = \dim R$, by (5) and (6).

Hence, $\dim \operatorname{gr}_{\mathfrak{m}} R = \dim R_{\mathfrak{m}}$, proving the first claim.

Assume now that $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim R_{\mathfrak{m}} =: n$. Then we obtain a surjective homomorphism

$$\rho\colon \mathbb{C}[y_1, \ldots, y_n] \to \operatorname{gr}_{\mathfrak{m}} R$$

by sending $y_1, \ldots, y_n$ to a $\mathbb{C}$-basis of $\mathfrak{m}/\mathfrak{m}^2$. But every proper residue class ring of $\mathbb{C}[y_1, \ldots, y_n]$ has dimension $< n$, and so the homomorphism ρ is an isomorphism.

On the other hand, if $\operatorname{gr}_{\mathfrak{m}} R$ is a polynomial ring, then $\dim R_{\mathfrak{m}} = \dim \operatorname{gr}_{\mathfrak{m}} R = \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2$. Moreover, $\bigcap_{j>0} \mathfrak{m}^j = (0)$ by Lemma 4.10.4, because every element from $R \setminus \mathfrak{m}$ is a nonzero divisor, and so R is a domain by Lemma 4.10.3(1). □

Corollary 4.10.6. *If X is an affine variety, then $X_{sing} \subseteq X$ is a closed subset whose complement is dense in X.*

Proof. Let $X = \bigcup_i X_i$ is the decomposition of X into irreducible components. A point $x \in X_i$ is a singular point of X if and only if it is either a singular point of X_i or it belongs to two different irreducible components. Thus

$$X_{sing} = \bigcup_i (X_i)_{sing} \cup \bigcup_{j \neq k} X_j \cap X_k,$$

and the claim follows easily. □

4.11. $\mathfrak{m}$-adic completion. Let us denote by $\hat{\mathcal{O}}_{X,x}$ the $\mathfrak{m}_x$*-adic completion* of the local ring $\mathcal{O}_{X,x}$. It is defined to be the *inverse limit*

$$\hat{\mathcal{O}}_{X,x} := \varprojlim \mathcal{O}(X)/\mathfrak{m}_x^k.$$

(We refer to [**Eis95**, I.7.1 and I.7.2] for more details and some basic properties.) Since $\bigcap \mathfrak{m}_x^k = \{0\}$ we have a natural embedding $\mathcal{O}_{X,x} \subseteq \hat{\mathcal{O}}_{X,x}$. Moreover, $\hat{\mathcal{O}}_{X,x}$ is *Noetherian*, and it is *flat* over $\mathcal{O}_{X,x}$ ([**Eis95**, Theorem 7.1 and 7.2]).

If $X = \mathbb{C}^n$ and $x = 0$, then the completion coincides with the *algebra of formal power series* in n variables:

$$\hat{\mathcal{O}}_{\mathbb{C}^n,0} = \mathbb{C}[\![x_1, \ldots, x_n]\!].$$

The next result is an easy consequence of Theorem 4.10.1 above.

Corollary 4.11.1. *The point $x \in X$ is nonsingular if and only if $\hat{\mathcal{O}}_{X,x}$ is isomorphic to the algebra of formal power series in $\dim_x X$ variables.*

Remark 4.11.2. A famous result of Auslander-Buchsbaum states that the local ring $\mathcal{O}_{X,x}$ in a nonsingular point of a variety X is always a *unique factorization domain*. For a proof we refer to [**Mat89**, §20, Theorem 20.3].

We might ask here which properties of a local ring $\mathcal{O}_{X,x}$ are carried over to the completion $\hat{\mathcal{O}}_{X,x}$. The following important result is due to Zariski. A proof can be found in [**ZS60**, Ch.VIII, §13, Theorem 32].

Proposition 4.11.3. *If $\mathcal{O}_{X,x}$ is normal, then so is $\hat{\mathcal{O}}_{X,x}$.*

As an application we have the following proposition about smoothness in the target of a morphism.

PROPOSITION 4.11.4 (ZARISKI). *Let $\varphi\colon X \to Y$ be a dominant morphism of irreducible varieties, and let $x \in X$ be a point with the properties that the fiber $F := \varphi^{-1}(\varphi(x))$ is reduced and smooth in x and that $\dim_x F = \dim X - \dim Y$. If Y is normal in y, then $y := \varphi(x)$ is a smooth point of Y.*

PROOF. Let $S \subseteq X$ be an irreducible transversal slice in $x \in X$ to the fiber F, i.e.,

(1) $\dim S = \dim Y$ and $\psi := \varphi|_S\colon S \to Y$ is dominant;
(2) x is a smooth point of S;
(3) The tangent map $d\psi_x\colon T_x S \to T_y Y$ is injective.

Then we have an inclusion of local rings $\psi^*\colon \mathcal{O}_{Y,y} \hookrightarrow \mathcal{O}_{S,x}$ of the same dimension, and $\mathfrak{m}_{S,x} = S\psi^*(\mathfrak{m}_y)$ by (3). This implies that the induced homomorphism $\hat{\psi}^*\colon \hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{S,x}$ is surjective. By (2) $\hat{\mathcal{O}}_{S,x}$ is an algebra of formal power series (Corollary 4.11.1), and by ZARISKI's proposition above the completion $\hat{\mathcal{O}}_{Y,y}$ is normal, hence an intergral domain. Since $\dim_x S = \dim_y Y$ by (1), it follows that the map $\hat{\psi}^*$ has a trivial kernel, i.e. $\hat{\mathcal{O}}_{Y,y}$ is an algebra of formal power series and so y is a smooth point of Y. □

REMARK 4.11.5. The normality assumption in the previous proposition is essential. Consider the normalization $\eta\colon \mathbb{C} \to C$ of the cusp $C := \mathcal{V}(y^2 - x^3 - x^2) \subseteq \mathbb{C}^2$. Then the fiber $\eta^{-1}(0)$ is reduced and smooth, but $0 \in C$ is a singular point.

5. Normal Varieties and Divisors

5.1. Normality.

DEFINITION 5.1.1. Let $A \subseteq B$ be rings. An element $b \in B$ is *integral over* A if b satisfies an equation of the form

$$b^n = \sum_{i=0}^{n-1} a_i b^i \quad \text{where } a_i \in A.$$

Equivalently, $b \in B$ is integral over A if and only if the subring $A[b] \subseteq B$ is a finite A-module.

If every element from B is integral over A we say that B *is integral over* A.

EXERCISE 5.1.2. Let $A \subseteq B$ be rings. If A is Noetherian and B finite over A, then B is integral over A.

LEMMA 5.1.3. *Let $A \subseteq B \subseteq C$ be rings and assume that A is Noetherian.*

(1) *If B is integral over A and C integral over B, then C is integral over A.*
(2) *The set*
$$B' := \{b \in B \mid b \text{ is integral over } A\}$$
is a subring of B.

PROOF. (1) Let $c \in C$. Then we have an equation $c^m = \sum_{j=0}^{m-1} b_j c^j$ with $b_j \in B$. In particular, the coefficients b_j are integral over A and so, by induction, $A_1 := A[b_0, b_1, \ldots, b_{m-1}]$ is a finitely generated A-module. Moreover, $A_1[c]$ is a finitely generated A_1-module, hence a finitely generated A-module. But then $A[c] \subseteq A_1[c]$ is also finitely generated.

(2) Let $b_1, b_2 \in B'$. Then $A[b_1]$ is integral over A and b_2 is integral over A, hence integral over $A[b_1]$, and so $A[b_1, b_2]$ is integral over $A[b_1]$. Thus, by (1), $A[b_1, b_2]$ is

integral over A which implies that $b_1 + b_2$ and $b_1 b_2$ are both integral over A, hence belong to B'. □

Exercise 5.1.4. Let $f \in \mathbb{C}[x]$ be a nonconstant polynomial. Then $\mathbb{C}[x]$ is integral over the subalgebra $\mathbb{C}[f]$.

Definition 5.1.5. Let A be a domain with field of fraction K. We call A *integrally closed* if the following holds:

If $x \in K$ is integral over A, then $x \in A$.

An affine variety X is *normal* if X is irreducible and $\mathcal{O}(X)$ is integrally closed. We say that X *is normal in* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is integrally closed.

Example 5.1.6. A unique factorization domain A is integrally closed. In particular, $\mathbb{C}^n$ is a normal variety.
(Let K be the field of fractions of A and $x \in K$ integral over A: $x^n = \sum_{i=0}^{n-1} a_i x^i$ where $a_i \in A$. Write $x = \frac{a}{b}$ where $a, b \in A$ have no common divisor. Then $a^n = b(\sum_{i=0}^{n-1} a_i b^{n-i-1} a^i)$ which implies that b is a unit in A and so $x \in A$.)

Exercise 5.1.7. If the domain A is integrally closed, then so is every ring of fraction A_S where $1 \in S \subseteq A$ is multiplicatively closed.

Lemma 5.1.8. *Let X be an irreducible variety. Then X is normal if and only if all local rings $\mathcal{O}_{X,x}$ are integrally closed.*

Proof. If X is normal, then $\mathcal{O}_{X,x} = \mathcal{O}(X)_{\mathfrak{m}_x}$ is integrally closed (see the Exercise above), and the reverse implication follows from $\mathcal{O}(X) = \bigcap_{x \in X} \mathcal{O}_{X,x}$ (Exercise 1.7.6). □

5.2. Integral closure and normalization.

Proposition 5.2.1. *Let A be a finitely generated $\mathbb{C}$-algebra with no zero-divisors $\neq 0$ and with field of fractions K, and let L/K be a finite field extension. Then*

$$A' := \{x \in L \mid x \text{ is integral over } A\} \supseteq A$$

is a finitely generated $\mathbb{C}$-algebra which is finite over A.

Proof. We already know that A' is a $\mathbb{C}$-algebra (Lemma 5.1.3(2)).

(a) We first assume that $A = \mathbb{C}[z_1, \dots, z_m]$ is a polynomial ring and $K = \mathbb{C}(z_1, \dots, z_m)$. Let $L = K[x]$ where x is integral over A and $[L : K] =: n$. Denote by $x_1 := x, x_2, \dots, x_n$ the conjugates of x in some Galois extension L' of K. Clearly, all x_j are integral over A, because they satisfy the same equation as x.

If $y = \sum_{i=0}^{n-1} c_i x^i$ ($c_i \in K$) is an arbitrary element of L we obtain the "conjugates" of y in L' in the form

$$y_j = \sum_{i=0}^{n-1} c_i x_j^i \quad \text{for } j = 1, \dots, n.$$

The $n \times n$-matrix $X := (x_j^i)$ has determinant $d = \prod_{j<k}(x_j - x_k)$ which is integral over A. Obviously, d^2 is symmetric, hence fixed under the Galois group of L'/K, and so $d^2 \in K$. Since d^2 is also integral over A we finally get $d^2 \in A$. From Cramer's rule we obtain

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = X^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \frac{1}{d} \operatorname{Adj}(X) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

This shows that if y is integral over A, then so is dc_i for all i, hence $d^2 c_i \in A$ for all i. This implies that $d^2 A' \subseteq \sum_{i=0}^{n-1} A x^i$, and so A' is a finitely generated A-module.

(b) For the general case we use Noether's Normalization Lemma (Theorem 3.2.12) which states that A contains a polynomial ring $A_0 = \mathbb{C}[x_1, \dots, x_m]$ such that A is finite over A_0. Thus A is integral over A_0 and therefore, by Lemma 5.1.3(1)

$$A' = \{x \in L \mid x \text{ is integral over } A_0\}.$$

It follows from part (a) that A' is a finitely generated A_0-module, hence also a finitely generated A-module. $\square$

Definition 5.2.2. Let A be a finitely generated $\mathbb{C}$-algebra with no zero-divisors $\neq 0$. If L is a finite field extension of the field of fractions of A, then

$$A' := \{x \in L \mid x \text{ is integral over } A\} \supseteq A$$

is called the *integral closure of A in L*. Clearly, A' is integrally closed.

Let X be an irreducible affine variety and denote by $\mathcal{O}(X)' \subseteq \mathbb{C}(X)$ the integral closure of $\mathcal{O}(X)$ in its field of fractions $\mathbb{C}(X)$. By Proposition 5.2.1 there is a normal variety $\tilde{X}$ and a finite birational morphism $\eta\colon \tilde{X} \to X$ such that $\mathcal{O}(\tilde{X}) \simeq \mathcal{O}(X)'$. More precisely, we have the following result.

Lemma 5.2.3. *Let X be an irreducible variety and $\eta\colon \tilde{X} \to X$ a morphism with the following two properties:*

(1) *$\tilde{X}$ is normal;*
(2) *η is finite and birational.*

Then $\mathcal{O}(\tilde{X})$ is the integral closure of $\eta^*(\mathcal{O}(X))$ in $\mathbb{C}(\tilde{X}) = \eta^*(\mathbb{C}(X))$, and we have the following universal property:

> *If Y is a normal affine variety, then every dominant morphism $\varphi\colon Y \to X$ factors through η: There is a uniquely determined $\tilde{\varphi}\colon Y \to \tilde{X}$ such that $\varphi = \eta \circ \tilde{\varphi}$:*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\eta"] \\ Y \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & X \end{tikzcd}$$

Proof. Since η is birational we have $\eta^*(\mathcal{O}(X)) \subseteq \mathcal{O}(\tilde{X}) \subseteq \mathbb{C}(\tilde{X}) = \eta^*(\mathbb{C}(X))$. By (2) $\mathcal{O}(\tilde{X})$ is finite, hence integral over $\eta^*(\mathcal{O}(X))$, and by (1) it is the integral closure of $\eta^*(\mathcal{O}(X))$.

If Y is normal affine variety and $\varphi\colon Y \to X$ a dominant morphism, then

$$\mathcal{O}(X) \xrightarrow{\sim} \varphi^*(\mathcal{O}(X)) \subseteq \mathcal{O}(Y) \subseteq \mathbb{C}(Y).$$

Denote by $\mathcal{O}(X)'$ the integral closure of $\mathcal{O}(X)$ in $\mathbb{C}(X)$. Since $\mathcal{O}(Y)$ is integrally closed it follows that $\varphi^*(\mathcal{O}(X)') \subseteq \mathbb{C}(Y)$ is contained in $\mathcal{O}(Y)$. Since η^* induces an isomorphism $\mathcal{O}(X)' \xrightarrow{\sim} \mathcal{O}(\tilde{X})$ there is a uniquely determined homomorphism $\rho\colon \mathcal{O}(\tilde{X}) \to \mathcal{O}(Y)$ which makes the following diagram commutative:

$$\begin{tikzcd} & \mathcal{O}(\tilde{X}) \arrow[ddl, "\rho"'] \\ & \mathcal{O}(X)' \arrow[u, "\simeq"'] \arrow[dl] \\ \mathcal{O}(Y) & \mathcal{O}(X) \arrow[l, "\varphi^*"'] \arrow[u, hook, "\subseteq"'] \arrow[uu, bend right, "\eta^*"'] \end{tikzcd}$$

Clearly, the corresponding morphism $\tilde{\varphi}\colon Y \to \tilde{X}$ is the unique morphism such that $\varphi = \eta \circ \tilde{\varphi}$. $\square$

DEFINITION 5.2.4. The morphism $\eta\colon \tilde{X} \to X$ constructed above is called *normalization of* X. It follows from Lemma 5.2.3 that it is unique up to a uniquely determined isomorphism.

EXERCISE 5.2.5. If $\varphi\colon X \to Y$ is a finite surjective morphism where X is irreducible and Y is normal, then $\#\varphi^{-1}(y) \leq \deg\varphi$ for all $y \in Y$. (See Proposition 3.6.1 and its proof.)

PROPOSITION 5.2.6. *Let X be an irreducible variety. Then the set*

$$X_{norm} := \{x \in X \mid X \text{ is normal in } x\}$$

is open and dense in X.

PROOF. Let $\mathcal{O}(X)' \subseteq \mathbb{C}(X)$ be the integral closure of $\mathcal{O}(X)$ and define

$$\mathfrak{a} := \{f \in \mathcal{O}(X) \mid f\mathcal{O}(X)' \subseteq \mathcal{O}(X)\}.$$

Then $\mathfrak{a}$ is a nonzero ideal of $\mathcal{O}(X)$, because $\mathcal{O}(X)'$ is finite over $\mathcal{O}(X)$, and $X_{\text{norm}} = X \setminus \mathcal{V}_X(\mathfrak{a})$. In fact, for $S := \mathcal{O}(X) \setminus \mathfrak{m}_x$ we have

$$\mathcal{O}_{X,x} = \mathcal{O}(X)_S \subseteq \mathcal{O}(X)'_S$$

and the latter is the integral closure of $\mathcal{O}_{X,x}$. On the other hand, $\mathcal{O}(X)_S = \mathcal{O}(X)'_S$ if and only if $S \cap \mathfrak{a} \neq \emptyset$ which is equivalent to $x \notin \mathcal{V}_X(\mathfrak{a})$. □

EXERCISE 5.2.7. Consider the morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^4$, $(x, y) \mapsto (x, xy, y^2, y^3)$.

(1) φ is finite and $\varphi\colon \mathbb{C}^2 \to Y := \varphi(\mathbb{C}^2)$ is the normalization.
(2) $0 \in Y$ is the only non-normal and the only singular point of Y.
(3) Find defining equations for $Y \subseteq \mathbb{C}^4$ and generators of the ideal $I(Y)$.

EXERCISE 5.2.8. If X is a normal variety, then so is $X \times \mathbb{C}^n$.

New part from 4.2.2015:

We know that for a dominant morphism $\varphi\colon X \to Y$ of finite degree d there is an open dense set $U \subseteq Y$ such that every fiber $\varphi^{-1}(y)$, $y \in U$, has exactly d points (Proposition 3.6.1). Under stronger assumptions this can be improved.

PROPOSITION 5.2.9. *Let $\varphi\colon X \to Y$ be a finite surjective morphism where X, Y are irreducible and Y is normal. Then $|\varphi^{-1}(y)| \leq \deg\varphi$ for all $y \in Y$. Moreover, the set*

$$\{y \in Y \mid |\varphi^{-1}(y)| = \deg\varphi\} \subseteq Y$$

is open and dense in Y.

PROOF. (a) Let $\varphi^{-1}(y_0) = \{x_1, \ldots, x_k\}$. Choose an $f \in \mathcal{O}(X)$ such that $f(x_i) \neq f(x_j)$ for $i \neq j$. Let $F = t^m + h_1 t^{m-1} + \cdots + h_m$ be the minimal equation of f over $\mathbb{C}(Y)$. Then $m \leq \deg\varphi$, and the coefficients h_i belong to $\mathcal{O}(Y)$ since they are integral over $\mathcal{O}(Y)$. It follows that $f(x_1), \ldots, f(x_k)$ are distinct roots of the polynomial $F(y_0, t)$, hence $k \leq m \leq \deg\varphi$, proving the first claim.

(b) Now assume that the fiber of y_0 has $d := \deg\varphi$ points. We know that such points exist, see Proposition 3.6.1. With the notation above we see that $F(y_0, t)$ has degree d and that $f(x_1), \ldots, f(x_d)$ are the d distinct roots of $F(y_0, t)$. In particular, the discriminant of F does not vanish in y_0, hence there is an open neighborhood U of y_0 such that $F(y, t)$ has d distinct roots for all $y \in U$. We will show that $|\varphi^{-1}(y)| = d$ for $y \in U$ which proves the second claim.

Consider the finite morphism $\varphi \times f\colon X \to Y \times \mathbb{C}$, and denote by $X' \subseteq Y \times \mathbb{C}$ its image. We have inclusions $\mathcal{O}(Y) \subseteq \mathcal{O}(X') \subseteq \mathcal{O}(X)$. Since f belongs to $\mathcal{O}(X')$ and has a minimal equation of degree d over $\mathbb{C}(Y)$ we get $\mathbb{C}(X') = \mathbb{C}(X)$, i.e. the induced morphism $\varphi'\colon X \to X'$ is birational. Moreover, $X' \subseteq \mathcal{V}_{Y\times\mathbb{C}}(F) \subseteq Y \times \mathbb{C}$, hence coincides with an irreducible component of the hypersurface $Z := \mathcal{V}_{Y\times\mathbb{C}}(F)$, because Z has codimension 1, by KRULL's Theorem 3.3.4.

We claim that Z is irreducible. Let $Z = Z_1 \cup \cdots \cup Z_k$ be the decomposition into irreducible components where $Z_1 = X'$. By KRULL's Theorem 3.3.4, all Z_i have the same dimension, namely $\dim Y$. Since $p := \mathrm{pr}_Y|_Z : Z \to Y$ is finite, we get $p(Z_i) = Y$ for all i. Moreover, $p^{-1}(y) = \{(y,a) \mid F(y,a) = 0\}$, hence $|p^{-1}(y)| \le d$ for all $y \in Y$. On the other hand, $p' := p|_{X'} : X' = Z_1 \to Y$ has degree d, and so there is a dense open set $U' \subseteq Y$ such that $|p'^{-1}(y)| = d$ for all $y \in U'$ (Proposition 3.6.1). Therefore, $p^{-1}(U') \subseteq Z_1$, hence $Z = Z_1$, because $p^{-1}(U')$ is dense in Z.

