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AUTOMORPHISM GROUPS OF AFFINE VARIETIES AND A CHARACTERIZATION OF AFFINE n-SPACE

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Dedicated to Ernest Vinberg
at the occasion of his 80th birthday

Abstract. We show that the automorphism group of affine n-space $\mathbb{A}^n$ determines $\mathbb{A}^n$ up to isomorphism: If X is a connected affine variety such that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $X \simeq \mathbb{A}^n$ as varieties.

We also show that every torus appears as $\mathrm{Aut}(X)$ for a suitable irreducible affine variety X, but that $\mathrm{Aut}(X)$ cannot be isomorphic to a semisimple group. In fact, if $\mathrm{Aut}(X)$ is finite-dimensional and if $X \not\simeq \mathbb{A}^1$, then the connected component $\mathrm{Aut}(X)^\circ$ is a torus.

Concerning the structure of $\mathrm{Aut}(\mathbb{A}^n)$ we prove that any homomorphism $\mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}$ of ind-groups either factors through $\mathrm{jac}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathbb{k}^*$ where jac is the Jacobian determinant, or it is a closed immersion. For $\mathrm{SAut}(\mathbb{A}^n) := \ker(\mathrm{jac}) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ we show that every nontrivial homomorphism $\mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ is a closed immersion.

Finally, we prove that every nontrivial homomorphism $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{SAut}(\mathbb{A}^n)$ is an automorphism, and that φ is given by conjugation with an element from $\mathrm{Aut}(\mathbb{A}^n)$.

1. Introduction and main results

Our base field $\mathbb{k}$ is algebraically closed of characteristic zero. For an affine variety X the automorphism group $\mathrm{Aut}(X)$ has the structure of an *affine ind-group*. We will shortly recall the basic definitions in §2. The classical example is $\mathrm{Aut}(\mathbb{A}^n)$, the group of automorphisms of affine n-space $\mathbb{A}^n = \mathbb{k}^n$.

A fundamental question is how much information about X can be retrieved from $\mathrm{Aut}(X)$. For example, Jelonek shows in [Jel15] that if $\mathrm{Aut}(X)$ is infinite, then X is *uniruled*. Our main result shows that $\mathbb{A}^n$ is completely determined by its automorphism group.

Theorem 1.1. *Let X be a connected affine variety. If $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $X \simeq \mathbb{A}^n$ as varieties.*

It is clear that X has to be connected since the automorphism group does not change if we form the disjoint union of $\mathbb{A}^n$ with a variety Y with trivial automorphism group. Some generalization of this result can be found in [Reg17].

The proof of the theorem will follow from a more general result (Theorem 5.5; see Remark 5.4) where the group $\mathrm{Aut}(\mathbb{A}^n)$ is replaced by the subgroup $\mathcal{U}(\mathbb{A}^n)$ generated by the unipotent elements.

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Another important question is which groups appear as automorphism groups of affine varieties. For finite groups we have the following result due to Jelonek.

Theorem 1.2 ([Jel15, Proposition 7.2]). *For every finite group G and every $n \geq 1$ there is an n-dimensional smooth connected affine variety X such that* $\mathrm{Aut}(X) \simeq G$.

Moreover, there exist surfaces with infinite discrete automorphism groups (see [FK17, Proposition 12.7.1]). As for algebraic groups, we have $\mathrm{Aut}(\mathbb{A}^1) = \mathrm{Aff}_1$, and we will give examples where $\mathrm{Aut}(X)$ is a torus (Example 7.4). But other groups cannot appear as the next result shows.

Theorem 1.3. *Let X be a connected affine variety. If $\dim \mathrm{Aut}(X) < \infty$, then either $X \simeq \mathbb{A}^1$ or the connected component $\mathrm{Aut}(X)^\circ$ is a torus.*

The last results concern the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of affine n-space. This group has a closed normal subgroup $\mathrm{SAut}(\mathbb{A}^n)$ consisting of those automorphisms $\mathbf{f} = (f_1, \dots, f_n)$ whose Jacobian determinant

$$\mathrm{jac}(\mathbf{f}) := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{(i,j)}$$

is equal to 1:

$$\mathrm{SAut}(\mathbb{A}^n) := \ker(\mathrm{jac}\colon \mathrm{Aut}(\mathbb{A}^n) \to \Bbbk^*).$$

For an ind-group $\mathcal{G}$ the tangent space $T_e\mathcal{G}$ carries a canonical structure of a Lie algebra which we denote by $\mathrm{Lie}\mathcal{G}$. For $\mathrm{SAut}(\mathbb{A}^n)$, the Lie algebra can be identified with $\mathrm{Vec}^0(\mathbb{A}^n)$, the vector fields ξ on $\mathbb{A}^n$ with divergence $\operatorname{div}\xi = 0$. This Lie algebra is simple, so one could expect that $\mathrm{SAut}(\mathbb{A}^n)$ is simple as an ind-group. This is claimed in [Sha66, Sha81], but the proofs turned out to be incorrect (see [FK17, section 15]). What we can show here is the following.