As a consequence, we obtain a factorization

$$\varphi\colon X \xrightarrow{\varphi'} Z = \mathcal{V}_{Y\times\mathbb{C}}(F) \xrightarrow{p} Y$$

where both maps φ' and p are finite and surjective. Since $|p^{-1}(y)| = d$ for $y \in U$, we get $|\varphi^{-1}(y)| \ge d$ for $y \in U$, hence $|\varphi^{-1}(y)| = d$ by (a), and the claim follows. $\square$

(end of new part)

5.3. Analytic normality. We might ask which properties of a local ring $\mathcal{O}_{X,x}$ are carried over to the completion $\hat{\mathcal{O}}_{X,x}$. The following important result is due to Zariski. A proof can be found in [**ZS60**, Ch.VIII, §13, Theorem 32].

Proposition 5.3.1. *If $\mathcal{O}_{X,x}$ is normal, then so is $\hat{\mathcal{O}}_{X,x}$.*

In general, a local ring is called *analytically normal* if the completion $\hat{R}$ is normal. It is not true that every normal local ring is analytically normal Nagata example

As an application we have the following proposition about smoothness in the target of a morphism.

Proposition 5.3.2 (Zariski). *Let $\varphi\colon X \to Y$ be a dominant morphism of irreducible varieties, and let $x \in X$ be a point with the properties that the fiber $F := \varphi^{-1}(\varphi(x))$ is reduced and smooth in x and that $\dim_x F = \dim X - \dim Y$. If Y is normal in y, then $y := \varphi(x)$ is a smooth point of Y.*

Proof. Let $S \subseteq X$ be an irreducible transversal slice in $x \in X$ to the fiber F, i.e.,

(1) $\dim S = \dim Y$ and $\psi := \varphi|_S : S \to Y$ is dominant;
(2) x is a smooth point of S;
(3) The tangent map $d\psi_x : T_x S \to T_y Y$ is injective.

Then we have an inclusion of local rings $\psi^*\colon \mathcal{O}_{Y,y} \hookrightarrow \mathcal{O}_{S,x}$ of the same dimension, and $\mathfrak{m}_{S,x} = S\psi^*(\mathfrak{m}_y)$ by (3). This implies that the induced homomorphism $\hat{\psi}^*\colon \hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{S,x}$ is surjective. By (2) $\hat{\mathcal{O}}_{S,x}$ is an algebra of formal power series (Corollary 4.11.1), and by Zariski's proposition above the completion $\hat{\mathcal{O}}_{Y,y}$ is normal, hence an intergral domain. Since $\dim_x S = \dim_y Y$ by (1), it follows that the map $\hat{\psi}^*$ has a trivial kernel, i.e. $\hat{\mathcal{O}}_{Y,y}$ is an algebra of formal power series and so y is a smooth point of Y. $\square$

Remark 5.3.3. The normality assumption in the previous proposition is essential. Consider the normalization $\eta\colon \mathbb{C} \to C$ of the cusp $C := \mathcal{V}(y^2 - x^3 - x^2) \subseteq \mathbb{C}^2$. Then the fiber $\eta^{-1}(0)$ is reduced and smooth, but $0 \in C$ is a singular point.

5.4. Discrete valuation rings and smoothness. Let K be a field.

Definition 5.4.1. A *discrete valuation of the field K* is a surjective map $\nu\colon K^* := K \setminus \{0\} \to \mathbb{Z}$ with the following properties:

(a) $\nu(xy) = \nu(x) + \nu(y)$;
(b) $\nu(x+y) \ge \min(\nu(x), \nu(y))$.

To simplify the notation one usually defines $\nu(0) := \infty$.

EXAMPLE 5.4.2. Let $K = \mathbb{Q}$ and $p \in \mathbb{N}$ a prime number. Define $\nu_p(x) := r \in \mathbb{Z}$ if p occurs with exponent r in the rational number $x \neq 0$. Then $\nu_p \colon \mathbb{Q}^* \to \mathbb{Z}$ is a discrete valuation of $\mathbb{Q}$.

The following lemma collects some facts about discrete valuations. The easy proofs are left to the reader.

LEMMA 5.4.3. *Let K be a field and $\nu \colon K^* \to \mathbb{Z}$ a discrete valuation.*

(1) $A := \{x \in K \mid \nu(x) \geq 0\}$ *is a subring of* K.
(2) $\mathfrak{m} := \{x \in K \mid \nu(x) > 0\} \subseteq A$ *is a maximal ideal of* A.
(3) $\{x \in K \mid \nu(x) = 0\}$ *are the units of* A.
(4) *For every nonzero* $x \in K$ *we have* $x \in A$ *or* $x^{-1} \in A$.
(5) $\mathfrak{m} = (x)$ *for every* $x \in K$ *with* $\nu(x) = 1$.
(6) $\mathfrak{m}^k = \{x \in K \mid \nu(x) \geq k\}$ *and these are all nonzero ideals of* A.
(7) *If* $\mathfrak{m} = (x)$, *then every* $z \in K$ *has a unique expression of the form* $z = tx^k$ *where* $k \in \mathbb{Z}$ *and* t *is a unit of* A.

DEFINITION 5.4.4. A domain A is called a *discrete valuation ring*, shortly *DVR*, if there is a discrete valuation ν of its field of fractions K such that $A = \{x \in K \mid \nu(x) \geq 0\}$. In particular, A has all the properties listed in Lemma 5.4.3 above. Clearly, ν is uniquely determined by A.

EXERCISE 5.4.5. Let A be a discrete valuation ring with field of fraction K. If $B \subseteq K$ is a subring containing A, then either $B = A$ or $B = K$.

In the sequel we will use the following characterization of a discrete valuation rings (see [**AM69**, Proposition 9.2]).

PROPOSITION 5.4.6. *Let A be a Noetherian local domain of dimension 1, i.e. the maximal ideal $\mathfrak{m} \neq (0)$ and (0) are the only prime ideals in A. Then the following statements are equivalent:*

(i) *A is a discrete valuation ring.*
(ii) *A is integrally closed.*
(iii) *The maximal ideal $\mathfrak{m}$ is principal.*
(iv) $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$.
(v) *Every nonzero ideal of A is a power of $\mathfrak{m}$.*
(vi) *There is an $x \in A$ such that every nonzero ideal of A is of the form (x^k).*

PROOF. (i)⇒(ii): If $x \in K$ and $x \notin A$, then $A[x] = K$ which is not finite over A.

(ii)⇒(iii): Let $a \in \mathfrak{m}$, $a \neq 0$. Then $\mathfrak{m}^k \subseteq (a)$ and $\mathfrak{m}^{k-1} \not\subseteq (a)$ for some $k > 0$. Choose an element $b \in \mathfrak{m}^{k-1} \setminus (a)$ and put $x := \frac{a}{b}$. Then $x^{-1}\mathfrak{m} = \frac{1}{a} b\mathfrak{m} \subseteq \frac{1}{a}\mathfrak{m}^k \subseteq A$. If $x^{-1}\mathfrak{m} \subseteq \mathfrak{m}$, then x^{-1} would be integral over A and so $x^{-1} \in A$, contradicting the construction. Thus $x^{-1}\mathfrak{m} = A$ and so $\mathfrak{m} = (x)$.

(iii)⇒(iv): If $\mathfrak{m} = (x)$, then $\mathfrak{m}/\mathfrak{m}^2 = A/\mathfrak{m} \cdot (x + \mathfrak{m}^2)$, and $\mathfrak{m}^2 \neq \mathfrak{m}$.

(iv)⇒(v): Let $\mathfrak{a} \subseteq A$ be a nonzero ideal. Then $\sqrt{\mathfrak{a}} = \mathfrak{m}$ and so $\mathfrak{m}^k \subseteq \mathfrak{a}$ for some $k \in \mathbb{N}$. Put $\bar{A} := A/\mathfrak{m}^k$ and denote by $\bar{\mathfrak{m}} \subseteq \bar{A}$ the image of $\mathfrak{m}$. Since $\mathfrak{m} = (x) + \mathfrak{m}^2$ we get $\mathfrak{m} = (x) + \mathfrak{m}^k$ for all $k \in \mathbb{N}$ and so $\bar{\mathfrak{m}} = (\bar{x}) \subseteq \bar{A}$. Now it is easy to see that $\bar{\mathfrak{a}} = \bar{\mathfrak{m}}^r$ for some $r \leq k$, and so $\mathfrak{a} = \mathfrak{m}^r$.

(v)⇒(vi): We have $\mathfrak{m} \neq \mathfrak{m}^2$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then, by assumption, $(x) = \mathfrak{m}^k$ for some $k \geq 1$, and so $\mathfrak{m} = (x)$.

(vi)⇒(i): By assumption, every element $a \in A$ has a unique expression of the form $a = tx^k$ where $k \in \mathbb{N}$ and t a unit of A. Define $\nu(a) := k$. This has a well-defined extension to K^* by setting $\nu(\frac{a}{b}) := \nu(a) - \nu(b)$ for $a, b \in A$, $b \neq 0$. One easily verifies that ν is a discrete valuation of K and that A is the corresponding valuation ring. $\square$

Let X be an irreducible variety and $H \subseteq X$ an irreducible hypersurface, i.e. $\operatorname{codim}_X H = 1$. The ideal $\mathfrak{p} := I(H)$ of H is a minimal prime ideal $\neq (0)$ and thus the localization $\mathcal{O}_{X,H} := \mathcal{O}(X)_{\mathfrak{p}}$ is a local Noetherian domain of dimension 1. If X is normal it follows from Proposition 5.4.6 that $\mathcal{O}_{X,H}$ is a discrete valuation ring which corresponds to a discrete valuation $\nu_H \colon \mathbb{C}(X)^* \to \mathbb{Z}$. In this case, ν_H vanishes on the nonzero constants, i.e. *ν_H is a discrete valuation of $\mathbb{C}(X)/\mathbb{C}$.*

Example 5.4.7. If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a nonconstant irreducible polynomial and $H := \mathcal{V}(f)$, then the valuation ν_H has the following description: *For a rational function $r \in \mathbb{C}(x_1, \ldots, x_n)$ we have $\nu_H(r) = m$ if f occurs with exponent m in a primary decomposition of r.*

Exercise 5.4.8. Let K/k be a finitely generated field extension, and let $A \subseteq K$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$, field of fraction K and containing k. Then $\operatorname{tdeg}_k A/\mathfrak{m} < \operatorname{tdeg}_k K$.
(Hint: If $\operatorname{tdeg}_k R/\mathfrak{m} = \operatorname{tdeg}_k K$, then R contains a field L with $\operatorname{tdeg}_k L = \operatorname{tdeg}_k K$. This implies that K is a finitely generated R-module which is impossible.)

5.5. The case of curves. If Y is an irreducible curve, then the local rings $\mathcal{O}_{Y,y} = \mathcal{O}(Y)_{\mathfrak{m}_y}$ satisfy the assumptions of the proposition above. The equivalence of (i), (ii) and (iv) then gives the following result. (In fact, we do not need to assume that Y is irreducible; cf. Theorem 4.10.1.)

Proposition 5.5.1. *Let Y be an affine variety and $y \in Y$ such that $\dim_y Y = 1$. Then the following statements are equivalent:*

(i) *The local ring $\mathcal{O}_{Y,y}$ is a discrete valuation ring.*
(ii) *Y is normal in y.*
(iii) *Y is smooth in y.*

In particular, a normal curve is smooth and an irreducible smooth curve is normal.

Now assume that C is a normal curve. Every point $c \in C$ determines a discrete valuation ν_c of the field of rational functions $\mathbb{C}(C)$, with corresponding DVR the local ring $A_c := \mathcal{O}_{C,c}$. Clearly, A_c contains the constants $\mathbb{C}$, and the point $c \in C$ is determined by A_c. Moreover, $\mathcal{O}(C) = \bigcap_{c\in Y} A_c$. On the other hand, if ν is a discrete valuation such that the corresponding DVR A contains $\mathcal{O}(C)$, then $\nu = \nu_c$ for a suitable point $c \in C$. (In fact, $A/\mathfrak{m} = \mathbb{C}$ (Exercise 5.4.8) and so $\mathfrak{m} \cap \mathcal{O}(C)$ is a maximal ideal $\mathfrak{m}_c$. It follows that $\mathcal{O}_{C,c} \subseteq A$, hence they are equal, by Exercise 5.4.5).

As a consequence, we get the following special case of Zariski's Main Theorem from section 5.6 below.

Proposition 5.5.2. *Let $\varphi \colon C \to D$ be a birational morphism of irreducible affine curves where D is normal. Then φ is an open immersion.*

Proof. (a) Let us first assume that φ is surjective and C is normal. Identifying $\mathbb{C}(D)$ with $\mathbb{C}(C)$ via φ^* we get $\mathcal{O}(D) \subseteq \mathcal{O}(C)$. For $c \in C$ and $d := \varphi(c) \in D$ we get $\mathcal{O}_{C,c} \subseteq \mathcal{O}_{D,d}$, hence $\mathcal{O}_{C,c} = \mathcal{O}_{D,d}$, by Exercise 5.4.5. Therefore, c is uniquely determined by d, and so φ is bijective. It follows that

$$\mathcal{O}(D) = \bigcap_{d\in D} \mathcal{O}_{D,d} = \bigcap_{c\in C} \mathcal{O}_{C,c} = \mathcal{O}(C),$$

i.e. φ is an isomorphism.

(b) In general, the image $\varphi(C) \subseteq D$ is open. Choose a special open set $C' \subseteq \varphi(C)$ and consider the morphism $\varphi' \colon D' \to C'$ where $D' \to \varphi^{-1}(C')$ is the normalization. Hence, by (a), φ' is an isomorphism, and the claim follows. $\square$

Let us describe now the discrete valuations of the field $\mathbb{C}(x)$ of rational functions on the affine line $\mathbb{C}$. For $a \in \mathbb{C}$ we get $\nu_a(f) := \operatorname{ord}_{(x-a)} f$, the order of the factor

$(x-a)$ in f, and the corresponding DVR is $A_a := \mathbb{C}[x]_{(x-a)}$. In addition, there is the discrete valuation $\nu_\infty\colon \mathbb{C}(x)^* \to \mathbb{Z}$ where $\nu_\infty(f) = -\deg f$, with corresponding DVR $A_\infty := \mathbb{C}[x^{-1}]_{(x^{-1})}$.

LEMMA 5.5.3. *The set of discrete valuations ν of the field $\mathbb{C}(x)$ which vanish on the nonzero constants $\mathbb{C}\setminus\{0\}$ is given by $\{\nu_a \mid a \in \mathbb{C}\cup\{\infty\}\}$. In particular, $\bigcap_\nu A_\nu = \mathbb{C}$.*

PROOF. Let $\nu\colon \mathbb{C}(x)^* \to \mathbb{Z}$ be a discrete valuation with valuation ring $A \supseteq \mathbb{C}$ and maximal ideal $\mathfrak{m} \subseteq A$.

(a) If $\nu(x) \geq 0$, then $\mathbb{C}[x] \subseteq A$ and $\mathfrak{m}\cap\mathbb{C}[x]$ is a maximal ideal of $\mathbb{C}[x]$, because $A/\mathfrak{m} = \mathbb{C}$ (Exercise 5.4.8). Thus $\mathfrak{m}\cap\mathbb{C}[x] = \mathfrak{m}_a$ for some $a \in \mathbb{C}$ and so $A_a \subseteq A$. This implies that $A = A_a$ (Exercise 5.4.5), hence $\nu = \nu_a$.

(b) If $\nu(x) < 0$, then, setting $y := x^{-1}$, we get $\nu(y) > 0$, hence $A = \mathbb{C}[y]_{(y)} = A_\infty$, by (a).

(c) The last statement is clear: $\bigcap_\nu A_\nu = \mathbb{C}[x]\cap\mathbb{C}[x^{-1}] = \mathbb{C}$. □

As a consequence, we can classify the smooth rational curves.

PROPOSITION 5.5.4. *Let C be a smooth rational curve. Then C is isomorphic to $\mathbb{C}\setminus F$ where $F \subseteq \mathbb{C}$ is a finite set.*

PROOF. By assumption, we have $\mathbb{C}(C) = \mathbb{C}(x)$. Denote by Ω the set of discrete valuations of $\mathbb{C}(x)$ corresponding to points of C. Since $\bigcap_{a\in\mathbb{C}} A_a = \mathcal{O}(C)$ it follows from Lemma 5.5.3 at least one discrete valuation ν_a does not belong to Ω.

If $\nu_\infty \notin \Omega$, then $\mathcal{O}(C) = \bigcap_{\nu\in\Omega} A_\nu \supseteq \bigcap_{a\in\mathbb{C}} A_a = \mathbb{C}[x]$. Thus we get a rational map $C \to \mathbb{C}$ which is an open immersion by Proposition 5.5.2.

If $\nu_a \notin \Omega$ for some $a \in \mathbb{C}$, then $y := \frac{1}{x-a} \in A_b$ for all $b \neq a$, hence $\mathbb{C}[y] \subseteq \bigcap_{b\in\Omega} A_b = \mathcal{O}(C)$, and the claim follows as above. □

EXAMPLE 5.5.5. Let C be a normal curve, and assume that there is a dominant morphism $\varphi\colon \mathbb{C}^n \to C$. Then $C \simeq \mathbb{C}$. In fact, C is a rational curve by LÜROTH's Theorem (see Proposition 2.4.1), hence $C \xrightarrow{\sim} \mathbb{C}\setminus F$. But every invertible function on C defines an invertible function on $\mathbb{C}^n$, and so F is empty.

5.6. Zariski's Main Theorem. We start with the following generalization of the previous result saying that normal curves are smooth (Proposition 5.5.1). Recall that the singular points X_{sing} of an affine variety form a closed subset with a dense complement (Proposition 4.10.6).

PROPOSITION 5.6.1. *Let X be a normal affine variety. Then* $\operatorname{codim}_X X_{\text{sing}} \geq 2$.

PROOF. (a) Let $H \subseteq X$ be an irreducible hypersurface and assume that $I(H) = (f)$. We claim that if $x \in H$ is a singular point of X, then x is a singular point of H, too. In fact, $\mathcal{O}(H) = \mathcal{O}(X)/(f)$ and $\mathfrak{m}_{H,x} = \mathfrak{m}_x/f\mathcal{O}(X)$. Thus $\mathfrak{m}_{H,x}/\mathfrak{m}_{H,x}^2 = (\mathfrak{m}_x/\mathfrak{m}_x^2)/\mathbb{C}\cdot\bar{f}$ and so $\dim T_xH \geq \dim T_xX - 1 > \dim X - 1 = \dim H$.

(b) Now assume that $\operatorname{codim}_X X_{sing} = 1$, and let $H \subseteq X_{sing}$ be an irreducible hypersurface of X. If $\mathfrak{p} := I(H)$ is a principal ideal it follows from (a) that H consists of singular points. But this contradicts the fact that the smooth points of an irreducible variety form a dense open set.

In general, the localization $\mathcal{O}_{X,H}$ is a discrete valuation ring and therefore its maximal ideal $\mathfrak{p}\mathcal{O}_{X,H}$ is principal (Proposition 5.4.6). This implies that we can find an element $s \in \mathcal{O}(X)\setminus\mathfrak{p}$ such that the ideal $\mathfrak{p}\mathcal{O}(X)_s \subseteq \mathcal{O}(X)_s = \mathcal{O}(X_s)$ is principal. Since $\mathfrak{p}\mathcal{O}(X)_s = I(H\cap X_s)$ we arrive again at a contradiction, namely that all points of $H\cap X_s$ are singular. □

Another important property of normal varieties is that regular functions can be extended over closed subset of codimension ≥ 2.

PROPOSITION 5.6.2. *Let X be a normal affine variety and $f \in \mathbb{C}(X)$ a rational function which is defined on an open set $U \subseteq X$. If $\operatorname{codim}_X X \setminus U \geq 2$, then f is a regular function on X.*

PROOF. Define the "ideal of denominators" $\mathfrak{a} := \{q \in \mathcal{O}(X) \mid q \cdot f \in \mathcal{O}(X)\}$. By definition $U \subseteq V \setminus \mathcal{V}_X(\mathfrak{a})$ and so, by assumption, $\operatorname{codim}_X \mathcal{V}_X(\mathfrak{a}) \geq 2$.

Using NOETHER'S Normalization Lemma (Theorem 3.2.12) we can find a finite surjective morphism $\varphi\colon X \to \mathbb{C}^n$. We have $\varphi(\mathcal{V}_X(\mathfrak{a})) = \mathcal{V}(\mathfrak{a} \cap \mathbb{C}[x_1, \ldots, x_n])$ and $\dim \varphi(\mathcal{V}_X(\mathfrak{a})) = \dim \mathcal{V}(\mathfrak{a} \cap \mathbb{C}[x_1, \ldots, x_n]) \leq n-2$. This implies that we can find two polynomials $q_1, q_2 \in \mathfrak{a} \cap \mathbb{C}[x_1, \ldots, x_n]$ with no common divisor (see the following Exercise 5.6.3). As a consequence, we have $f = \dfrac{p_1}{q_1} = \dfrac{p_2}{q_2}$ for suitable $p_1, p_2 \in \mathcal{O}(X)$.

If $f^{(1)} := f, f^{(2)}, \ldots, f^{(d)}$ are the conjugates of f in some finite field extension $L/\mathbb{C}(x_1, \ldots, x_n)$ of degree d containing $\mathbb{C}(X)$ we have

$$f^{(i)} = \frac{p_1^i}{q_1} = \frac{p_2^i}{q_2} \quad \text{for } i = 1, \ldots, d$$

where the $p_1^{(i)}$ are the conjugates of p_1 and the $p_2^{(i)}$ the conjugates of p_2. The element $f \in \mathbb{C}(X)$ satisfies the equation

$$\prod_{i=1}^{d}(t - f^{(i)}) = t^d + \sum_{j=1}^{d} b_j t^{n-j} = 0$$

where the coefficients $b_j \in \mathbb{C}(x_1, \ldots, x_n)$ are given by the elementary symmetric functions σ_j in the following form:

$$b_j = \pm\sigma_j(f^{(1)}, \ldots, f^{(d)}) = \pm\frac{1}{q_1^j}\sigma_j(p_1^{(1)}, \ldots, p_1^{(d)}) = \pm\frac{1}{q_2^j}\sigma_j(p_2^{(1)}, \ldots, p_2^{(d)}).$$

Since $p_1, p_2 \in \mathcal{O}(X)$ are integral over $\mathbb{C}[x_1, \ldots, x_n]$ we see that both $\sigma_j(p_1^{(1)}, \ldots, p_1^{(d)})$ and $\sigma_j(p_2^{(1)}, \ldots, p_2^{(d)})$ belong to $\mathbb{C}[x_1, \ldots, x_n]$. Since q_1 and q_2 have no common factor this implies that $b_j \in \mathbb{C}[x_1, \ldots, x_n]$. As a consequence, f is integral over $\mathbb{C}[x_1, \ldots, x_n]$ and thus belongs to $\mathcal{O}(X)$. $\square$

EXERCISE 5.6.3. Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal with the property that any two elements $f_1, f_2 \in \mathfrak{a}$ have a non-trivial common divisor. Then there is a nonconstant h which divides every element of $\mathfrak{a}$.

COROLLARY 5.6.4. *If X is a normal variety, then $\mathcal{O}(X) = \bigcap_{\mathfrak{p}} \mathcal{O}(X)_{\mathfrak{p}}$ where $\mathfrak{p}$ runs through the minimal prime ideals $\neq (0)$.*

PROOF. Let $r \in \bigcap_{\mathfrak{p}} \mathcal{O}(X)_{\mathfrak{p}}$ and define $\mathfrak{a} := \{q \in \mathcal{O}(X) \mid q \cdot r \in \mathcal{O}(X)\}$. It follows that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for all minimal primes $\mathfrak{p} \neq 0$, and so $\mathcal{V}_X(\mathfrak{a})$ does not contain an irreducible hypersurface. This implies that $\operatorname{codim}_X \mathcal{V}_X(\mathfrak{a}) \geq 2$ and so r is regular by the Proposition 5.6.2 above. $\square$

We thus have the following characterization of normal varieties. *An irreducible variety X is normal if and only if the following two condition hold:*

(a) *For every minimal prime $\mathfrak{p} \neq (0)$ the local ring $\mathcal{O}(X)_{\mathfrak{p}}$ is a discrete valuation ring;*
(b) *$\mathcal{O}(X) = \bigcap_{\mathfrak{p}} \mathcal{O}(X)_{\mathfrak{p}}$ where $\mathfrak{p}$ runs through the minimal prime ideals $\neq (0)$.*

We have seen in examples that there are bijective morphisms which are not isomorphisms. This cannot happen if the target variety is normal, as the following result due to IGUSA shows (cf. [**Igu73**, Lemma 4, page 379]).

LEMMA 5.6.5 (IGUSA's Lemma). *Let X be an irreducible and Y a normal affine variety and let $\varphi\colon X \to Y$ be a dominant morphism. Assume*

(a) $\operatorname{codim}_Y \overline{Y \setminus \varphi(X)} \geq 2$, *and*
(b) $\deg \varphi = 1$.

Then φ is an isomorphism.

PROOF. By assumption (b), we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}(Y) \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & \mathcal{O}(X) \arrow[d, "\subseteq"'] \\ \mathbb{C}(Y) \arrow[r, equal] & \mathbb{C}(X) \end{tikzcd}$$

If $H \subseteq Y$ is an irreducible hypersurface, then, by assumption (a), H meets the image $\varphi(X)$ in a dense set and so $\overline{\varphi(\varphi^{-1}(H))} = H$. This implies that there is an irreducible hypersurface $H' \subseteq X$ such that $\overline{\varphi(H')} = H$. If we denote by $\mathfrak{p} := I(H) \subseteq \mathcal{O}(Y)$ and $\mathfrak{p}' := I(H') \subseteq \mathcal{O}(X)$ the corresponding minimal prime ideals we get $\mathfrak{p}' \cap \mathcal{O}(Y) = \mathfrak{p}$. Thus

$$\mathcal{O}(Y)_{\mathfrak{p}} \subseteq \mathcal{O}(X)_{\mathfrak{p}'} \subsetneqq \mathbb{C}(Y) = \mathbb{C}(X).$$

Since $\mathcal{O}(Y)_{\mathfrak{p}}$ is a discrete valuation ring this implies $\mathcal{O}(Y)_{\mathfrak{p}} = \mathcal{O}(X)_{\mathfrak{p}'}$ (see Exercise 5.4.5). Thus, by Corollary 5.6.4,

$$\mathcal{O}(X) \subseteq \bigcap_{\mathfrak{p}'} \mathcal{O}(X)_{\mathfrak{p}'} = \bigcap_{\mathfrak{p}} \mathcal{O}(Y)_{\mathfrak{p}} = \mathcal{O}(Y),$$

and the claim follows. $\square$

EXAMPLE 5.6.6. Let X be an irreducible variety and $\varphi\colon X \to \mathbb{C}^n$ a dominant morphism of degree 1 with finite fibers. Then $\varphi(X) \subseteq \mathbb{C}^n$ is a special open set and $\varphi\colon X \xrightarrow{\sim} \varphi(X)$ is an isomorphism.

PROOF. Let $Y := \overline{\mathbb{C}^n \setminus \varphi(X)} \subseteq \mathbb{C}^n$. If $H \subseteq Y$ is an irreducible hypersurface, $H = \mathcal{V}(f)$, then $\varphi^{-1}(H)$ has codimension ≥ 2 in X. Since $\varphi^{-1}(H) = \mathcal{V}_X(\varphi^*(f))$, it follows from KRULL's Principal Ideal Theorem A.3.3.4 that $\varphi^{-1}(H) = \emptyset$, and so $\varphi(X) \subseteq \mathbb{C}^n_f$. Repeating this we finally end up with a special open set $U \subseteq \mathbb{C}^n$ such that $\varphi(X) \subseteq U$ and $\operatorname{codim} \overline{U \setminus \varphi(X)} \geq 2$. Now the claim follows from IGUSA's Lemma 5.6.5 above. $\square$

This example generalizes to the following result called ZARISKI's Main Theorem.

THEOREM 5.6.7. *Let X be an irreducible affine variety and $\varphi\colon X \to Y$ a dominant morphism with finite fibers. Then there is a finite morphism $\eta\colon \tilde{Y} \to Y$ and an open immersion $\iota\colon X \hookrightarrow \tilde{Y}$ such that $\varphi = \eta \circ \iota$:*

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & \tilde{Y} \arrow[d, "\eta"] \\ & Y \end{tikzcd}$$

In particular, if Y is normal and $\deg \varphi = 1$, then φ is an open immersion.

PROOF. Replacing Y by its normalization $\tilde{Y}$ in the field extension $\mathbb{C}(X)/\mathbb{C}(Y)$ we can assume that $\deg \varphi = 1$, and have to show that φ is an open immersion. Let $H \subseteq Y$ be an irreducible hypersurface such that $\overline{H \cap \varphi(X)}$ has codimension ≥ 2 in H. The ideal of H is a minimal prime $\mathfrak{p} \subseteq \mathcal{O}(Y)$ and $\mathcal{O}(Y)_{\mathfrak{p}}$ is a discrete valuation ring. Since $\varphi^{-1}(H)$ has codimension ≥ 2 in X we see that $\mathcal{V}_X(\mathfrak{p}) = \emptyset$, and so $\varphi(X) \subseteq Y \setminus H$. It follows that there are finitely many hypersurfaces $H_i \subseteq Y$ such that $\varphi(X) \subseteq Y' := Y \setminus \bigcup_i H_i$ and that $\overline{Y' \setminus \varphi(X)}$ has codimension ≥ 2. Now we apply IGUSA's Lemma 5.6.5 to a covering of Y' by special open sets to see that $\varphi(X) = Y'$ and that $\varphi\colon X \xrightarrow{\sim} Y'$ is an isomorphism. □

5.7. Complete intersections. There is a partial converse of Proposition 5.6.1 which is a special case of SERRE*'s-Criterion* (see below) which we now formulate without giving a proof.

PROPOSITION 5.7.1. *Let $H \subseteq \mathbb{C}^n$ be an irreducible hypersurface. If the singular points H_{sing} have codimension ≥ 2 in H, then H is normal.*

EXAMPLE 5.7.2. Let $Q_n := \mathcal{V}(x_1^2 + x_2^2 + \cdots + x_n^2) \subseteq \mathbb{C}^n$. Then $\dim Q_n = n - 1$ and $0 \in Q_n$ is the only singular point. Thus Q_n is normal for $n \geq 3$.

EXERCISE 5.7.3. Show that the nilpotent cone $N := \{A \in \mathrm{M}_2 \mid A \text{ nilpotent}\}$ is a normal variety.

DEFINITION 5.7.4. A closed subvariety $X \subseteq \mathbb{C}^n$ of codimension d is called a *complete intersection* if the ideal $I(X)$ can be generated by d elements.

Note that every irreducible component of the zero set $X := I(f_1, \ldots, f_d)$ of d polynomials $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ has codimension $\leq d$ (Proposition 5.5.2), but even if $\operatorname{codim} X = d$ this does not imply that X is a complete intersection. Such a variety is called a *set-theoretic complete intersection*. The first part of the following result gives a criterion to test if such a zero set is a complete intersecion.

PROPOSITION 5.7.5 (SERRE's Criterion). *Let $X \subseteq \mathbb{C}^n$ be the zero set of $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$: $X := \mathcal{V}(f_1, \ldots, f_r)$. Define*

$$X' := \{x \in X \mid \operatorname{rk} \operatorname{Jac}(f_1, \ldots, f_r)(x) < r\}.$$

(1) *If $X \setminus X'$ is dense in X, then $I(X) = (f_1, \ldots, f_r)$ and $X' = X_{\text{sing}}$. In particular, X is a complete intersection.*
(2) *If $\operatorname{codim}_X X \setminus X' \geq 2$, then X is normal.*

EXAMPLE 5.7.6. Let $N := \{A \in \mathrm{M}_n \mid A \text{ nilpotent}\}$ be the nilpotent cone in M_n. Then N is a normal variety. This result goes back to a fundamental paper of KOSTANT, see [**Kos63**].

PROOF. Consider the morphism $\pi\colon \mathrm{M}_n \to \mathbb{C}^n$, $\pi(A) := (\operatorname{tr} A, \operatorname{tr} A^2, \ldots, \operatorname{tr} A^n)$. Then $N = \pi^{-1}(0)$. If $P \in N$ is a nilpotent element of rank $n-1$, then $\operatorname{rk} d\pi_P = n$. In fact, $\operatorname{tr}(P + \varepsilon X)^k = \operatorname{tr}(P^k + \varepsilon k P^{k-1} X) = \varepsilon k \operatorname{tr}(P^{k-1} X)$. Taking P in Jordan normal form one easily sees that $d\pi_P\colon X \mapsto (\operatorname{tr} X, \operatorname{tr} PX, \operatorname{tr} P^2 X, \ldots, \operatorname{tr} P^{n-1} X)$ is surjective. It follows that $\operatorname{rk} \operatorname{Jac}(f_1, \ldots, f_n)(P) = n$ for the functions $f_j(A) := \operatorname{tr} A^j$ and for $P \in N' := \{\text{nilpotent matrices of rank } n-1\}$. Now one shows that $\operatorname{codim}_N N \setminus N' = 2$. □

5.8. Divisors. Let X be a normal affine variety. Define

$$\mathcal{H} := \{H \subseteq X \mid H \text{ irreducible hypersurface}\}.$$

DEFINITION 5.8.1. A *divisor on X* is a finite formal linear combination

$$D = \sum_{H \in \mathcal{H}} n_H \cdot H \quad \text{where } n_H \in \mathbb{Z}.$$

We write $D \geq 0$ if $n_H \geq 0$ for all $H \in \mathcal{H}$. The set of divisors forms the *divisor group*

$$\operatorname{Div} X = \bigoplus_{H \in \mathcal{H}} \mathbb{Z} \cdot H.$$

Recall that for any irreducible hypersurface $H \in \mathcal{H}$ we have defined a discrete valuation $\nu_H : \mathbb{C}(X)^* \to \mathbb{Z}$ whose discrete valuation ring is the local ring $\mathcal{O}_{X,H}$ (see section 5.4).

DEFINITION 5.8.2. For $f \in \mathbb{C}(X)^*$ we define the *divisor of* (f) by

$$(f) := \sum_{H \in \mathcal{H}} \nu_H(f) \cdot H.$$

Such a divisors is called a *principal divisor*.

REMARKS 5.8.3.
(1) (f) is indeed a divisor, i.e. $\nu_H(f) \neq 0$ only for finitely many $H \in \mathcal{H}$.
(This is clear for $f \in \mathcal{O}(X) \setminus \{0\}$, because $\nu_H(f) > 0$ if and only if $H \subseteq \mathcal{V}(f)$, and follows for a general $f = \frac{p}{q}$, because $(f) = (p) - (q)$, by definition.)
(2) $(f \cdot h) = (f) + (h)$ for all $f, h \in \mathbb{C}(X)$.
(3) $(f) \geq 0$ if and only if $f \in \mathcal{O}(X)$.
(We have $\nu_H(f) \geq 0$ if and only if $f \in \mathcal{O}_{X,H}$. Since $\bigcap_{H \in \mathcal{H}} \mathcal{O}_{X,H} = \mathcal{O}(X)$ the claim follows.)
(4) $(f) = 0$ if and only if f is invertible in $\mathcal{O}(X)$.
(If $(f) = 0$, then, by (3), $f \in \mathcal{O}(X)$ and $f^{-1} \in \mathcal{O}(X)$.)

DEFINITION 5.8.4. Two divisors $D, D' \in \operatorname{Div} X$ are called *linearly equivalent*, written $D \sim D'$, if $D - D'$ is a principal divisor. The set of equivalence classes is the *divisor class group of* X:

$$\operatorname{Cl} X := \operatorname{Div} X / \{\text{principal divisors}\}$$

It follows that we have an exact sequence of commutative groups

$$1 \to \mathcal{O}(X)^* \to \mathbb{C}(X)^* \to \operatorname{Div} X \to \operatorname{Cl} X \to 0$$

REMARK 5.8.5. We have $\operatorname{Cl} X = 0$ if and only if $\mathcal{O}(X)$ is a unique factorization domain. In fact, a unique factorization domain is characterized by the condition that all minimal prime ideals $\mathfrak{p} \neq (0)$ are principal.

EXAMPLE 5.8.6. Let $C \subseteq \mathbb{C}^2$ be a smooth curve. If $f \in \mathcal{O}(C)$ and $\tilde{f} \in \mathbb{C}[x, y]$ a representative of f, then

$$(f) = \sum_{P \in C \cap \mathcal{V}(\tilde{f})} m_P \cdot P,$$

and the integers $m_P > 0$ can be understood as the *intersection multiplicity* of C and $\mathcal{V}(\tilde{f})$ in P. For example, if the intersection is transversal, i.e., if $T_P C \cap T_P \mathcal{V}(\tilde{f}) = (0)$, then $m_P = 1$ (see the following Exercise 5.8.7).

EXERCISE 5.8.7. Let $C, E \subseteq \mathbb{C}^2$ be two irreducible curves, $I(C) = (f)$ and $I(E) = (h)$. If $P \in C \cap E$ define $m_P := \dim_{\mathbb{C}} \mathbb{C}[x, y]/(f, h)$. Show that

(1) If C is smooth and $\bar{h} = h|_C \in \mathcal{O}(C)$, then $(\bar{h}) = \sum_{P \in C \cap E} m_P \cdot P$
(2) If $P \in C \cap E$ and $T_P C \cap T_P E = (0)$, then $m_P = 1$.

EXERCISE 5.8.8.
(1) For the parabola $C = \mathcal{V}(y - x^2)$ we have $\operatorname{Cl} C = (0)$.
(2) For an elliptic curve $E = \mathcal{V}(y^2 - x(x^2 - 1))$ every divisor D is linearly equivalent to 0 or to P for a suitable point $P \in E$.

Exercises

For the convenience of the reader we collect here all exercises from Appendix A.

Exercise. For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ denote by $\mathrm{ev}_a \colon \mathcal{O}(\mathbb{C}^n) \to \mathbb{C}$ the *evaluation map* $f \mapsto f(a)$. Then the kernel of ev_a is the maximal ideal

$$\mathfrak{m}_a := (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n).$$

Exercise. Let $W \subseteq \mathcal{O}(V)$ a finite dimensional subspace. Then the linear functions $\mathrm{ev}_v|_W$ for $v \in V$ span the dual space W^*.

Exercise. Show that the subset $A := \{(n, m) \in \mathbb{C}^2 \mid n, m \in \mathbb{Z} \text{ and } m \geq n \geq 0\}$ is Zariski-dense in $\mathbb{C}^2$.

Exercise. A regular function $f \in \mathcal{O}(V)$ is called *homogeneous of degree d* if $f(tv) = t^d f(v)$ for all $t \in \mathbb{C}$ and all $v \in V$.

(1) A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree d as a regular function on $\mathbb{C}^n$ if and only if all monomials occurring in f have degree d.
(2) Assume that the ideal $\mathfrak{a} \subseteq \mathcal{O}(V)$ is generated by homogeneous functions. Then the zeros set $\mathcal{V}(\mathfrak{a}) \subseteq V$ is a cone.
(3) Conversely, if $X \subseteq V$ is a cone, then the ideal $I(X)$ can be generated by homogeneous functions. More precisely, if $f|_X = 0$, then $f_d|_X = 0$ for every homogeneous component f_d of f.

Exercise. Show that every non-empty open set in $\mathbb{C}^n$ is dense in the $\mathbb{C}$-topology. (Hint: Reduce to the case $n = 1$ where the claim follows from Example 1.2.6(4).)

Exercise. Let $U, U' \subseteq \mathbb{C}^n$ be two non-empty open sets. Then $U \cap U'$ is non-empty, too. In particular, the Zariski topology is not Hausdorff.

Exercise. Consider a polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ of the form $f = x_0 - p(x_1, \ldots, x_n)$, and let $X = \mathcal{V}(f)$ be its zero set. Then $I(X) = (f)$ and $\mathcal{O}(X) \simeq \mathbb{C}[x_1, \ldots, x_n]$.

Exercise. If $X \subseteq V$ is a closed subset and $\mathfrak{m} \subseteq \mathcal{O}(X)$ a maximal ideal, then $\mathcal{O}(X)/\mathfrak{m} = \mathbb{C}$. Moreover, $\mathfrak{m} = \ker(\mathrm{ev}_x \colon f \mapsto f(x))$ for a suitable $x \in X$.

Exercise. Let $\mathfrak{a} \subseteq R$ be an ideal of a (commutative) ring R. Then $\mathfrak{a}$ is perfect if and only if the residue class ring $R/\mathfrak{a}$ has no nilpotent elements different from 0.