Theorem 1.4. *Let $n \geq 2$.*

(1) *Let $\varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups. Then either φ factors through $\mathrm{jac}\colon \mathrm{Aut}(\mathbb{A}^n) \to \Bbbk^*$, or φ is a closed immersion, i.e., the image is closed and isomorphic to $\mathrm{Aut}(\mathbb{A}^n)$ under φ.*

(2) *Every nontrivial homomorphism $\mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ of ind-groups is a closed immersion.*

This theorem has the following interesting applications. By definition, a *representation of an ind-group $\mathcal{G}$ on a vector space V of countable dimension* is a homomorphism $\mathcal{G} \to \mathrm{GL}(V)$ such that the corresponding map $\mathcal{G} \times V \to V$ is a morphism of ind-varieties (see §3). An *action of an ind-group $\mathcal{G}$ on an affine variety X* is a homomorphism $\mathcal{G} \to \mathrm{Aut}(X)$ of ind-groups.

Corollary 1.5. *Assume that $n \geq 2$.*

(1) *The ind-group $\mathrm{SAut}(\mathbb{A}^n)$ does not have a nontrivial finite-dimensional representation.*

(2) *Assume that $\mathrm{SAut}(\mathbb{A}^n)$ acts nontrivially on a connected affine variety X. Then the action is faithful, and there are no fixed points.*

Proof. (1) Let $\rho\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{GL}(V)$ be a finite-dimensional representation. If ρ is nontrivial, then it is a closed immersion, by Theorem 1.4(2). This is impossible, because $\mathrm{GL}(V)$ is finite-dimensional.

(2) We have a nontrivial homomorphism $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{Aut}(X)$ which is a closed immersion, by Theorem 1.4(2). Thus the action is faithful, and the same is true for the induced action of $\mathrm{SL}_n \subseteq \mathrm{SAut}(\mathbb{A}^n)$. Since X is connected, it follows that SL_n acts nontrivially on every irreducible component of X. This implies that for every fixed

point $x \in X^{\mathrm{SL}_n}$ the tangent representation of SL_n on $T_x X$ is nontrivial. Hence, the tangent representation of $\mathrm{SAut}(\mathbb{A}^n)$ on every fixed point of $\mathrm{SAut}(\mathbb{A}^n)$ is also nontrivial, contradicting (1). $\square$

It is shown in [BKY12] that every automorphism of the ind-group $\mathrm{Aut}(\mathbb{A}^n)$ is inner, i.e., given by conjugation with a suitable $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$ (cf. [FK17, Theorem 12.5.2]). This can be generalized in the following way.

Theorem 1.6.
(1) *Every injective homomorphism* $\varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}(\mathbb{A}^n)$ *is an isomorphism, and* $\varphi = \mathrm{Int}\,\mathbf{g}$ *for a well-defined* $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$.
(2) *Every nontrivial homomorphism* $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{SAut}(\mathbb{A}^n)$ *is an isomorphism, and* $\varphi = \mathrm{Int}\,\mathbf{g}$ *for a well-defined* $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$.

Remark 1.7. The analogue of Theorem 1.6 for vector fields, namely that *every injective homomorphism* $\varphi\colon \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ *of Lie algebras is an automorphism*, would imply the Jacobian Conjecture in dimension n; see [KR17, Corollary 4.4].