Exercise. Let $C \subseteq \mathbb{C}^2$ be the plane curve defined by $y - x^2 = 0$. Then $I(C) = (y - x^2)$ and $\mathcal{O}(C)$ is a polynomial ring in one variable.

Exercise. Let $D \subseteq \mathbb{C}^2$ be the zero set of $xy - 1$. Then $\mathcal{O}(D)$ is not isomorphic to a polynomial ring, but there is an isomorphism $\mathcal{O}(D) \xrightarrow{\sim} \mathbb{C}[t, t^{-1}]$.

Exercise. Consider the "parametric curve"

$$Y := \{(t, t^2, t^3) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}.$$

Then Y is closed in $\mathbb{C}^3$. Find generators for the ideal $I(Y)$ and show that $\mathcal{O}(Y)$ is isomorphic to the polynomial ring in one variable.

Exercise. Let $X \subseteq V$ be a closed subset and $f \in \mathcal{O}(X)$ a regular function such that $f(x) \neq 0$ for all $x \in X$. Then f is invertible in $\mathcal{O}(X)$, i.e. the $\mathbb{C}$-valued function $f^{-1} \colon x \mapsto f(x)^{-1}$ is regular on X.

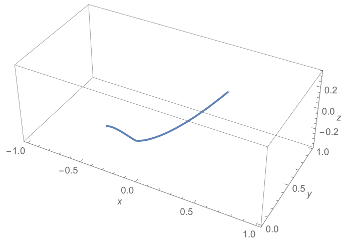
Exercise. Every closed subset $X \subseteq \mathbb{C}^n$ is quasi-compact, i.e., every covering of X by open sets contains a finite covering. Is this also true for open or even locally closed subsets of $\mathbb{C}^n$?

Exercise. Let $X \subseteq \mathbb{C}^n$ be a closed subset. Assume that there are no nonconstant invertible regular function on X. Then every nonconstant $f \in \mathcal{O}(X)$ attains all values in $\mathbb{C}$, i.e. $f \colon X \to \mathbb{C}$ is surjective.

Exercise. Consider the curve

$$Y := \{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$$

cf. Exercise 1.3.14. Then Y is closed in $\mathbb{C}^3$. Find generators for the ideal $I(Y)$ and show that $I(Y)$ cannot be generated by two polynomials.



(Hint: Define the *weight* of a monomial in x, y, z by $\mathrm{wt}(x) := 3, \mathrm{wt}(y) := 4, \mathrm{wt}(z) := 5$. Then the ideal $I(Y)$ is linearly spanned by the differences $m_1 - m_2$ of two monomials of the same weight. This occurs for the first time for the weight 8, and then also for the weights 9 and 10. Now show that for any generating system of $I(Y)$ these three differences have to occur in three different generators.)

Exercise. Let Z be an affine variety with coordinate ring $\mathcal{O}(Z)$. Then every polynomial $f \in \mathcal{O}(Z)[t]$ with coefficients in $\mathcal{O}(Z)$ defines a function on the product $Z \times \mathbb{C}$ in the usual way:

$$f = \sum_{k=0}^{m} f_k t^k : (z, a) \mapsto \sum_{k=0}^{m} f_k(z) a^k \in \mathbb{C}$$

Show that $Z \times \mathbb{C}$ together with $\mathcal{O}(Z)[t]$ is an affine variety.
(Hint: For any ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ there is a canonical isomorphism $\mathbb{C}[x_1, \ldots, x_n, t]/(\mathfrak{a}) \xrightarrow{\sim} (\mathbb{C}[x_1, \ldots, x_n]/\mathfrak{a})[t]$.)

Exercise. For any affine variety Z there is a inclusion-reversing bijection

$$\{A \subseteq Z \text{ closed}\} \xrightarrow{\sim} \{\mathfrak{a} \subseteq \mathcal{O}(Z) \text{ perfect ideal}\}$$

given by $A \mapsto I(A) := \{f \in \mathcal{O}(Z) \mid f|_A = 0\}$ (cf. Corollary 1.3.15).

Exercise. Denote by C_n the n-tuples of complex numbers up to sign, i.e., $C_n := \mathbb{C}^n/\sim$ where $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ if $a_i = \pm b_i$ for all i. Then every polynomial in $\mathbb{C}[x_1^2, x_2^2, \ldots, x_n^2]$ is a well-defined function on C_n. Show that C_n together with these functions is an affine variety.
(Hint: Consider the map $\Phi \colon \mathbb{C}^n \to \mathbb{C}^n$, $(a_1, \ldots, a_n) \mapsto (a_1^2, \ldots, a_n^2)$ and proceed like in Example 1.4.3.)

Exercise. Let X be an affine variety. Show that every choice of a generating system $f_1, f_2, \ldots, f_n \in \mathcal{O}(X)$ of the algebra $\mathcal{O}(X)$ consisting of n elements defines an identification of X with a closed subset $\mathcal{V}(\mathfrak{a}) \subseteq \mathbb{C}^n$.
(Hint: Consider the map $X \to \mathbb{C}^n$ given by $x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$.)

Exercise. Let R be an arbitrary $\mathbb{C}$-algebra. For any element $s \in R$ define $R_s := R[x]/(s \cdot x - 1)$.

(1) Describe the kernel of the canonical homomorphism $\iota \colon R \to R_s$.
(2) Prove the universal property: *For any homomorphism $\rho \colon R \to A$ such that $\rho(s)$ is invertible in A there is a unique homomorphism $\bar{\rho} \colon R_s \to A$ such that $\bar{\rho} \circ \iota = \rho$.*
(3) What happens if s is a zero divisor and what if s is invertible?

Exercise. The closed subvariety $X := \mathcal{V}(x^2 - yz, xz - x) \subseteq \mathbb{C}^3$ has three irreducible components. Describe the corresponding prime ideals in $\mathbb{C}[x, y, z]$.

Exercise. Let $X = X_1 \cup X_2$ where $X_1, X_2 \subseteq X$ are closed and disjoint. Then one has a canonical isomorphism $\mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(X_1) \times \mathcal{O}(X_2)$.

Exercise. Let $X = \bigcup_i X_i$ be the decomposition into irreducible components. Let $U_i \subseteq X_i$ be open subsets and put $U := \bigcup_i U_i \subseteq X$.

(1) Show that U is not necessarily open in X.
(2) Find sufficient conditions to ensure that U is open in X.
(3) Show that U is dense in X if and only if all U_i are non-empty.

Exercise. If $f \in \mathbb{C}(\mathbb{C}^2) = \mathbb{C}(x, y)$ is defined in $\mathbb{C}^2 \setminus \{(0, 0)\}$, then f is regular.

Exercise. Let $f \in \mathbb{C}(V)$ be a nonzero rational function on the vector space V. Then $\operatorname{Def}(f)$ is a special open set in V.

Exercise. If X is irreducible, then $\mathcal{O}(X) = \bigcap_{x \in X} \mathcal{O}_{X,x}$.

Exercise. Let X be an affine variety, $x \in X$ a point and $X' \subseteq X$ the union of irreducible components of X passing through x. Then the restriction map induces a natural isomorphism $\mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{X',x}$.

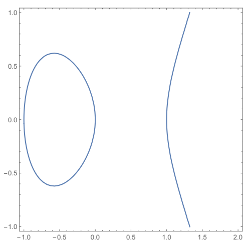
Exercise. Let R be an algebra and $\mu\colon R \to R_S$ the canonical map $r \mapsto \frac{r}{1}$ where R_S is the localization at a multiplicatively closed subset $1 \in S \subseteq R$ $(0 \notin S)$.

(1) If $\mathfrak{a} \subseteq R$ is an ideal and $\mathfrak{a}_S := R_S\,\mu(\mathfrak{a}) \subseteq R_S$, then
$$\mu^{-1}(\mu(\mathfrak{a})) = \mu^{-1}(\mathfrak{a}_S) = \{b \in R \mid sb \in \mathfrak{a} \text{ for some } s \in S\}.$$
Moreover, $(R/\mathfrak{a})_{\bar{S}} \xrightarrow{\sim} R_S/\mathfrak{a}_S$ where $\bar{S}$ is the image of S in $R/\mathfrak{a}$.
(Hint: For the second assertion use the universal property of the localization.)

(2) If $\mathfrak{m} \subseteq R$ is a maximal ideal and $S := R \setminus \mathfrak{m}$, then $\mathfrak{m}_S$ is the maximal ideal of R_S and the natural maps $R/\mathfrak{m}^k \xrightarrow{\sim} R_S/\mathfrak{m}_S^k$ are isomorphisms for all $k \geq 1$.
(Hint: The image $\bar{S}$ in $R/\mathfrak{m}^k$ consists of invertible elements.)

Exercise. Let $p < q$ be positive integers with no common divisor and define $C_{p,q} := \{(t^p, t^q) \mid t \in \mathbb{C}\} \subseteq \mathbb{C}^2$. Then $C_{p,q}$ is a closed irreducible plane curve which is rational, i.e. $\mathbb{C}(C_{p,q}) \simeq \mathbb{C}(t)$. Moreover, $\mathcal{O}(C_{p,q})$ is a polynomial ring if and only if $p = 1$.

Exercise. Consider the *elliptic curve* $E := \mathcal{V}(y^2 - x(x^2 - 1)) \subseteq \mathbb{C}^2$. Show that E is not rational, i.e. that $\mathbb{C}(E)$ is not isomorphic to $\mathbb{C}(t)$.
(Hint: If $\mathbb{C}(E) = \mathbb{C}(t)$, then there are rational functions $f(t), h(t)$ which satisfy the equation $f(t)^2 = h(t)(h(t)^2 - 1)$.)



Exercise. Let $g \in \mathrm{GL}_n$ be an invertible matrix. Then left multiplication $A \mapsto gA$, right multiplication $A \mapsto Ag$ and conjugation $A \mapsto gAg^{-1}$ are automorphisms of M_n.

Exercise. Let $\varphi\colon X \to Y$ be a morphism. If $X' \subseteq X$ and $Y' \subseteq Y$ are closed subvarieties such that $\varphi(X') \subseteq Y'$, then the induced map $\varphi'\colon X' \to Y'$, $x \mapsto \varphi(x)$, is again a morphism. The same holds if X' and Y' are special open sets.

Exercise. (1) Every morphism $\mathbb{C} \to \mathbb{C}^*$ is constant.
(2) Describe all morphisms $\mathbb{C}^* \to \mathbb{C}^*$.
(3) Every nonconstant morphism $\mathbb{C} \to \mathbb{C}$ is surjective.
(4) An injective morphism $\mathbb{C} \to \mathbb{C}$ is an isomorphism, and the same holds for injective morphisms $\mathbb{C}^* \to \mathbb{C}^*$.

Exercise. Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism and define
$$\Gamma_\varphi := \{(a, \varphi(a)) \in \mathbb{C}^{n+m}\}.$$
which is called the *graph* of the morphism φ. Show that Γ_φ is closed in $\mathbb{C}^{n+m}$, that the projection $\mathrm{pr}_{\mathbb{C}^n}\colon \mathbb{C}^{n+m} \to \mathbb{C}^n$ induces an isomorphism $p\colon \Gamma_\varphi \xrightarrow{\sim} \mathbb{C}^n$ and that $\varphi = \mathrm{pr}_{\mathbb{C}^m} \circ p^{-1}$.

Exercise. Show that for an affine variety X the morphisms $X \to \mathbb{C}^*$ correspond bijectively to the invertible functions on X.

Exercise. Let X, Y be affine varieties and $\varphi\colon X \to Y$, $\psi\colon Y \to X$ morphisms such that $\psi \circ \varphi = \mathrm{Id}_X$. Then $\varphi(X) \subseteq Y$ is closed and $\varphi\colon X \xrightarrow{\sim} \varphi(X)$ is an isomorphism.

EXERCISE. If $\varphi_1, \varphi_2 \colon X \to Y$ are two morphisms, then the "kernel of coincidence"

$$\ker(\varphi_1, \varphi_2) := \{x \in X \mid \varphi_1(x) = \varphi_2(x)\} \subseteq X$$

is closed in X

EXERCISE. Let $\varphi \colon X \to Y$ be a morphism of affine varieties.

(1) If X is irreducible, then $\overline{\varphi(X)}$ is irreducible.
(2) Every irreducible component of X is mapped into an irreducible component of Y.
(3) If $U \subseteq Y$ is a special open set, then so is $\varphi^{-1}(U)$.

EXERCISE. Let $\varphi \colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism, $\varphi = (f_1, f_2, \ldots, f_m)$ where $f_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$, and let $Y := \overline{\varphi(\mathbb{C}^n)}$ be the closure of the image of φ. Then

$$I(Y) = (y_1 - f_1, y_2 - f_2, \ldots, y_m - f_m) \cap \mathbb{C}[y_1, y_2, \ldots, y_m]$$

where both sides are considered as subsets of $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. So $I(Y)$ is obtained from the ideal $(y_1 - f_1, \ldots, y_m - f_m)$ by *eliminating the variables* $x_1, \ldots, x_n$.
(Hint: Use the graph Γ_φ defined in Exercise 2.1.8 and show that the ideal $I(\Gamma_\varphi)$ is generated by $\{y_j - f_j \mid j = 1, \ldots, m\}$.)

EXERCISE. Let $\varphi \colon X \xrightarrow{\sim} X$ be an automorphism and $Y \subseteq X$ a closed subset such that $\varphi(Y) \subseteq Y$. Then $\varphi(Y) = Y$ and $\varphi|_Y \colon Y \to Y$ is an automorphism, too.
(Hint: Look at the descending chain $Y \supseteq Y_1 := \varphi(Y) \supseteq Y_2 := \varphi(Y_1) \supseteq \cdots$. If $Y_n = Y_{n+1}$, then $\varphi(Y_{n-1}) = Y_n = \varphi(Y_n)$ and so $Y_{n-1} = Y_n$.)

EXERCISE. Let $\varphi \colon X \to Y$ and $\psi \colon Y \to Z$ be morphisms, and assume that the composition $\psi \circ \varphi$ is a closed immersion. Then φ is a closed immersion.

EXERCISE. Describe the fibers of the morphism $\varphi \colon \mathrm{M}_2 \to \mathrm{M}_2$, $A \mapsto A^2$.
(Hint: Use the fact that $\varphi(gAg^{-1}) = g\varphi(A)g^{-1}$ for $g \in \mathrm{GL}_2$.)

EXERCISE. Show that all fibers of the morphism $\psi \colon \mathbb{C} \to D := \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{C}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$, are reduced and that ψ induces an isomorphism $\mathbb{C} \setminus \{1, -1\} \xrightarrow{\sim} D \setminus \{(0, 0)\}$.

EXERCISE. Consider the morphism $\varphi \colon \mathrm{SL}_2 \to \mathbb{C}^3$, $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) := (ab, ad, cd)$.

(1) The image of φ is a closed hypersurface $H \subseteq \mathbb{C}^3$ defined by $xz - y(y - 1) = 0$.
(2) The fibers of φ are the left cosets of the subgroup $T := \{\begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \mid t \in \mathbb{C}^*\}$.
(3) All fibers are reduced.

(Hint: Show that the left multiplication with some $g \in \mathrm{SL}_2$ induces an automorphism λ_g of H and isomorphisms $\varphi^{-1}(y) \xrightarrow{\sim} \varphi^{-1}(\lambda_g(y))$ for all $y \in H$. This implies that it suffices to study just one fiber, e.g. $\varphi^{-1}(\varphi(E))$.)

EXERCISE. Consider the morphism $\varphi \colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $\varphi(x, y) := (x, xy)$.

(1) $\varphi(\mathbb{C}^2) = \mathbb{C}^2 \setminus \{(0, y) \mid y \neq 0\}$ which is not locally closed.
(2) What happens with the lines parallel to the x-axis or parallel to the y-axis?
(3) $\varphi^{-1}(0) = y$-axis. Is this fiber reduced?
(4) φ induces an isomorphism $\mathbb{C}^2 \setminus y$-axis $\xrightarrow{\sim} \mathbb{C}^2 \setminus y$-axis.

EXERCISE. Let $\varphi \colon \mathbb{C} \to \mathbb{C}$ be a nonconstant morphism. Then φ has finite degree d, and there is a non-empty open set $U \subseteq \mathbb{C}$ such that $\#\varphi^{-1}(x) = d$ for all $x \in U$.

EXERCISE. Show that the ideal of the diagonal $\Delta(X) \subseteq X \times X$ is generated by the function $f \cdot 1 - 1 \cdot f$, $f \in \mathcal{O}(X)$ (see Example 2.5.2(4)).

EXERCISE. Show that $\mathcal{O}(X \times_S Y) \simeq (\mathcal{O}(X) \otimes_{\mathcal{O}(S)} \mathcal{O}(Y))_{\mathrm{red}}$ where $R_{\mathrm{red}} := R/\sqrt{(0)}$.

EXERCISE. Let K be a field of characteristic zero which contains the roots of unity. Let $d \in \mathbb{N}$ and assume that $a \in K \setminus \bigcup_{p|d} K^p$. Then the polynomial $z^d - a \in K[z]$ is irreducible.
(Hint: If $b^d = a$, then $z^d - a = \prod_j (z - \zeta^j b)$ where $\zeta \in K$ is a primitive d-th root of unity.

It follows that $K[b]/K$ is a Galois extension, and that the Galois group G embeds into the group $\mu_d \subseteq K$ of d-th roots of unity by $\sigma \mapsto \frac{\sigma(b)}{b}$. Thus G is cyclic, and if the order is $m|d$, then the power of b^m is fixed by G.)

Exercise. If $\varphi\colon X \xrightarrow{\sim} Y$ is an isomorphism, then $\dim_x X = \dim_{\varphi(x)} Y$ for all $x \in X$.

Exercise. Let $G \subseteq \mathrm{GL}_n$ be a closed subgroup. Then $\dim_g G = \dim G$ for all $g \in G$. (Hint: Use the fact that left multiplication with g is an isomorphisms $G \xrightarrow{\sim} G$.)

Exercise. Let X be an affine variety. Assume that $\mathcal{O}(X)$ is generated by r elements. Then $\dim X \leq r$, and if $\dim X = r$, then $X \simeq \mathbb{C}^r$.

Exercise. The function $x \mapsto \dim_x X$ is upper semi-continuous on X. (This means that for all $\alpha \in \mathbb{R}$ the set $\{x \in X \mid \dim_x X < \alpha\}$ is open in X.)

Exercise. Let A be a finitely generated algebra. Then the following statements are equivalent.

(i) A is finite dimensional.
(ii) $A_{\text{red}} := A/\sqrt{(0)}$ is finite dimensional.
(iii) The number of maximal ideals in A is finite.

Exercise. Let $U \subseteq X$ be a dense open set. Then $\dim X \setminus U < \dim X$.

Exercise. Every nonconstant morphism $\varphi\colon \mathbb{C} \to \mathbb{C}$ is finite, and the same holds for the nonconstant morphisms $\psi\colon \mathbb{C}^* \to \mathbb{C}^*$.

Exercise. Define $\varphi\colon \mathbb{C}^* \to \mathbb{C}$ by $t \mapsto t + \frac{1}{t}$. Show that his morphism is closed, has finite fibers, but is not finite. Thus the converse statement of the Proposition 3.2.4 above is not true.

Exercise. Let X be an affine variety and $x \in X$. Assume that $f_1, \dots, f_r \in \mathfrak{m}_x$ generate the ideal $\mathfrak{m}_x$ modulo $\mathfrak{m}_x^2$, i.e., $\mathfrak{m}_x = (f_1, \dots, f_r) + \mathfrak{m}_x^2$. Then $\{x\}$ is an irreducible component of $\mathcal{V}_X(f_1, \dots, f_r)$.
(Hint: If $C \subseteq \mathcal{V}_X(f_1, \dots, f_r)$ is an irreducible component containing x and $\mathfrak{m} \subseteq \mathcal{O}(C)$ the maximal ideal of x, then $\mathfrak{m}^2 = \mathfrak{m}$. Hence $\mathfrak{m} = 0$ by the Lemma of Nakayama above.)

Exercise. Let $\varphi\colon X \to Y$ be a finite surjective morphism. Then $\dim X = \dim Y$.

Exercise. Let X be an affine variety and $X = \bigcup_i X_i$ the irreducible decomposition. A morphism $\varphi\colon X \to Y$ is finite if and only if the restrictions $\varphi|_{X_i}\colon X_i \to Y$ are finite for all i.

Exercise. Assume that the morphism $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ is given by nonconstant homogeneous polynomials $f_1, \cdots, f_m$. If $\varphi^{-1}(0)$ is finite, then $\varphi^{-1}(0) = \{0\}$ and φ is a finite morphism.
(Hint: Use the example above together with Exercise 3.1.11.)

Exercise. Let $X \subseteq \mathbb{C}^n$ be a closed cone and $\lambda\colon \mathbb{C}^n \to \mathbb{C}^m$ a linear map. If $X \cap \ker \lambda = \{0\}$, then $\lambda|_X\colon X \to \mathbb{C}^m$ is finite. Moreover, the set of linear maps $\lambda\colon \mathbb{C}^n \to \mathbb{C}^m$ such that $\lambda|_X$ is finite is open in $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) = \mathrm{M}_{m,n}(\mathbb{C})$.

Exercise. Let $r \in \mathbb{C}(x_1, \dots, x_n)$ satisfy an equation of the form

$$r^m + p_1 r^{m-1} + \cdots + p_m = 0 \text{ where } p_j \in \mathbb{C}[x_1, \dots, x_n].$$

Then $r \in \mathbb{C}[x_1, \dots, x_n]$. In particular, if $A \subseteq \mathbb{C}(a_1, \dots, a_n)$ is a subalgebra which is finite over $\mathbb{C}[a_1, \dots, a_n]$, then $A = \mathbb{C}[a_1, \dots, a_n]$.

Exercise. Let X be an affine variety and $f \in \mathcal{O}(X)$ a nonzero divisor. For any $x \in \mathcal{V}_X(f)$ we have $\dim_x \mathcal{V}_X(f) = \dim_x X - 1$.
(Hint: If f is a nonzero divisor, then f is nonzero on every irreducible component X_i of X and so $\mathcal{V}_{X_i}(f)$ is either empty or every irreducible component has codimension 1. Now the claim follows easily.)

Exercise. Work out the decomposition of Theorem 3.4.1 for the morphisms $\varphi\colon \mathrm{SL}_2 \to \mathbb{C}^3$, $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) := (ab, ad, cd)$ (see Exercise 2.2.14). What is the degree of the finite morphism ρ?

Exercise. Consider the morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $(x, y) \mapsto (x, xy)$. Show that the image $\varphi(\mathbb{C}^2)$ is not locally closed in $\mathbb{C}^2$ and that the map $a \mapsto \dim \varphi^{-1}(a)$ is not upper-semicontinuous.

Exercise. Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. If $D \subseteq Y$ is a dense subset such that $\dim \varphi^{-1}(y) = d$ for all $y \in D$, then $\dim X = \dim Y + d$.

Exercise.
1. Finite unions, finite intersections and complements of constructible sets are again constructible.
2. If C is a constructible, then C contains a set U which is open and dense in $\bar{C}$.

Exercise. Let X be an irreducible affine variety and $C \subseteq X$ a dense constructible subset. Then C can written in the form

$$C = C_0 \cup \bigcup_{j=1}^{m} C_j$$

where $C_0 \subseteq X$ is open and dense, C_j is locally closed, $\overline{C_j}$ is irreducible of codimension ≥ 1, and $\overline{C_j} \cap C_0 = \emptyset$.

Exercise. What is the degree of the morphism $\mathrm{M}_n \to \mathrm{M}_n$ given by $A \mapsto A^k$?

Exercise. Let $\varphi\colon X \to Y$ be a dominant morphism where X and Y are irreducible. If there is an open dense set $U \subseteq X$ such that $\varphi|_U$ is injective, then φ is birational.

Exercise. Let $\varphi\colon X \to Y$ be a *quasi-finite* morphism, i.e. all fibers are finite. Then $\dim \overline{\varphi(X)} = \dim X$.

Exercise. Let $\delta \in T_x X$ be a tangent vector in x. Then
1. $\delta(c) = 0$ for every constant $c \in \mathcal{O}(X)$.
2. If $f \in \mathcal{O}(X)$ is invertible, then $\delta(f^{-1}) = -\dfrac{\delta f}{f(x)^2}$.

Exercise. The canonical homomorphism $\mathcal{O}(X) \to \mathcal{O}_{X,x}$ induces an isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\sim} \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ is the maximal ideal.

Exercise. If $Y \subseteq X$ is a closed subvariety and $x \in Y$, then $\dim T_x Y \leq \dim T_x X$. (Hint: The surjection $\mathcal{O}(X) \to \mathcal{O}(Y)$ induces a surjection $\mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \to \mathfrak{m}_{x,Y}/\mathfrak{m}_{x,Y}^2$.)

Exercise. Calculate the tangent spaces of the plane curves $C_1 := \mathcal{V}(y - x^2)$ and $C_2 = \mathcal{V}(y^2 - x^2 - x^3)$ in arbitrary points (a, b).

Exercise. If $X, Y \subseteq \mathbb{C}^n$ are closed subvarieties and $z \in X \cap Y$, then $T_z(X \cap Y) \subseteq T_z X \cap T_z Y \subseteq \mathbb{C}^n$. Give an example where $T_z(X \cap Y) \subsetneqq T_z X \cap T_z Y$.

Exercise. If X is an affine variety such that all irreducible components have the same dimension. Then X_{sing} is closed and has a dense complement.

Exercise. The hypersurface $H = \mathcal{V}(xz - y(y-1)) \subseteq \mathbb{C}^3$ from Exercise 2.2.14 is nonsingular.

Exercise. Let $q \in \mathbb{C}[x_1, \ldots, x_n]$ be a quadratic form and $Q := \mathcal{V}(q) \subseteq \mathbb{C}^n$. Then 0 is a singular point of Q. It is the only singular point if and only if q is nondegenerate.

Exercise. Determine the singular points of the plane curves

$$E_p := \mathcal{V}(y^2 - p(x))$$

where $p(x)$ is an arbitrary polynomial, and deduce a necessary and sufficient condition for E_p to be smooth.

Exercise. Let $X \subseteq \mathbb{C}^n$ be a closed cone (see Exercise 1.2.9). Then X_{sing} is a cone, too. Moreover, $0 \in X$ is a nonsingular point if and only if X is subspace.

Exercise. Let X be an affine variety such that the group of automorphisms acts transitively on X. Then X is smooth.

EXERCISE. Determine the vector fields on the curve $D := \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{C}^2$. Do they all vanish in the singular point of D?

EXERCISE. Determine the vector fields on the curves $D_1 := \{(t, t^2, t^3) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$ and $D_2 := \{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$.
(Hint: For D_2 one can use that $\mathcal{O}(D_2) \simeq \mathbb{C}[t^3, t^4, t^5] = \mathbb{C} \oplus \bigoplus_{i \geq 3} \mathbb{C}t^i$.)

EXERCISE. Let A be an arbitrary associative $\mathbb{C}$-algebra. Then A is a Lie algebra with Lie bracket $[a, b] := ab - ba$, i.e., the bracket $[\ ,\ ]$ satisfies the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \text{ for all } a, b, c \in A.$$

EXERCISE. Let R be an associative $\mathbb{C}$-algebra. If $\xi, \eta \colon R \to R$ are both $\mathbb{C}$-derivations, then so is the commutator $\xi \circ \eta - \eta \circ \xi$. This means that the derivations $\operatorname{Der}(R)$ form a Lie subalgebra of $\operatorname{End}_{\mathbb{C}}(R)$.

EXERCISE. Let $X \subseteq \mathbb{C}^n$ be a closed and irreducible. Then $\dim TX \geq 2 \dim X$. If X is smooth, then TX is irreducible and smooth of dimension $\dim TX = 2 \dim X$.
(Hint: If $I(X) = (f_1, \dots f_m)$, then $TX \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is defined by the equations

$$f_j = 0 \text{ and } \sum_{i=1}^{n} y_i \frac{\partial f_j}{\partial x_i}(x) = 0 \text{ for } j = 1, \dots, m.$$

The Jacobian matrix of this system of $2m$ equations in $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$ has the following block form

$$\begin{bmatrix} \operatorname{Jac}(f_1, \dots, f_m) & 0 \\ * & \operatorname{Jac}(f_1, \dots, f_m) \end{bmatrix}$$

and thus has rank $\geq 2 \cdot \operatorname{rk} \operatorname{Jac}(f_1, \dots, f_m) = 2(n - \dim X)$.)

EXERCISE. Let $\varphi \colon X \to Y$ and $\psi \colon Y \to Z$ be morphisms of affine varieties and let $x \in X$. Then

$$d(\psi \circ \varphi)_x = d\psi_y \circ d\varphi_x$$

where $y := \varphi(x) \in Y$.

EXERCISE. Consider the multiplication $\mu \colon \mathrm{M}_2 \times \mathrm{M}_2 \to M_2$ and show:

(1) $d\mu_{(A,B)}$ is surjective, if A or B is invertible.
(2) If $\operatorname{rk} A = \operatorname{rk} B = 1$, then $d\mu_{(A,B)}$ has rank 3.
(3) We have $\operatorname{rk} d\mu_{(A,0)} = \operatorname{rk} d\mu_{(0,A)} = 2 \operatorname{rk} A$.

EXERCISE. Calculate the differential of the morphism $\varphi \colon \operatorname{End}(V) \times V \to V$ given by $(\rho, v) \mapsto \rho(v)$, and determine the pairs (ρ, v) where $d\varphi_{(\rho,v)}$ is surjective.

EXERCISE. For every point $(x, y) \in X \times Y$ we have $T_x X = \ker d(\mathrm{pr}_Y)_{(x,y)}$ and $T_y X = \ker d(\mathrm{pr}_X)_{(x,y)}$ where $\mathrm{pr}_X, \mathrm{pr}_Y$ are the canonical projections (see Proposition 4.1.9).

EXERCISE. For the closed subset $N \subseteq \mathrm{M}_2$ of nilpotent 2×2-matrices we have $I(N) = (\operatorname{tr}, \det)$.

EXERCISE. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a nonconstant polynomial. Then $\mathcal{V}(f - \lambda)$ is a smooth hypersurface for almost all $\lambda \in \mathbb{C}$.

EXERCISE. Let $A \subseteq B$ be rings. If A is Noetherian and B finite over A, then B is integral over A.

EXERCISE. Let $f \in \mathbb{C}[x]$ be a nonconstant polynomial. Then $\mathbb{C}[x]$ is integral over the subalgebra $\mathbb{C}[f]$.

EXERCISE. If the domain A is integrally closed, then so is every ring of fraction A_S where $1 \in S \subseteq A$ is multiplicatively closed.

EXERCISE. If $\varphi \colon X \to Y$ is a finite surjective morphism where X is irreducible and Y is normal, then $\#\varphi^{-1}(y) \leq \deg \varphi$ for all $y \in Y$. (See Proposition 3.6.1 and its proof.)

EXERCISE. Consider the morphism $\varphi \colon \mathbb{C}^2 \to \mathbb{C}^4$, $(x, y) \mapsto (x, xy, y^2, y^3)$.

(1) φ is finite and $\varphi \colon \mathbb{C}^2 \to Y := \varphi(\mathbb{C}^2)$ is the normalization.

(2) $0 \in Y$ is the only non-normal and the only singular point of Y.
(3) Find defining equations for $Y \subseteq \mathbb{C}^4$ and generators of the ideal $I(Y)$.

Exercise. If X is a normal variety, then so is $X \times \mathbb{C}^n$.

Exercise. Let A be a discrete valuation ring with field of fraction K. If $B \subseteq K$ is a subring containing A, then either $B = A$ or $B = K$.

Exercise. Let K/k be a finitely generated field extension, and let $A \subseteq K$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$, field of fraction K and containing k. Then $\operatorname{tdeg}_k A/\mathfrak{m} < \operatorname{tdeg}_k K$.
(Hint: If $\operatorname{tdeg}_k R/\mathfrak{m} = \operatorname{tdeg}_k K$, then R contains a field L with $\operatorname{tdeg}_k L = \operatorname{tdeg}_k K$. This implies that K is a finitely generated R-module which is impossible.)

Exercise. Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal with the property that any two elements $f_1, f_2 \in \mathfrak{a}$ have a non-trivial common divisor. Then there is a nonconstant h which divides every element of $\mathfrak{a}$.

Exercise. Show that the nilpotent cone $N := \{A \in \mathrm{M}_2 \mid A \text{ nilpotent}\}$ is a normal variety.

Exercise. Let $C, E \subseteq \mathbb{C}^2$ be two irreducible curves, $I(C) = (f)$ and $I(E) = (h)$. If $P \in C \cap E$ define $m_P := \dim_{\mathbb{C}} \mathbb{C}[x, y]/(f, h)$. Show that

(1) If C is smooth and $\bar{h} = h|_C \in \mathcal{O}(C)$, then $(\bar{h}) = \sum_{P \in C \cap E} m_P \cdot P$
(2) If $P \in C \cap E$ and $T_P C \cap T_P E = (0)$, then $m_P = 1$.

Exercise. (1) For the parabola $C = \mathcal{V}(y - x^2)$ we have $\operatorname{Cl} C = (0)$.
(2) For an elliptic curve $E = \mathcal{V}(y^2 - x(x^2 - 1))$ every divisor D is linearly equivalent to 0 or to P for a suitable point $P \in E$.

APPENDIX B

The Strong Topology on Complex Affine Varieties

In this appendix we define the $\mathbb{C}$-topology (or strong topology) on affine varieties and prove some basic properties, e.g. that finite morphisms are proper in the $\mathbb{C}$-topology and that irrreducible varieties are connected in the $\mathbb{C}$-topology. Then we show the complete reducibility of the representations of the classical groups $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}_n, \mathrm{SO}_n, \mathrm{Sp}_n$ by using Weyl's "unitary trick" ([**Wey39**, Chap. VIII B]).

The classical groups G contain subgroups K consisting of unitary matrices which are Zariski dense and are compact with respect to the $\mathbb{C}$-topology. By means of a Haar measure one gets the complete reducibility of the representations of the compact groups K (Theorem of HURWITZ-SCHUR [**?**]). Since K is Zariski dense in G, this implies the linear reductivity of G. At the end we give a brief description of the Cartan and Iwasawa decompositions of reductive groups.

Contents

1. $\mathbb{C}$-Topology on Varieties

The affine n-space $\mathbb{C}^n$ carries a natural topology coming from the standard metric on $\mathbb{C}^n = \mathbb{R}^{2n}$. We call this the *strong topology* or the $\mathbb{C}$-*topology* in order to distinguish from the Zariski-topology. Since polynomials are continuous in the strong topology we see that every affine variety X carries a strong topology, independent of the embedding into affine n-space. As the name suggests, the $\mathbb{C}$-topology is stronger then the Zariski-topology: Zariski-open sets are open in the $\mathbb{C}$-topology, shortly $\mathbb{C}$-open, and Zariski-closed sets are closed in the $\mathbb{C}$-topology, shortly $\mathbb{C}$-closed. The same argument shows that morphisms are continuous in the $\mathbb{C}$-topology, shortly $\mathbb{C}$-continous.

1.1. Smooth points. A first result we have in this context is the following.

PROPOSITION 1.1.1. *Let X be an affine variety and $x \in X$ a smooth point. Then X is a complex manifold in a neighborhood of x, i.e. there is a $\mathbb{C}$-open neighborhood U of x which is $\mathbb{C}$-homeomorphic to $\mathbb{C}^d$ where $d = \dim_x X$.*

PROOF. Let X be a closed subset of $\mathbb{C}^n$, and let $f_1, \cdots, f_m$ be generators of the ideal $I(X) \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Then, by assumption, the Jacobian matrix $J(f_1, \ldots, f_m)$ has rank d in x. We can assume that the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_d}{\partial x_1} & \frac{\partial f_d}{\partial x_2} & \cdots & \frac{\partial f_d}{\partial x_d} \end{bmatrix}$$

is invertible. By the complex implicit function theorem this implies that the projection $p\colon \mathbb{C}^n \to \mathbb{C}^d$ onto the first d coordinates induces local homeomorphism $p\colon X \to \mathbb{C}^d$ in a neighborhood of $x \in X$. □

REMARK 1.1.2. The complex implicit function theorem is much stronger than what we used above. It implies that the open set of smooth points of an algebaic variety is a complex manifold.

1.2. Proper morphisms. Next we want to show that finite morphisms are closed in the $\mathbb{C}$-topology, shortly $\mathbb{C}$-closed. For this we use the fact that $\mathbb{C}^n$ is a (Zariski-) open subset of the n-dimensional projective space $\mathbb{P}^n$ which is compact in the $\mathbb{C}$-topology. (The $\mathbb{C}$-topology on $\mathbb{P}^n$ is obtained as usual from a covering by $n+1$ copies of $\mathbb{C}^n$.)

LEMMA 1.2.1. *Let $\varphi\colon X \to Y$ be a finite morphism and $\psi\colon X \to Z$ an arbitrary morphism. Then the image of X under that map $(\varphi, \psi)\colon X \to Y \times Z$ is Zariski-closed.*

PROOF. This follows immediately from the following commutative diagram, because the upper horizontal map has a retraction and thus is a closed immersion, and the vertical map on the right is finite.

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id},\varphi,\psi)} & X \times Y \times Z \\ \downarrow & & \downarrow{\scriptstyle \varphi \times \mathrm{id}_{Y \times Z}} \\ Y \times Z & \xrightarrow[\subseteq]{\text{closed}} & Y \times Y \times Z \end{array}$$

□

PROPOSITION 1.2.2. *A finite morphism $\varphi\colon X \to Y$ is a closed map with respect to the $\mathbb{C}$-topology.*

PROOF. We can assume that X is a closed subset of $\mathbb{C}^n$. Now consider the embedding $\psi\colon X \hookrightarrow \mathbb{C}^n \hookrightarrow \mathbb{P}^n$. By the previous lemma, the image of X in $Y \times \mathbb{P}^n$ is closed, and the claim follows, because the projection $Y \times \mathbb{P}^n \to Y$ is a closed map with respect to the $\mathbb{C}$-topology. □

1.3. Connectedness.

Here we want to prove the following result.

PROPOSITION 1.3.1. *An irreducible variety X is connected in the $\mathbb{C}$-topology.*

In his book [**Sha94b**, Chap. VII, section 2] SHAFAREVICH gives two proofs of this result (pages 123–130). We believe that our proof is more elementary, although there is some overlap with SHAFAREVICH's second proof.

PROOF. (a) If we have a surjective morphism $\varphi\colon Y \to X$ and if Y is $\mathbb{C}$-connected, then so is X. Moreover, every non-empty Z-open set of X meets every $\mathbb{C}$-connected component of X. This follows because the Z-closure and the $\mathbb{C}$-closure of a Z-open set are equal (see ???).

(b) By Noether's normalization we can find a finite surjective morphism $\varphi\colon X \to \mathbb{A}^n$. Replacing X by a Z-open set of a suitable ramified finite covering we can assume that X is smooth and that we have a finite unramified Galois-covering $\psi\colon X \to U$ where $U \subseteq \mathbb{A}^n$ is a Z-open set. In particular, X is a complex manifold (see ???), and for every $\mathbb{C}$-connected component X_i of X the induced map $\psi_i\colon X_i \to U$ is an unramified covering. It follows that the Galois group Γ permutes the $\mathbb{C}$-connected components transitively. Denote by $\Gamma_0 \subseteq \Gamma$ the normalizer of X_i and set $Y := X/\Gamma_0$. Then Y is smooth, has the same number of $\mathbb{C}$-connected components Y_i as X, and the induced morphism $\bar\psi\colon Y \to U$ is again an unramified Galois-covering with Galois group $\bar\Gamma := \Gamma/\Gamma_0$. By construction, each Y_i is mapped bijectively, hence biholomorphically, onto U under $\bar\psi$, i.e. $\bar\psi_i := \bar\psi|_{Y_i}\colon Y_i \xrightarrow{\sim} U$ is biholomorphic. Thus the number of connected components of X and Y is equal to $d := [\mathbb{C}(Y) : \mathbb{C}(U)]$.

(c) Choose a regular function $f \in \mathcal{O}(Y)$ which generates the field extension $\mathbb{C}(Y)/\mathbb{C}(U)$. Then f satisfies an equation $f^d + \sum_{i=0}^{d-1} r_i f^i = 0$ where $r_i \in \mathbb{C}(U)$. Clearly, the holomorphic functions $f_i := f|_{Y_i}$ satisfy the same equation, as well as the pull-backs $h_i := (\bar\psi^*)^{-1}(f_i)$ which are holomorphic functions on U. By Proposition 1.4.1 below the h_i are rational functions on U. Since $t^d + \sum_{i=0}^{d-1} r_i t^i$ is an irreducible polynomial, we see that $d = 1$, hence X is $\mathbb{C}$-connected. □

1.4. Holomorphic functions satisfying an algebraic equation.

PROPOSITION 1.4.1. *Let $U \subseteq \mathbb{C}^n$ be a Z-open set, and let h be a holomorphic function on U. If h satisfies an algebaic equation $h^d + \sum_{i=0}^{d-1} r_i h^i = 0$ with rational coefficents $r_i \in \mathbb{C}(U)$, then h is rational.*

PROOF. We can assume that $\mathbb{C}^n \setminus U = \mathcal{V}(f)$ for some polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$. Multiplying h with a suitable rational function we can also assume that the coefficients r_i of the equation are polynomials. Since $h(x)$ is an eigenvalue of the $d \times d$-matrix

$$B(x) := \begin{bmatrix} 0 & 0 & \cdots & 0 & -r_0(x) \\ 1 & 0 & \cdots & 0 & -r_1(x) \\ 0 & 1 & \cdots & 0 & -r_2(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -r_{d-1}(x) \end{bmatrix}$$

we see that $|h(x)|^2 \le \|B(x)\|^2 = \sum_{i=0}^{d-1} |r_i(x)|^2 + (d-1)$ by Lemma 1.4.2 below. In particular, $|f(x)h(x)| \le |f(x)| \cdot \|B(x)\|$ which implies that fh extends to a continuous function on $\mathbb{C}^n$ with value zero on the complement $\mathcal{V}(f)$ of U in $\mathbb{C}^n$. Now we use CAUCHY's integral theorem on polydisks to see that fh defines a holomorphic function on $\mathbb{C}^n$. Then Lemma 1.4.3 implies that fh is a polynomial. $\square$

The next lemma is well-known.