We finally mention the following example showing that bijective homomorphisms of ind-groups are not necessarily isomorphisms. The details can be found in [FK17, section 8]; cf. [BW00, section 11, last paragraph]. Denote by $\Bbbk\langle x, y\rangle$ the free associative $\Bbbk$-algebra in two generators. Then $\mathrm{Aut}(\Bbbk\langle x, y\rangle)$ is an ind-group, and we have a canonical homomorphism

$$\pi\colon \mathrm{Aut}(\Bbbk\langle x, y\rangle) \to \mathrm{Aut}(\Bbbk[x, y]).$$

Proposition 1.8. *The map* $\pi\colon \mathrm{Aut}(\Bbbk\langle x, y\rangle) \to \mathrm{Aut}(\Bbbk[x, y])$ *is a bijective homomorphism of ind-groups, but it is not an isomorphism, because it is not an isomorphism on the Lie algebras.*

Note that $\mathrm{Aut}(\Bbbk\langle x, y\rangle)$ is generated by the closed algebraic subgroups $G \subseteq \mathrm{Aut}(\Bbbk\langle x, y\rangle)$, and that $\pi\colon G \xrightarrow{\sim} \pi(G)$ is an isomorphism for these subgroups.

2. Notation and preliminary results

The notion of an ind-group goes back to Shafarevich who called these objects *infinite-dimensional groups*; see [Sha66, Sha81]. We refer to [Kum02] and the notes [FK17] for basic notation in this context.

Definition 2.1. An *ind-variety* $\mathcal{V}$ is a set together with an ascending filtration $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots \subseteq \mathcal{V}$ such that the following holds:
(1) $\mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}_k$;
(2) each $\mathcal{V}_k$ has the structure of an algebraic variety;
(3) for all $k \in \mathbb{N}$ the inclusion $\mathcal{V}_k \hookrightarrow \mathcal{V}_{k+1}$ is closed immersion.

A *morphism* between ind-varieties $\mathcal{V} = \bigcup_k \mathcal{V}_k$ and $\mathcal{W} = \bigcup_m \mathcal{W}_m$ is a map $\varphi\colon \mathcal{V} \to \mathcal{W}$ such that, for every k, there is an m with the properties that $\varphi(\mathcal{V}_k) \subseteq \mathcal{W}_m$ and that the induced map $\mathcal{V}_k \to \mathcal{W}_m$ is a morphism of varieties. *Isomorphisms* of ind-varieties are defined in the usual way.

Two filtrations $\mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}_k$ and $\mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}'_k$ are called *equivalent* if, for any k, there is an m such that $\mathcal{V}_k \subseteq \mathcal{V}'_m$ is a closed subvariety as well as $\mathcal{V}'_k \subseteq \mathcal{V}_m$. Equivalently, the identity map $\mathrm{id}\colon \mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}_k \to \mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}'_k$ is an isomorphism of ind-varieties.

An ind-variety $\mathcal{V}$ has a natural *topology* where $S \subseteq \mathcal{V}$ is open, respectively, closed, if $S_k := S \cap \mathcal{V}_k \subseteq \mathcal{V}_k$ is open, respectively, closed for all k. Obviously, a locally closed subset $S \subseteq \mathcal{V}$ has a natural structure of an ind-variety. It is called an *ind-subvariety*. An ind-variety $\mathcal{V}$ is called *affine* if all $\mathcal{V}_k$ are affine. A subset $X \subseteq \mathcal{V}$ is called *algebraic* if it is locally closed and contained in some $\mathcal{V}_k$. Such an X has a natural structure of an algebraic variety.

Example 2.2. (1) Any $\Bbbk$-vector space V of countable dimension carries the structure of an (affine) ind-variety by choosing an increasing sequence of finite-dimensional subspaces V_k such that $V = \bigcup_k V_k$. Clearly, all these filtrations are equivalent.

(2) If R is a commutative $\Bbbk$-algebra of countable dimension, $\mathfrak{a} \subseteq R$ a subspace, e.g., an ideal, and $S \subseteq \Bbbk[x_1, \ldots, x_n]$ a set of polynomials, then the subset

$$\big\{(a_1, \ldots, a_n) \in R^n \mid f(a_1, \ldots, a_n) \in \mathfrak{a} \text{ for all } f \in S\big\} \subseteq R^n$$

is a closed ind-subvariety of R^n.

For any ind-variety $\mathcal{V} = \bigcup_{k\in\mathbb{N}} \mathcal{V}_k$ we can define the *tangent space* in $x \in \mathcal{V}$ in the obvious way. We have $x \in \mathcal{V}_k$ for $k \geq k_0$, and $T_x\mathcal{V}_k \subseteq T_x\mathcal{V}_{k+1}$ for $k \geq k_0$, and then define

$$T_x\mathcal{V} := \varinjlim_{k\geq k_0} T_x\mathcal{V}_k$$

which is a vector space of countable dimension. A morphism $\varphi\colon \mathcal{V} \to \mathcal{W}$ induces linear maps $d\varphi_x\colon T_x\mathcal{V} \to T_{\varphi(x)}\mathcal{W}$ for every $x \in X$. Clearly, for a $\Bbbk$-vector space V of countable dimension and for any $v \in V$ we have $T_vV = V$ in a canonical way.