LEMMA 1.4.2. *Let A be a complex $n \times n$-matrix. For any eigenvalue λ of A we have $|\lambda| \leq \|A\|$ where $\|A\| := \sqrt{\sum_{i,j} |a_{ij}|^2}$.*

PROOF. Choose an eigenvector v of length $\|v\| = 1$. Then $Av = \lambda v$ implies that

$$|\lambda| = \|\lambda v\| = \|Av\| \leq \|A\| \cdot \|v\| = \|A\|.$$

$\square$

LEMMA 1.4.3. *Let h be a holomorphic function on $\mathbb{C}^n$ which is polynomially bounded, i.e. there is an $m > 0$ and a $C > 0$ such that $|h(x)| \leq C(\|x\|^m + 1)$ for all $x \in \mathbb{C}^n$. Then h is a polynomial.*

PROOF. For $z \in \mathbb{C}$ and $x \in \mathbb{C}^n$ define the holomorphic function $\tilde{h}(z,x) := h(zx)$. Then we have

$$\tilde{h}(z,x) = \sum_{k=0}^{\infty} h_k(x) z^k.$$

Since

$$h_k(x) = \sum_{i_1,\ldots,i_k} \frac{1}{k!} \left. \frac{\partial^k \tilde{h}}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \right|_{z=0} x_{i_1} x_{i_2} \cdots x_{i_k}$$

we see that the coefficients $h_k(x)$ are homogenous polynomials of degree k. We claim that $h_k(x) = 0$ for $k > m$. In fact,

$$|h_k(x)| = \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} h(e^{i\theta}x) d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |e^{-ik\theta} h(e^{i\theta}x)| d\theta \leq C(\|x\|^m + 1),$$

and the claim follows. $\square$

1.5. Closures in Zariski- and $\mathbb{C}$-topology. The main result here is the following.

PROPOSITION 1.5.1. *Let X be an affine variety and $C \subseteq X$ a constructible subset. Then the closure $\overline{C}^c$ of C in the $\mathbb{C}$-topology is equal to the closure $\overline{C}$ in the Zariski-topology.*

PROOF. Since every constructible subset C contains a set $U \subseteq C$ which is open and dense in $\bar{C}$ we can assume that $X = \bar{C}$ and that$C \subseteq X$ is open. Moreover, one easily reduces to the case of an irreducible X. If $\dim X = 1$, then $C = X \setminus F$ for a finite set F.

$\square$

2. Reductivity of the Classical Groups

2.1. Maximal compact subgroups.

Exercises

For the convenience of the reader we collect here all exercises from Appendix B.

APPENDIX C

Fiber Bundles, Slice Theorem and Applications

Contents

1. Introduction: Local Cross Sections and Slices

LUNA's famous Slice Theorem gives a "local description" of an action of a reductive group G on an affine variety. It is modeled on the case of compact transformation groups, but one has to take into account the existence of non-closed orbits. Also one has to modify the concept of "local" and of "open neighborhoods" which make the whole story much more complicated. We first describe the situation of a compact group acting continuously on a nice topological space.

1.1. Free actions and cross sections. Let K be a compact group and let X be a K-*space*, i.e. a Hausdorff topological space with a continuous action of G. Then the orbit space X/G is again Hausdorff and the quotient map $\pi\colon X \to X/G$ is open, closed and proper.

Assume that the point $x \in X$ has a trivial stabilizer. Then one might expect that in a suitable neighborhood of the orbit Kx the action is free and X looks like $K \times U$ where K acts by left multiplication on K. This is indeed the case under very mild assumptions, e.g. if K is a compact Lie group and X is locally compact.

A *cross section* is a continuous map $\sigma\colon X/G \to X$ such that $\pi \circ \sigma$ is the identity on X/G. A *local cross section* defined on $U \subseteq X/G$ is a cross section of $\pi^{-1}(U) \to U$. A first result for compact transformation groups in this setting is the following, see [**Bre72**, Chap. II, Theorem 5.4].

PROPOSITION 1.1.1. *Assume that K is a compact Lie group and that X is locally compact. If $x \in X$ has a trivial stabilizer, $K_x = \{e\}$, then there is a local cross section σ in a neighborhood U of $\pi(x)$ such that $\pi^{-1}(U) \simeq K \times U$. Thus a free action of K on X looks locally like $K \times U$.*

EXAMPLE 1.1.2. Let us look at an algebraic example. Take the finite group $G = \mathbb{Z}/2$ acting on $X := \mathbb{C}$ by $\pm\,\mathrm{id}$. Then the orbit space X/G can be identified with $\mathbb{C}$ where the quotient map $\pi\colon X \to \mathbb{C}$ is given by $\pi(z) := z^2$. Removing the origin $\{0\} \in X$, the action is free and the quotient $\pi\colon \dot{X} := X \setminus \{0\} \to \dot{\mathbb{C}} := \mathbb{C} \setminus \{0\}$ is a 2-fold covering. This is clearly locally trivial in the $\mathbb{C}$-topology, but not locally trivial in the ZARISKI-topology. However, looking at the two fiber products

$$\begin{array}{ccc} F = \mathbb{C} \cup \mathbb{C} & \longrightarrow & X \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{C} & \xrightarrow{z \mapsto z^2} & \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} \dot{F} = \dot{\mathbb{C}} \cup \dot{\mathbb{C}} & \longrightarrow & \dot{X} \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ \dot{\mathbb{C}} & \xrightarrow{z \mapsto z^2} & \dot{\mathbb{C}} \end{array}$$

we find that $F \simeq \mathcal{V}(xy) \subseteq \mathbb{C}^2$, the union of two lines intersecting in the origin, and that $\dot{F}$ is the disjoint union of two copies of $\dot{\mathbb{C}}$, interchanged by G and each one mapped isomorphically to $\dot{\mathbb{C}}$ under $\tilde{\pi}$. Thus the quotient π can be trivialized, not with an open covering of $\dot{\mathbb{C}}$, but with the "étale" surjective map $\dot{\mathbb{C}} \to \dot{\mathbb{C}}, z \mapsto z^2$.

1.2. Associated bundles and slices. Assume again that K is a compact group and X a K-space. What can we say if the action is not free? More precisely, how does X look like in a neighborhood of an orbit $O \simeq K/H$? In order to explain this we make the following construction. Consider an H-space Y and define

$$X := K \times^H Y := (K \times Y)/H$$

where H acts freely on the product $K \times Y$ by $h(g, y) := (gh^{-1}, hy)$. We will denote the orbit of (g, y) by $[g, y] \in K \times^H Y$. This space is called *twisted product* or *associated bundle*. It has a number of remarkable properties. First of all, we have an action of K on $K \times^H Y$ induced by the left multiplication on K: $g'[g, y] := [g'g, y]$. Then, there is a natural closed embedding $Y \hookrightarrow K \times^H Y$, $y \mapsto [e, y]$.

PROPOSITION 1.2.1. (1) *There is a canonical bijection between the K-orbits in $K \times^H Y$ and the H-orbits in Y given by $O = K[g,y] \mapsto Hy = O \cap Y$. This map induces a homeomorphism of orbit spaces $(G \times^H Y)/G \xrightarrow{\sim} Y/H$, the inverse map is given by $Hy \mapsto G[e,y]$.*

(2) *The projection $K \times Y \to K$ induces a K-equivariant map $p\colon K \times^H Y \to K/H$ which is a locally trivial bundle with fiber Y: $p^{-1}(gH) = gY$.*

Except for the last statement, the proofs are easy exercises and are left to the reader. For the last statement, one has to use the fact that the projection $K \to K/H$ admits local cross sections.

EXAMPLE 1.2.2. Let us give again an algebraic example. Take $G := \mathbb{C}^*$ and $H := \{\pm 1\} \subseteq \mathbb{C}^*$, and consider the action of H on $Y := \mathbb{C}$ by $\pm\,\mathrm{id}$ as in the example above. Then the associated bundle $G \times^H Y$ has the following description:

$$\mathbb{C}^* \times^H \mathbb{C} \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C},\ [t,z] \mapsto (t^2, tz),$$

the $\mathbb{C}^*$-action on $\mathbb{C}^* \times \mathbb{C}$ is given by $t(s,x) = (t^2 s, tx)$, and the closed embedding $\mathbb{C} \hookrightarrow \mathbb{C}^* \times \mathbb{C}$ is $z \mapsto (1,z)$. Thus $\mathbb{C}^*(s,x) \cap \mathbb{C} = \{\pm x\}$ and $(\mathbb{C}^* \times \mathbb{C})/\mathbb{C}^* \simeq \mathbb{C}$ where the quotient map $\pi\colon \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$ is given by $(s,x) \mapsto x^2$. Finally, $p\colon \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^*/H \simeq \mathbb{C}^*$ is the projection $\mathrm{pr}_{\mathbb{C}^*}$ and so p is a trivial bundle with fiber $\mathbb{C}$.

REMARK 1.2.3. There is an easy criterion to show that a given K-space X is an associated bundle. Assume that there is a K-equivariant map $p\colon X \to K/H$ with some closed subgroup $H \subseteq K$. Then $Y := p^{-1}(eH)$ is an H-space, and we have a canonical homeomorphism

$$\varphi\colon K \times^H Y \xrightarrow{\sim} X,\ [g,y] \mapsto gy.$$

In fact, φ is continuous and bijective, and the inverse map is given by $x \mapsto [p(x), \widetilde{p(x)}^{-1} x]$ where $\widetilde{p(x)} \in K$ is a representative of $p(x)$. We use here again the fact that $K \to K/H$ has local cross sections.

Now we can formulate the local structure theorem for actions of compact groups, see [**Bre72**, Chap. II, Theorem 5.4].

THEOREM 1.2.4. *Let K be a compact Lie groups and X a locally compact K-space. For any $x \in X$ there is a locally closed and K_x-stable subset $S \subseteq X$ containing x such that*

(1) *KS is an open neighborhood of Kx,*
(2) *$K \times^{K_x} Y \to KY$, $[g,y] \mapsto gy$, is a homeomorphism.*

Such an $S \subseteq X$ is called a *slice in* x, and KS is called a *tube about* Kx. The theorem together with Proposition 1.2.1 above tells us that the action of K in a neighborhood of an orbit $O = Kx$ is completely determined by the action of H_x on a slice in x.

2. Flat and Étale Morphisms

In this section we discuss the concept of "local" in algebraic geometry. Since there are no "small" open neighborhoods in the Zariski-topology we will replace them by so-called "étale neighborhoods". For this we have to define étale morphisms and to describe their basic properties. In the smooth case, a morphism is étale in a point if and only if its differential is an isomorphism. In general, one has to ask in addition that the morphism is flat.

In this section, we will use some results from the literature, and we refer to [**Har77**, III.9], [**Mat89**, 3.7 and 8], and [**Eis95**, Section 6] for more details and proofs. Our approach is based on "standard étale morphisms" (Example 2.2.1)

2.1. Unramified and étale morphisms. Let $\varphi\colon X \to Y$ be a morphism, let $x \in X$ and set $y := \varphi(x) \in Y$. Then the morphism φ induces a homomorphism $\varphi^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of local rings, i.e. $\varphi^*(\mathfrak{m}_y) \subseteq \mathfrak{m}_x$.

DEFINITION 2.1.1. The morphism φ is *unramified in* $x \in X$ if $\mathfrak{m}_x = \varphi^*(\mathfrak{m}_y)\mathcal{O}_{X,x}$. More geometrically, this means that x is an isolated point of the fiber $F := \varphi^{-1}(y)$ and F is reduced in x.

Recall that the differential $d\varphi_x\colon T_xX \to T_yY$ vanishes on $T_xF \subseteq T_xX$, and that $T_xF = \ker d\varphi_x$ in case the fiber is reduced in x. It follows that φ is unramified in x if and only if the differential $d\varphi_x$ is injective. A immediate consequence is that an unramified morphism $\varphi\colon X \to Y$ has finite reduced fibers.

EXERCISE 2.1.2. Show that the subset $\{x \in X \mid \varphi \text{ is unramified in } x\} \subseteq X$ is open. (Hint:)

Another important concept is flatness. It will play a central rôle in all what follows. Unfortunately, there is no easy "geometric meaning" of flatness; it is a purely algebraic concept.

DEFINITION 2.1.3. If R is a ring, then an R-module M is called *flat* if the functor $N \mapsto N \otimes_R M$, N an R-module, is left exact. A morphism $\varphi\colon X \to Y$ is called *flat in* $x \in X$ if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\varphi(x)}$-module (with respect to $\varphi^*\colon \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$).

We have the following "Local Criterion for Flatness", see [**Eis95**, Theorem 6.8].

LEMMA 2.1.4. *Let $\varphi\colon X \to Y$ be a morphism, let $x \in X$ and set $y := \varphi(x) \in Y$. Then φ is flat in x if and only if the map $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is injective.*

EXERCISE 2.1.5. Show that the projection $\mathrm{pr}_Y\colon X \times Y \to Y$ is flat.

EXERCISE 2.1.6. If $\varphi\colon X \to Y$ is flat in $x \in X$, then $\varphi^*\colon \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is injective. (Hint: For $h \in \mathfrak{m}_{\varphi(x)}$ denote by $\mathfrak{a}_h \subseteq \mathcal{O}_{Y,\varphi(x)}$ the kernel of $\mu_h : f \mapsto hf$. Then we get an exact sequence $0 \to \mathfrak{a}_h \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \xrightarrow{\mu_h} \mathcal{O}_{X,x}$. Hence $\mu_h|_{\mathcal{O}_{X,x}} = 0$ if and only if $h = 0$.)

Finally, we define étale morphisms which will be the algebraic-geometric replacement for local isomorphisms.

DEFINITION 2.1.7. The morphism $\varphi\colon X \to Y$ is *étale* in $x \in X$ if φ is unramified and flat in x. Equivalently, φ^* induces an isomorphism $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \xrightarrow{\sim} \mathfrak{m}_x$ where $y := \varphi(x)$.

EXAMPLES 2.1.8.
(1) An open immersion $X \hookrightarrow Y$ is étale. (This is clear since $\mathcal{O}_{X,x} = \mathcal{O}_{Y,x}$ for all $x \in X$.)
(2) If $\varphi\colon X \to Y$ is étale in $x \in X$, then the differential $d\varphi_x\colon T_xX \to T_{\varphi(x)}Y$ is an isomorphism.
(Since $\mathfrak{m}_y^n \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \xrightarrow{\sim} \mathfrak{m}_x^n$ for all n (see the following exercise) it follows that $\mathfrak{m}_y/\mathfrak{m}_y^2 \xrightarrow{\sim} \mathfrak{m}_x/\mathfrak{m}_x^2$ is an isomorphism.)
(3) If $\varphi\colon X \to Y$ is étale in $x \in X$ and $y := \varphi(x)$, then X is smooth in x if and only if Y is smooth in y.
(The following exercise implies that the canonical maps $\mathfrak{m}_y^n/\mathfrak{m}_y^{n+1} \to \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1}$ are isomorphisms for all $n \geq 0$. Hence $\mathrm{gr}_{\mathfrak{m}_y} \mathcal{O}_{Y,y} \simeq \mathrm{gr}_{\mathfrak{m}_x} \mathcal{O}_{X,x}$, and the claim follows from Theorem A.4.10.1.)

EXERCISE 2.1.9. If $\varphi\colon X \to Y$ is étale in $x \in X$, then the maps $\mathfrak{m}_y^n \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathfrak{m}_x^n$ are isomorphisms for all $n \geq 0$.

In case X and Y are smooth, there is simple criterion for φ to be étale, see [**Har77**, III. Proposition 10.4 and Exercise 10.3].

Proposition 2.1.10. *Assume that X is smooth in x and Y is smooth in y. Then φ is étale in x if and only if the differential $d\varphi_x\colon T_xX \to T_yY$ is an isomorphism.*

Using the implicit function theorem it follows that an étale morphism between smooth varieties is a local homeomorphism in the $\mathbb{C}$-topology. We will see that this holds in general for any étale morphism, as a consequence of Proposition 2.2.2.

Let us recall some basic properties of flat and étale morphisms. We refer to [**Har77**, III.9], [**Mat89**, 3.7 and 8], and [**Eis95**, Section 6] for more details and proofs.

Lemma 2.1.11.
(1) *Let $\psi\colon X \xrightarrow{\eta} Y \xrightarrow{\varphi} Z$ be a composition. If η and φ are flat (resp. étale), then ψ is flat (resp. étale). If ψ and η are flat (resp. étale) and η is surjective, then φ is flat (resp. étale).*
(2) *If $\varphi\colon X \to Y$ is flat in $x \in X$, then $\varphi^*\colon \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is injective, and for every ideal $\mathfrak{a} \subseteq \mathcal{O}_{Y,\varphi(x)}$ we have $\mathfrak{a}\mathcal{O}_{X,x} \cap \mathcal{O}_{Y,\varphi(x)} = \mathfrak{a}$. In particular, $\mathcal{O}_{X,x}/\mathfrak{a}\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,\varphi(x)}/\mathfrak{a}$ and $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,\varphi(x)}$.*
(3) *For an arbitrary morphism $\varphi\colon X \to Y$ the set of points $x \in X$ where φ is flat (resp. étale) is open in X.*
(4) *A flat morphism $\varphi\colon X \to Y$ is open and equidimensional, i.e., if φ is flat in $x \in X$, then $\dim_x X = \dim_{\varphi(x)} Y + \dim_x \varphi^{-1}(\varphi(x))$.*

Proof. (1) This is an easy exercise which we leave to the reader.

(2) This follows immediately from the definition, see [**Mat89**, Theorem 7.5].

(3) For flatness this is [**Mat89**, Theorem 24.3]. For the étaleness one remarks that the set of points $x \in X$ where the differential $d\varphi_x$ is injective is open, see Exercise 2.1.2.)

(4) See [**Har77**, Chap. III, Exercise 9.1 and Proposition 9.5] or [**Mat89**, Theorem 15.1]. $\square$

A morphism $\varphi\colon X \to Y$ is called *faithfully flat* if it is flat and surjective. If X and Y are affine this is equivalent to the following condition: *A homomorphism $N \to M$ of $\mathcal{O}(Y)$-modules is injective if and only if $\mathcal{O}(X)\otimes_{\mathcal{O}(Y)}N \to \mathcal{O}(X)\otimes_{\mathcal{O}(Y)}M$ is injective.* Here is a useful application, on the level of rings.

Lemma 2.1.12. *Let A be a ring, and let R be an A-algebra. Let B/A be faithfully flat and assume that $B \otimes_A R$ is a finitely generated B-algebra. Then R is finitely generated over A.*

This behavior is usually expressed in the following way. *If an A-algebra R becomes finitely generated under a faithfully flat base change, then R is finitely generated.* We might ask here which other properties of an A-algebra behave in a similar way. E.g. being an integral domain or being reduced are such properties.

Proof. The ring R is the union of finitely generated A-subalgebras R_ν. Since the tensor product commutes with direct limits, $\varinjlim(B \otimes_A R_\nu) \xrightarrow{\sim} B \otimes_A \varinjlim R_\nu = B \otimes_A R$, there is a ν such that $B \otimes_A R_\nu \xrightarrow{\sim} B \otimes_A R$. Since B/A is faithfully flat, this implies that $R_\nu = R$. $\square$

2.2. Standard étale morphisms. The following example gives a general construction of an étale morphism reflecting what we usually have in mind. Unfortunately, the proof is not easy and needs some work.

Example 2.2.1 (standard étale morphism). Let U be an affine variety, and let $F \in \mathcal{O}(U)[t]$ be a monic polynomial. Then the projection onto U induces a morphism $\eta\colon \mathcal{V}_{U\times\mathbb{C}}(F) \to U$, and the following holds:

(1) *The morphism* η *is étale in any* $(u,a) \in \mathcal{V}_{U\times\mathbb{C}}(F)$ *such that* $F'(u,a) \neq 0$, *where* $F' := \frac{dF}{dt} \in \mathcal{O}(U)[t]$.
(2) *Define* $Z := \mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$. *Then* $\mathcal{O}(U)[t]_{F'}/(F) \xrightarrow{\sim} \mathcal{O}(Z)$ *is an isomorphism, i.e. the ideal* $(F) \subseteq \mathcal{O}(U)[t]_{F'}$ *is perfect.*

Proof. (a) The morphism $p\colon \mathcal{V}_{U\times\mathbb{C}}(F) \to U$ is finite and surjective, and $Z \subseteq \mathcal{V}_{U\times\mathbb{C}}(F)$ is open. Set $R := \mathcal{O}(U)[t]_{F'}/(F)$ so that $\mathcal{O}(Z) = R/\sqrt{(0)}$. For any $u \in p(Z)$ we get $R/\mathfrak{m}_u R = \mathbb{C}[t]/(F(u,t)$ and this is a product of copies of $\mathbb{C}$. It follows that $R/\mathfrak{m}_u R = \mathcal{O}(Z)/\mathfrak{m}_u\mathcal{O}(Z)$. Hence $p\colon Z \to U$ has discrete and reduced fibers, and so $p\colon Z \to U$ is unramified. Clearly, R is flat over $\mathcal{O}(U)$. So if we show that R is reduced, then (2) follows, and $\eta\colon Z \to U$ is flat, hence (1).

(b) From (b) we see that $\tilde{\mathfrak{m}}_z := \mathfrak{m}_z R \subseteq R$ is a maximal ideal, and that we get surjective homomorphisms

$$\mathfrak{m}_{p(z)}/\mathfrak{m}_{p(z)}^2 \twoheadrightarrow \mathfrak{m}_z/\mathfrak{m}_z^2 \twoheadrightarrow \tilde{\mathfrak{m}}_z/\tilde{\mathfrak{m}}_z^2.$$

This implies that $R_{\tilde{\mathfrak{m}}_z}$ is a regular local ring in case $p(z) \in U$ is a smooth point. Hence $R_{\tilde{\mathfrak{m}}_z} = \mathcal{O}_{Z,z}$, because a regular local ring is an integral domain.

(c) Now we look at the canonical map $\varphi\colon R \to \prod_{z\in Z'} R_{\tilde{\mathfrak{m}}_z}$ where $Z' := \{z \in Z \mid p(z) \text{ smooth in } U\}$. We want to show that φ is injective which implies that R is reduced. If $r \in \ker\varphi$, then, for every $z \in Z'$, there is an $s_z \notin \tilde{\mathfrak{m}}_z$ such that $s_z r = 0$. This implies that $\operatorname{Ann}(r) \not\subseteq \tilde{\mathfrak{m}}_z$ for all $z \in Z'$. If $r \neq 0$, then $\operatorname{Ann}(r)$ is contained in an associated prime of R. Since every irreducible component of Z contains smooth points, it follows that every minimal prime of R is contained in $\tilde{\mathfrak{m}}_z$ for some $z \in Z'$. So it remains to see that R has no embedded primes, i.e. every zero divisor is contained in a minimal prime.

(d) It suffices to prove this for the algebra $A := \mathcal{O}(U)[t]/(F)$. Let $\mathfrak{p} \subseteq A$ be an associated prime which is not minimal, and let $\mathfrak{p}' \subseteq \mathfrak{p}$ be a minimal prime. Then $\mathfrak{p}' \cap \mathcal{O}(U) \subsetneqq \mathfrak{p} \cap \mathcal{O}(U)$. If $a \in \mathfrak{p} \cap \mathcal{O}(U) \setminus \mathfrak{p}' \cap \mathcal{O}(U)$, then multiplication with a is injective on $\mathcal{O}(U)$, but has a kernel on A. This contradicts the fact that A is flat over $\mathcal{O}(U)$. $\square$

A morphism of the form $\eta\colon Z \to U$ as above is called *a standard étale morphism*. These morphisms have many nice properties, e.g. *a standard étale morphism is a local homeomorphism in the* $\mathbb{C}$*-topology*. In fact, this is obvious for $U = \mathbb{C}^n$ by the implicit function theorem, and using a closed embedding $U \hookrightarrow \mathbb{C}^n$ one gets a fiber product of the form

$$\begin{array}{ccc}
\mathcal{V}_{U\times\mathbb{C}}(F)_{F'} & \xrightarrow{\subseteq} & \mathcal{V}_{\mathbb{C}^n\times\mathbb{C}}(\tilde{F})_{\tilde{F}'} \\
\big\downarrow & & \big\downarrow \\
U & \xrightarrow{\subseteq} & \mathbb{C}^n
\end{array}$$

where $\tilde{F} \in \mathcal{O}(\mathbb{C}^n)[t]$ is a lift of $F \in \mathcal{O}(U)[t]$. Another point is the following. If $F \in \mathcal{O}(U)[t]$ has degree d as a polynomial in t, then the standard étale morphism $\eta\colon \mathcal{V}_{U\times\mathbb{C}}(F)_{F'} \to U$ has also degree d. In particular, if η is injective, then $d = 1$, hence F is linear, and so η is an open immersion. We will see below that this holds for every étale morphism.

The next result shows that every étale morphism is "locally standard".

Proposition 2.2.2. *Let* $\varphi\colon X \to Y$ *be a morphism, and assume that* φ *is étale in* $x_0 \in X$. *Then there is an affine open neighborhood* U *of* $\varphi(x_0)$, *a standard étale morphism* $\eta\colon Z \to U$ *and an open immersion of a neighborhood* V *of* x_0 *into* Z

such that $\varphi(V) \subseteq U$ *and* $\varphi|_V = \eta|_V : V \to U$*:*

$$\begin{tikzcd} X \arrow[d, "\varphi"] & V \arrow[l, hook', "\supseteq"', "open"] \arrow[r, hook, "\subseteq", "open"'] \arrow[d, "\varphi|_V = \eta|_V"'] & Z \arrow[d, "\eta"] \\ Y & U \arrow[l, hook', "\supseteq"', "open"] \arrow[r, equal] & U \end{tikzcd}$$

Let us first recall the following "Local Criterion for Flatness", see [**Eis95**, Theorem 6.8].

LEMMA 2.2.3. *Let* $\varphi\colon X \to Y$ *be a morphism, let* $x \in X$ *and set* $y := \varphi(x) \in Y$*. Then* φ *is flat in* x *if and only if the map* $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ *is injective.*

As a consequence we get the next lemma which will be used in the final part of the proof of Proposition 2.2.2.

LEMMA 2.2.4. *Let* $\mu\colon Z_1 \to Z_2$ *and* $\eta_2\colon Z_2 \to U$ *be morphisms. Assume that* $\eta_1 := \eta_2 \circ \mu$ *is flat in* $z_1 \in Z_1$ *and that* η_2 *is étale in* $z_2 := \mu(z_1)$*. Then* μ *is flat in* z_1*.*

$$\begin{tikzcd} Z_1 \arrow[r, "\mu"] \arrow[dr, "\eta_1"'] & Z_2 \arrow[d, "\eta_2"] \\ & U \end{tikzcd}$$

If η_1 *is étale in* z_1 *(and* η_2 *étale in* z_2*), then* μ *is étale in* z_1*.*

PROOF. Since η_2 is étale in z_2, we get $\mathfrak{m}_u\mathcal{O}_{Z_2,z_2} = \mathfrak{m}_{z_2}$ where $u := \eta_2(z_2)$. It follows that the first map in the composition

$$\mathfrak{m}_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{Z_1,z_1} \to \mathfrak{m}_{z_2} \otimes_{\mathcal{O}_{Z_2,z_2}} \mathcal{O}_{Z_1,x_1} \to \mathcal{O}_{Z_1,z_1}$$

is surjective. Since η_1 is flat in z_1 the composition is injective, hence the second map is injective, and this implies, by the lemma above, that μ is flat in z_1. The second claim follows, because μ is unramified in z_1 in case η_1 is unramified in z_1. $\square$

REMARK 2.2.5. Lemma 2.2.3 has the following generalization, see [**Mat89**, Theorem 22.3]. *Let* $\varphi\colon X \to Y$ *be a morphism, let* $x \in X$ *and set* $y := \varphi(x) \in Y$*, and let* $I \subseteq \mathcal{O}_{Y,y}$ *be an ideal. Then* φ *is flat in* x *if and only if the following holds: (i)* $\mathcal{O}_{X,x}/I\mathcal{O}_{X,x}$ *is flat over* $\mathcal{O}_{Y,y}/I$*, and (ii) the map* $I \otimes \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ *is injective.* This has the following nice application, generalizing Lemma 2.2.4.

PROPOSITION 2.2.6. *Consider the diagram*

$$\begin{tikzcd} X_1 \arrow[r, "\mu"] \arrow[dr, "\varphi_1"'] & X_2 \arrow[d, "\varphi_2"] \\ & Y \end{tikzcd}$$

where φ_1 *is flat. Assume that for every* $y \in Y$ *the induced morphism of the (schematic) fibers* $\varphi_1^{-1}(y) \to \varphi_2^{-1}(y)$ *is flat. Then* μ *is flat.*

PROOF. Choose $x_1 \in X_1$, and put $x_2 := \eta(x_1)$ and $y := \varphi_1(x_1) = \varphi_2(x_2)$. Set $I := \mathfrak{m}_y\mathcal{O}_{X_2,x_2} \subseteq \mathcal{O}_{X_2,x_2}$. Then the local ring of the schematic fiber $\varphi_2^{-1}(y)$ in x_1 is $\mathcal{O}_{X_1,x_1}/I\mathcal{O}_{X_1,x_1}$ which is flat over the local ring $\mathcal{O}_{X_2,x_2}/I$ of the schematic fiber $\varphi_2^{-1}(y)$ in x_2, by assumption. Moreover, $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X_1,x_1} \to I \otimes_{\mathcal{O}_{X_2,x_2}} \mathcal{O}_{X_1,x_1}$ is surjective, and the composition with $\iota\colon I \otimes_{\mathcal{O}_{X_2,x_2}} \mathcal{O}_{X_1,x_1} \to \mathcal{O}_{X_1,x_1}$ is injective, because φ_1 is flat in x_1. Thus ι is injective, and the claim follows from the remark above. $\square$

Proof of Proposition 2.2.2. We can assume that every irreducible component of X contains x_0.

(a) There is an open embedding $X \hookrightarrow \tilde{X}$ and a finite morphism $\tilde{\varphi}\colon \tilde{X} \to Y$ such that $\tilde{\varphi}|_X = \varphi$. Thus we can assume that φ is finite and surjective.

(b) There exists an affine open neighborhood $U \subseteq Y$ of $y_0 := \varphi(x_0)$ and a closed embedding $\rho\colon V := \varphi^{-1}(U) \hookrightarrow U \times \mathbb{C}$ of the form $x \mapsto (\varphi(x), h(x))$ where $h(x_0) = 1$.

(c) There is an $F \in \mathcal{O}(U)[t]$ with the following properties: (i) F vanishes on the image of Y; (ii) $F'(y_0, 1) \neq 0$; (iii) the leading term of F does not vanish in y_0. Localizing U at the leading term of F we can assume that F is monic.

Now we can finish the proof. By (c) we have a closed immersion $V \hookrightarrow \mathcal{V}_{U\times\mathbb{C}}(F)$. Since $F'(y_0, 1) \neq 0$ we can replace V by the open set $V' = V \cap \mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$ containing x_0, and we get a closed immersion $V' \hookrightarrow \mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$. Moreover, the induced morphism $\mathcal{V}_{U\times\mathbb{C}}(F)_{F'} \to U$ is a standard étale map. Thus we are in the situation of Lemma 2.2.4 which implies that the image of V' is open in $\mathcal{V}_{U\times\mathbb{C}}(F)_{F'}$. $\square$

Let us draw some important consequences.

Proposition 2.2.7. (1) *Consider the following fiber product.*

$$\begin{tikzcd} U \times_Y X \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\varphi}"] & X \arrow[d, "\varphi"] \\ U \arrow[r, "\eta"] & Y \end{tikzcd}$$

If η is étale, then the fiber product is reduced and $\tilde{\eta}$ is étale.

(2) *An injective étale morphism is an open immersion.*

Proof. (1) We can assume that X, Y and U are affine. If $X \to Y$ is a standard étale morphism, $\mathcal{O}(X) = \mathcal{O}(Y)[t]_{F'}/(F)$ where $F \in \mathcal{O}(Y)[t]$ is a monic polynomial, then $\mathcal{O}(U) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \simeq \mathcal{O}(U)[t]_{G'}/(G)$ where $G = \eta^*(F)$, hence $U \times_Y X$ is reduced and $U \times_Y X \to U$ is also a standard étale morphism. Now the claim follows from Proposition 2.2.2 above.

(2) We have seen above that an injective standard étale morphism is an open immersion. Hence the claim follows from Proposition 2.2.2 $\square$

2.3. Étale base change. The situation of the first statement of Proposition 2.2.7 above is a special case of the following setup. Let S be a variety, let $p\colon X \to S$ and $q\colon Y \to S$ two S-varieties, and let $\varphi\colon X \to Y$ be an S-morphism, i.e. $q \circ \varphi = p$. If $\eta\colon S' \to S$ is a morphism we obtain S'-varieties $X' := S' \times_S X$ and $Y' := S' \times_S Y$ and an induced S'-morphism $\varphi'\colon X' \to Y'$, as shown in the following diagram:

$$(*) \qquad \begin{tikzcd} X' \arrow[rr, "\eta_X"] \arrow[dr, "\varphi'"] \arrow[ddr, "p'"'] & & X \arrow[dr, "\varphi"] \arrow[ddr, "p"'] & \\ & Y' \arrow[rr, "\eta_Y"] \arrow[d, "q'"] & & Y \arrow[d, "q"] \\ & S' \arrow[rr, "\eta"] & & S \end{tikzcd}$$

This is usually expressed by saying that $\varphi'\colon X' \to Y'$ *is obtained from* $\varphi\colon X \to Y$ *by the base change* $\eta\colon S' \to S$. A basic question is what happens in case of a flat or étale base change. E.g. the first statement of Proposition 2.2.7 above says that for a an étale base change $\eta\colon S' \to S$, the fiber products X' and Y' are reduced and

η_X, η_Y are again étale. We will have more statements of this form later, but let us first prove the following useful result.

LEMMA 2.3.1. *Let $\varphi\colon X \to Y$ be an abstract map between varieties. If $\eta\colon X' \to X$ is an étale and surjective morphism such that the composition $\varphi\circ\eta$ is a morphism, then φ is a morphism.*

$$\begin{tikzcd} X' \arrow[r, "\eta"] \arrow[dr, "\eta\circ\varphi"'] & X \arrow[d, "\varphi"] \\ & Y \end{tikzcd}$$

PROOF. Denote by $\Gamma_\varphi \subseteq X \times Y$ the graph of the map φ. We have to show that Γ_φ is closed and that the induced map $p\colon \Gamma_\varphi \to X$ is an isomorphism. By assumption, the composition $\psi := \eta \circ \varphi$ is a morphism, and we get the following commutative diagram:

$$\begin{tikzcd} \Gamma_\psi \arrow[r, "\subseteq"] \arrow[d, "\gamma:=", "(\eta\times\mathrm{id}_Y)|_{\Gamma_\psi}"'] & X' \times Y \arrow[r, "\mathrm{pr}"] \arrow[d, "\eta\times\mathrm{id}_Y"] & X' \arrow[d, "\eta"] \\ \Gamma_\varphi \arrow[r, "\subseteq"] & X \times Y \arrow[r, "\mathrm{pr}"] & X \end{tikzcd} \tag{12}$$

Since η is surjective we see that $(\eta \times \mathrm{id}_Y)^{-1}(\Gamma_\varphi) = \Gamma_\psi$. It follows that $X \times Y \setminus \Gamma_\varphi$ is the image of the open set $X' \times Y \setminus \Gamma_\psi$ which is open, because $\eta \times \mathrm{id}_Y$ is flat. Hence Γ_φ is closed. Now the outer diagram of (12) is a fiber product, hence γ is étale and surjective, and the induced horizontal map $\Gamma_\psi \to X'$ is an isomorphism. Therefore, $\Gamma_\varphi \to X$ is a bijective étale morphism, by Lemma 2.1.11(1), and thus an isomorphism, by Proposition 2.2.7(2). □

EXAMPLES 2.3.2.
(1) If an S-variety X becomes smooth under an étale surjective base change $S' \to S$, then X is also smooth (see Example 2.1.8(3)).
(2) If an S-morphism $\varphi\colon X \to Y$ becomes an isomorphism under an étale surjective base change $S' \to S$, then φ is an isomorphism. (This follows from the lemma above applied to the map φ^{-1}.)
(3) If a variety X becomes affine under an étale surjective base change, then X is also affine. (The proof is base on the following result. If X is a variety, $S' \to S$ an étale morphism of affine varieties, and $X' := S' \times_S X$ the fiber product, then the canonical map $\mathcal{O}(S')\otimes_{\mathcal{O}(S)}\mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(X')$ is an isomorphism. If X' is affine, then $\mathcal{O}(X)$ is finitely generated (Lemma 2.1.12), and we have a canonical morphism $\varphi\colon X \to \bar{X}$ where $\bar{X}$ is the affine variety with ccordinate ring $\mathcal{O}(X)$. Now the claim follows from (2).)

EXERCISE 2.3.3. Proof the following special case of part (3) of the Example above. Assume that $X = X_1 \cup X_2$ where $X_1, X_2, X_1 \cap X_2 \subseteq X$ are affine open subsets. If $S' \to S$ is a flat morphism of affine varieties, and $X' := S' \times_S X$, then $\mathcal{O}(S')\otimes_{\mathcal{O}(S)}\mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(X')$ is an isomorphism.
(Hint: There is an exact sequence of $\mathcal{O}(S)$-modules

$$0 \to \mathcal{O}(X) \to \mathcal{O}(X_1) \times \mathcal{O}(X_2) \to \mathcal{O}(X_1 \cap X_2)$$

which remains exact after tensoring with $\mathcal{O}(S')$. Since $\mathcal{O}(S') \otimes_{\mathcal{O}(S)} \mathcal{O}(U) \xrightarrow{\sim} \mathcal{O}(S' \times_S U)$ for every affine open set $U \subseteq X$, the claim follows.)

The next example is a special case of the Slice Theorem for finite groups.

EXAMPLE 2.3.4. Let G be a finite group acting on the affine variety X, and denote by $\pi\colon X \to X/G$ the quotient. Define $X' := \{x \in X \mid G_x = \{e\}\}$. Then

(1) *X' is open in X and $\pi(X')$ is open in X/G.*

(2) *The map* $(g,x) \mapsto (x, gx) \colon G \times X' \to X' \times_{\pi(X')} X'$ *is a* G*-equivariant isomorphism:*

$$\begin{array}{ccccc} G \times X' & \xrightarrow[(g,x)\mapsto(x,gx)]{\simeq} & X' \times_{\pi(X')} X' & \longrightarrow & X' \\ \Big\downarrow \mathrm{pr}_{X'} & & \Big\downarrow p & & \Big\downarrow \pi|_{X'} \\ X' & =\!=\!= & X' & \xrightarrow{\pi|_{X'}} & X/G \end{array}$$

(3) *The induce morphism* $\pi|_{X'} \colon X' \to X/G$ *is étale.*

Proof. (1) The first statement is clear since $X \setminus X' = \bigcup_{g \neq e} X^g$.

(2) For any $g \in G$ the morphism $\iota_g \colon X' \to X' \times_{\pi(X')} X'$, $x \mapsto (x, gx)$, is a closed immersion, because $p \circ \iota_g = \mathrm{id}_{X'}$. Hence the fiber product $X' \times_{\pi(X')} X'$ is the disjoint union of copies of X', proving (2).

(3) For the last statement we can embed X as a closed G-stable subset into a representation V of G and thus assume that $X = V$. The following argument was indicated to us by G.W. Schwarz. We claim that for any $x \in V'$ there exist $n := \dim V$ invariant functions $p_1, \ldots, p_n$ vanishing in x such that the differentials $(dp_1)_x, \ldots, (dp_n)_x$ form a basis of the cotangent space $(T_x V)^* = \mathfrak{m}_x/\mathfrak{m}_x^2$. In fact, the following Exercise 2.3.6 shows that for a given cotangent vector $\xi \in (T_x V)^*$ there is an $f \in \mathcal{O}(V)$ vanishing on Gx such that $df_{gx} = g\xi$ for all $g \in G$. It is easy to that gf has the same property for all $g \in G$ and so the invariant $p := 1/|G| \sum_g gf$ vanishes at x and satisfies $dp_x = \xi$.

It follows that the G-invariant morphism $p := (p_1, \ldots, p_n) \colon V \to \mathbb{C}^n$ is unramified in gx for all $g \in G$. Replacing V by a suitable G-stable affine open neighborhood U of x we can assume that the fiber $p^{-1}(p(x))$ is equal to Gx and is therefore reduced. This means that the ideal $I(Gx) \subseteq \mathcal{O}(U)$ is generated by the invariants $p_1, \ldots, p_n$. But then, the maximal idea $\mathfrak{m}_{\pi(x)} = I(Gx) \cap \mathcal{O}(U)^G$ is also generated by $p_1, \ldots, p_n$, showing that $\pi(x)$ is a smooth point of U/G and that $d\pi_x \colon T_x U \to T_{\pi(x)} U/G$ is an isomorphism. Now the claim follows from Proposition 2.1.10. ☐

Exercise 2.3.5. (1) Let $\mathfrak{a} \subseteq \mathcal{O}(X)$ be an ideal. For any $x \notin \mathcal{V}_X(\mathfrak{a})$ we have $\mathfrak{m}_x \cap \mathfrak{a} + \mathfrak{m}_x^2 = \mathfrak{m}_x$, i.e. the map $\mathfrak{m}_x \cap \mathfrak{a} \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is surjective.

(2) Let $x_1, \ldots, x_n \in X$ be n different points. Then the canonical map

$$\mathfrak{m}_{x_1} \cap \mathfrak{m}_{x_2} \cap \cdots \cap \mathfrak{m}_{x_n} \to \bigoplus_i \mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2$$

is surjective.

(Hint: Use (1) with $\mathfrak{a} := \mathfrak{m}_{x_2}^2 \cap \cdots \cap \mathfrak{m}_{x_n}^2$ to show that the image of this map contains $\mathfrak{m}_{x_1}/\mathfrak{m}_{x_1}^2 \oplus (0) \cdots \oplus (0)$.)

Exercise 2.3.6. Use the previous exercise to show that for a finite set of points $x_1, \ldots, x_n \in X$ and cotangent vectors $\xi_i \in (T_{x_i} X)^*$ there is an $f \in \mathcal{O}(X)$ such that $df_{x_i} = \xi_i$ for all $i = 1, \ldots n$.

3. Fiber Bundles and Principal Bundles

Fiber bundles with fiber F are morphisms $\varphi \colon B \to X$ which look locally like $U \times F$. In order to get a useful concept, one has to replace the Zariski-open neighborhoods of a point $x \in X$ by *étale neighborhoods* which are defined as étale morphisms $\eta \colon U \to X$ such that $x \in \eta(U)$. One can define intersections of étale neighborhoods by taking the fiber product, and one can even introduce an *étale topology*.

3.1. Additional structures, s-varieties. In many applications we are dealing with varieties with an additional structure, e.g. a vector space, a quadratic space (i.e. a vector space with a nondegenerate quadratic form), an affine space, an algebraic group G, a G-variety (i.e. a variety with an action of an algebraic group G), or a G-module. These objects will be called *s-varieties*. We will not give a formal definition, but we will need that in all examples it is clear what an isomorphism between two such s-varieties is. In particular, for every s-variety F the automorphism group $\mathrm{Aut}(F)$ is a well-defined subgroup of $\mathrm{Aut}(|F|)$ where $|F|$ denotes the underlying variety.

In the examples above, we see that $\mathrm{Aut}(F) \subseteq \mathrm{Aut}(|F|)$ is a closed subgroup in case $|F|$ is affine. E.g., for a vector space V we have $\mathrm{Aut}(V) = \mathrm{GL}(V)$, for a quadratic space (Q, q) we have $\mathrm{Aut}(Q, q) = O(Q, q)$, and for an affine space A we have get $\mathrm{Aut}(A) = \mathrm{Aff}(A)$, the group of affine transformations. For an affine G-variety X we have $\mathrm{Aut}(X) = \mathrm{Aut}_G(|X|) = \mathrm{Aut}(|X|)^G$, the group of G-equivariant automorphisms of X, and for a G-module M we get $\mathrm{Aut}(M) = \mathrm{GL}(M)^G$.

Remark 3.1.1. In many cases, the s-variety F is determined by the pair $(|F|, \mathrm{Aut}(F))$. This means the following: F is isomorphic to E as an s-variety if and only if there is an isomorphism $\varphi\colon |F| \xrightarrow{\sim} |E|$ of varieties which induces an isomorphism $\mathrm{Aut}(F) \xrightarrow{\sim} \mathrm{Aut}(E)$ by $g \mapsto \varphi \circ g \circ \varphi^{-1}$. A necessary and sufficient condition for this is that $\mathrm{Aut}(F) \subseteq \mathrm{Aut}(|F|)$ is self-normalizing. As an exercise, the interested reader might check that the following subgroups of $\mathrm{Aut}(\mathbb{C}^n)$ are self-normalizing: $\mathrm{GL}_n, \mathrm{O}_n, \mathbb{C}^*$.

Lemma 3.1.2. *Let G be a reductive group acting on an affine variety X. If all invariants are constant, then $\mathrm{End}_G(X)$ and $\mathrm{Dom}_G(X)$ are affine algebraic semigroups, and $\mathrm{Aut}_G(X)$ is an affine algebraic group. Moreover, $\mathrm{Aut}_G(X)$ is closed in $\mathrm{Dom}_G(X)$ and $\mathrm{Dom}_G(X)$ is open in $\mathrm{End}_G(X)$.*

(Here $\mathrm{Dom}(X)$ denotes the semigroups of dominant endomorphisms.)

Proof. First it is clear that $\mathrm{End}_G(X) \subseteq \mathrm{End}(X)$ and $\mathrm{Aut}_G(X) \subseteq \mathrm{Aut}(X)$ are both closed. Since there are no invariants the isotypic components of $\mathcal{O}(X)$ are finite dimensional. This implies that we can find a finite direct sum $W \subseteq \mathcal{O}(X)$ of isotypic components of $\mathcal{O}(X)$ which generates $\mathcal{O}(X)$. Thus, we get an injective morphism $\iota\colon \mathrm{End}_G(X) \hookrightarrow \mathrm{End}_G(W)$, and a commutative diagram

$$\begin{array}{ccccc}
\mathrm{End}_G(X) & \xrightarrow{\iota} & \mathrm{End}_G(W) & \xrightarrow{\subseteq} & \mathrm{Hom}_G(W, \mathcal{O}(X)) \\
\downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} \\
\mathrm{End}(X) & \xrightarrow{\hookrightarrow} & \mathrm{Hom}(W, \mathcal{O}(X)) & = & \mathrm{Hom}(W, \mathcal{O}(X))
\end{array}$$

which shows that ι is a closed immersion. Hence $\mathrm{End}_G(X)$ an algebraic semigroup. Similarly, we see that $\mathrm{Dom}_G(X)$ is an algebraic semigroup and that $\mathrm{Aut}_G(X)$ is an algebraic group. For the remaining claims we use [**FK16**, Proposition 3.2.1] which shows that, for any affine variety X, $\mathrm{Aut}(X)$ is closed in $\mathrm{Dom}(X)$ and $\mathrm{Dom}(X)$ is open in $\mathrm{End}(X)$. $\square$

3.2. Fiber bundles. Let F be an affine s-variety.

Definition 3.2.1. A *fiber bundle over Y with fiber F* is a morphism $p\colon B \to Y$ with the following properties:

(1) Every fiber $p^{-1}(y)$ is an s-variety isomorphic to F;
(2) For every point $y \in Y$ there is an étale neighborhood $\eta\colon U \to Y$ such that $U \times_Y B$ is U-isomorphic to $U \times F$. This means that there is an isomorphism $\varphi_U\colon U \times F \xrightarrow{\sim} U \times_X B$ such that the induced morphisms

$F \xrightarrow{\sim} \{u\} \times F \xrightarrow{\varphi_U} p^{-1}(\eta(u))$ are isomorphisms of s-varieties, for every $u \in U$.

The set of isomorphism classes of fiber bundles over X with fiber F is denoted by $H^1(X, F)$. This is a *pointed set* where the distinguished element $\star$ is the class of the trivial bundle. Note that (2) implies that p is an open morphism and that the fibers of p are reduced.

In our definition, every fiber of p has given the structure of F, by (1), and condition (2) makes sure that this structure is "locally trivial" in the étale topology. This also implies that the s-structure is defined on the bundle B, as one can see in the following examples.

EXAMPLE 3.2.2. (1) If V is a (finite dimensional) vector space and $\mathcal{V} \to Y$ a fiber bundle with fiber V, then the additon $\mathcal{V} \times_Y \mathcal{V} \to \mathcal{V}$ and the scalar multiplication $\mathbb{C} \times \mathcal{V} \to \mathcal{V}$ are morphisms. These fiber bundles are called *vector bundles*. We will see later that the vector bundles are locally trivial in the Zariski-topology.

(2) If G is an algebraic group and $\mathfrak{G} \to Y$ a fiber bundle with fiber G, then the multiplication $\mathfrak{G} \times_Y \mathfrak{G} \to \mathfrak{G}$ and the inverse $\mathfrak{G} \to \mathfrak{G}$ are morphisms. This means that such a fiber bundle is a *group scheme* over Y.

(3) Let F be the affine n-space $\mathbb{A}^n (= \mathbb{C}^n)$ with associated vector space $\mathbb{C}^n$. The usual definition of an affine map between affine spaces shows that $\mathrm{Aut}(\mathbb{A}^n) = \mathrm{Aff}_n$, the groups of affine transformations $x \mapsto Bx + c$ where $B \in \mathrm{GL}_n$ and $c \in \mathbb{C}^n$. A fiber bundle with fiber $\mathbb{A}^n$ will be called an *affine space bundle*.

A stronger condition would be that a fiber bundle is locally trivial in the Zariski-topology. We denote by $H^1_{Zar}(Y, F) \subseteq H^1(Y, F)$ the subset of isomorphism classes of those fiber bundles which are locally trivial in the Zariski-topology.

REMARK 3.2.3. (1) It is clear from the definition that for $b \in B$ and $y := p(b) \in Y$ the tangent map $dp_b\colon T_bB \to T_yY$ is surjective with kernel $\ker dp_b = T_b p^{-1}(y)$. In particular, B is smooth in b if and only if the fiber $p^{-1}(y) \simeq F$ is smooth in b and Y is smooth in y.

(2) If F and Y are both affine varieties, then a fiber bundle $P \to X$ with fiber F is also an affine variety, see Example 2.3.2(3).

EXAMPLE 3.2.4. Let F be an s-variety such that $\mathrm{Aut}(F)$ is trivial. Then every fiber bundle $\varphi\colon B \to Y$ with fiber F is trivial. In fact, for every fiber $p^{-1}(y)$ there is a unique isomorphism $\psi_y\colon p^{-1}(y) \xrightarrow{\sim} F$, and this collection $(\psi_y)_{y \in Y}$ defines a map $\psi\colon B \to F$. We claim that ψ is a morphism and that $(\psi, \varphi)\colon B \to F \times Y$ is an isomorphism. If the bundle $B \to Y$ is trivial over the étale neighborhood $\eta\colon U \to Y$, we get the following commutative diagram:

$$\begin{tikzcd}
F \arrow[r, equal] & F \arrow[r, equal] & F \\
U \times F \arrow[u, "\mathrm{pr}_F"] \arrow[r, "\simeq"] \arrow[d, "\mathrm{pr}_U"'] & U \times_X B \arrow[u, "\psi \circ \eta_B"] \arrow[r, "\eta_B"] \arrow[d] & B \arrow[u, "\psi"'] \arrow[d, "\varphi"] \\
U \arrow[r, equal] & U \arrow[r, "\eta"] & X
\end{tikzcd}$$

Thus, $\psi \circ \eta_B$ is a morphism, and so $\psi|_{\varphi^{-1}(U)}$ is a morphism, by Lemma 2.3.1. The claim follows.

EXAMPLE 3.2.5. Take $F = \mathbb{C}^n$ considered as an Aff_n-variety with the standard action of Aff_n. Then $\mathrm{Aut}(F) = \mathrm{Aut}_{\mathrm{Aff}_n}(\mathbb{C}^n)$ is trivial, because a regular automorphism of $\mathbb{C}^n$ commuting with all affine transformations is trivial.
(To see this, one first shows that a regular automorphism of $\mathbb{C}^n$ commuting with the scalar multiplications is linear. From that the claim follows immediately.)
As a consequence, *every fiber bundle with fiber the* Aff_n*-variety* $\mathbb{C}^n$ *is trivial.*

EXAMPLE 3.2.6. We have mentioned above (Example 5.5.2) that a vector bundle is locally trivial in the Zariski-topology. The same is true if $F = \mathbb{A}^n$ considered as affine n-space. If Y is affine, then every affine space bundle over Y has the structure of a vector bundle, but this does not hold in general. E.g., define $B := \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ where $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal, and let $p\colon B \to \mathbb{P}^1$ be the morphism induced by the projection onto the first factor. Then B is an affine line bundle, trivial over $\mathbb{P}^1 \setminus \{0\}$ and $\mathbb{P}^1 \setminus \{\infty\}$, but it cannot be a line bundle, because B is an affine variety and so p has no section.

A weaker concept is that of a *fibration with fiber* F by what we mean a flat surjective morphism $p\colon B \to X$ with the condition that every fiber is (reduced and) isomorphic to F. A famous unsolved problem here is whether every fibration with fiber $\mathbb{C}^n$ is a fiber bundle, see [**KR14**, Section 5]. This is not the case if the base X is not normal. It is known to be true for $n = 1$ and X normal, and for $n = 2$ and X a smooth curve. In these cases, the bundle is even locally trivial in the Zariski-topology.

REMARK 3.2.7. Assume that the fiber F is a G-variety. *Then one has a canonical G-action on the total space B of every fiber bundle $B \to X$ with fiber F.*
In fact, there is an action of G on every fiber, and therefore a well-defined "abstract" action of G on B which becomes a regular action under an étale base change, by condition (2). Hence, the claim follows from the next lemma.

LEMMA 3.2.8. *Let Z be a variety with an "abstract" action of an algebraic group G. Assume that there is G-variety $\tilde{Z}$ and a surjective étale and G-equivariant morphism $\xi\colon \tilde{Z} \to Z$. Then the action of G on Z is regular.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccc}
\tilde{Y} := G \times \tilde{Z} & \xrightarrow[\simeq]{\tilde{\varphi}} & G \times \tilde{Z} \\
\Big\downarrow{\scriptstyle \mathrm{id}_G \times \xi} & & \Big\downarrow{\scriptstyle \mathrm{id}_G \times \xi} \\
Y := G \times Z & \xrightarrow[\text{bijective}]{\varphi} & G \times Z
\end{array}$$

where $\varphi(g, z) := (g, gz)$ and $\tilde{\varphi}(g, \tilde{z}) := (g, g\tilde{z})$. Then $\mathrm{id}_G \times \xi$ is étale and surjective, and the composition $\varphi \circ (\mathrm{id}_G \times \xi)$ is a morphism, and so the claim follows from Lemma 2.3.1. □

3.3. Principal bundles. An important special case is the following. Take $F := G$, an algebraic group considered as a G-variety where G acts by *right multiplication*. A fiber bundle with fiber G is called a *principal G-bundle*. The usual definition is the following which is equivalent, by Lemma 3.2.8 above.

DEFINITION 3.3.1. Let G be an algebraic group. A *principal G-bundle over X* is a variety P together with a right action by G and a G-invariant morphism $\rho\colon P \to X$ with the following property: For every $x \in X$ there is an étale neighborhood $\eta\colon U \to X$ such that the fiber product $U \times_X P$ is G-isomorphic to $U \times G$ over U.

We denote by $H^1(X, G)$ the set of isomorphism classes of principal G-bundles over X and by $H^1_{Zar}(X, G) \subseteq H^1(X, G)$ the subset of those which are locally trivial in the Zariski-topology.

Note that the principal G-bundle $\rho\colon P \to X$ is trivial, i.e. G-isomorphic to $\mathrm{pr}_X\colon X \times G \to X$ if and only if ρ has a section.

Example 3.3.2. A typical example is the following. Let H be an algebraic group and let $G \subseteq H$ be a closed subgroup. It is known that the left cosets $H/G := \{hG \mid h \in H\}$ form a smooth quasi-projective variety with the usual universal properties, see [**Bor91**, Chap. II, Theorem 6.8]. It follows that the projection $\pi\colon H \to H/G$ is a principal G-bundle. In fact, we have

$$\begin{tikzcd} H \times G \arrow[r, "\simeq", "(h,g)\mapsto[h,hg]"'] \arrow[d, "\mathrm{pr}_H"] & H \times_{H/G} H \arrow[r, "\tilde\pi", "{[h,h']\mapsto h'}"'] \arrow[d] & H \arrow[d, "\pi"] \\ H \arrow[r, equal] & H \arrow[r, "\pi"] & H/G \end{tikzcd}$$

i.e., the fiber product $H \times_{H/G} H$ is G-isomorphic to $H \times G$, hence a trivial principal G-bundle over H. Since the differential $d\pi_h$ is surjective for all $h \in H$, the next lemma shows that for every $h \in H$ there is a locally closed smooth subvariety $S \subseteq H$ such that $p|_S\colon S \to H/G$ is étale. Clearly, $S \times_{H/G} H \simeq S \times G$, and the claim follows.

Lemma 3.3.3. *Let $\varphi\colon X \to Y$ be a morphism of smooth varieties. Assume that $d\varphi_x\colon T_xX \to T_{\varphi(x)}Y$ is surjective for some $x \in X$. Then there is a closed subvariety $S \subseteq X$, containing x and smooth in x, such that $\varphi|_S\colon S \to Y$ is étale in x.*

Proof. We can assume that X and Y are both affine. By assumption, φ^* induces an injection $\mathfrak{m}_y/\mathfrak{m}_y^2 \hookrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$. Thus we can find a subspace $W \subseteq \mathfrak{m}_x$ of dimension $r = \dim X - \dim Y$ such that $\mathfrak{m}_x = W \oplus \varphi^*(\mathfrak{m}_y) \oplus \mathfrak{m}_x^2$. Define $S := \mathcal{V}_X(W) \subseteq X$. Then $\dim S \geq \dim X - r = \dim Y$, by Krull's Theorem, and $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_{x,S}/\mathfrak{m}_{x,S}^2$ is surjective, because $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_{x,S}/\mathfrak{m}_{x,S}^2$ is surjective and W is in the kernel. Since $\dim \mathfrak{m}_y/\mathfrak{m}_y^2 = \dim Y$, it follows that $\dim S = \dim Y$, that S is smooth in x and that $\mathfrak{m}_y/\mathfrak{m}_y^2 \xrightarrow{\sim} \mathfrak{m}_{x,S}/\mathfrak{m}_{x,S}^2$ is an isomorphism. This shows that $\varphi|_S\colon S \to Y$ is étale in x, by Proposition 2.1.10. $\square$

Proposition 3.3.4. *Every principal $\mathbb{C}^+$-bundle over an affine variety is trivial.*

Proof. Let $p\colon P \to X$ be a principal $\mathbb{C}^+$-bundle where X is an affine variety. We know that P is also an affine variety and that the $\mathbb{C}^+$-action on P corresponds to a locally nilpotent derivation $\delta\colon \mathcal{O}(P) \to \mathcal{O}(P)$ such that $\ker \delta = \mathcal{O}(X)$. In particular, δ is an $\mathcal{O}(X)$-module homomorphism. The bundle is trivial, if and only if there is an $f \in \mathcal{O}(P)$ such that $\delta(f) = 1$. Since the principal bundle P becomes trivial over a surjective étale morphism $X' \twoheadrightarrow X$ we can find a faithfully flat extension $R/\mathcal{O}(X)$ such that the induced derivation δ_R on $R \otimes_{\mathcal{O}(X)} \mathcal{O}(P)$ contains the element 1 in its image. Now the claim follows from the next lemma. $\square$

Lemma 3.3.5. *Let $B \to A$ be a faithfully flat homomorphism of finitely generated $\mathbb{C}$-algebras. Let $\varphi\colon M \to N$ be a homomorphism of A-modules and let $N' \subseteq N$ be a submodule. If $B \otimes_A N' \subseteq \varphi(B \otimes_A M)$, then $N' \subseteq \varphi(M)$.*

Proof. By assumption we have

$$\begin{aligned} B \otimes_A ((N' + \varphi(M))/\varphi(M)) &= (B \otimes_A N' + B \otimes_A \varphi(M))/B \otimes_A \varphi(M) \\ &= (B \otimes_A N' + \varphi(B \otimes_A M))/\varphi(B \otimes_A M) = (0), \end{aligned}$$

hence $(N' + \varphi(M))/\varphi(M) = (0)$ by faithful flatness, and the claim follows. $\square$

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Index