The *product* of two ind-varieties is defined in the obvious way. This allows us to define an *ind-group* as an ind-variety $\mathcal{G}$ with a group structure such that multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G}\colon (g,h) \mapsto g\cdot h$, and inverse $\mathcal{G} \to \mathcal{G}\colon g \mapsto g^{-1}$, are both morphisms.

Remark 2.3. Let $G \subseteq \mathcal{G}$ be a subgroup. If G is an algebraic subset, i.e., locally closed and contained in $\mathcal{G}_k$ for some k, then G is an algebraic group and is closed in $\mathcal{G}$. We will call such a G an *algebraic subgroup.*

Conversely, if G is an algebraic group and $\varphi\colon G \to \mathcal{G}$ a homomorphism of ind-groups, then $\varphi(G) \subseteq \mathcal{G}$ is a closed subgroup and an algebraic subset. The easy proofs are left to the reader.

If $\mathcal{G}$ is an affine ind-group, then $T_e\mathcal{G}$ has a natural structure of a Lie algebra which will be denoted by $\operatorname{Lie}\mathcal{G}$. The structure is obtained by showing that every $A \in T_e\mathcal{G}$ defines a unique left-invariant vector field δ_A on $\mathcal{G}$; see [Kum02, Proposition 4.2.2, p. 114].

Definition 2.4. An ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *discrete* if $\mathcal{G}_k$ is finite for all k. Clearly, $\mathcal{G}$ is discrete if and only if $\operatorname{Lie}\mathcal{G}$ is trivial.

The next result can be found in [FK17, sections 4.1 and 4.6]. Here $\operatorname{Vec}(X)$ denotes the Lie algebra of (algebraic) vector fields on X, i.e., $\operatorname{Vec}(X) = \operatorname{Der}(\mathcal{O}(X))$, the Lie algebra of derivations of $\mathcal{O}(X)$.

Proposition 2.5. *Let X be an affine variety. Then $\operatorname{Aut}(X)$ has a natural structure of an affine ind-group, and there is a canonical embedding $\xi\colon \operatorname{Lie}\operatorname{Aut}(X) \hookrightarrow \operatorname{Vec}(X)$ of Lie algebras.*

Remark 2.6. For $X = \mathbb{A}^n$ the embedding ξ identifies $\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$ with $\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)$, the vector fields

$$\delta = \sum_i f_i \frac{\partial}{\partial x_i}$$

with constant divergence

$$\operatorname{div}\delta := \sum_i \frac{\partial f_i}{\partial x_i} \in \Bbbk;$$

see [FK17, Proposition 4.9.1].

The Jacobian determinant

$$\operatorname{jac}(\mathbf{f}) := \det\left(\frac{\partial f_i}{\partial x_j}\right)_{(i,j)}$$

of an automorphism $\mathbf{f} = (f_1, \dots, f_n)$ of $\mathbb{A}^n$ defines a homomorphism

$$\operatorname{jac}\colon \operatorname{Aut}(\mathbb{A}^n) \to \Bbbk^*$$

of ind-groups. Setting $\operatorname{SAut}(\mathbb{A}^n) := \ker \operatorname{jac}$ one sees that ξ identifies $\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n)$ with $\operatorname{Vec}^0(\mathbb{A}^n)$, the vector fields δ with $\operatorname{div}\delta = 0$; see [FK17, Remark 4.9.3].

It is known that for $n \geq 2$ the Lie algebra $\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n)$ is simple and that $\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n) \subseteq \operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$ is the only proper ideal; see [Sha81, Lemma 3]. Moreover, both Lie algebras are generated by the subalgebras $\operatorname{Lie} G$ where G is an algebraic subgroup.

Another result which we will need is proved in [FK17, Proposition 2.7.6].

Proposition 2.7. *Let $\varphi, \psi\colon \mathcal{G} \to \mathcal{H}$ be two homomorphisms of ind-groups. Assume that $\mathcal{G}$ is connected and that $d\varphi_e = d\psi_e\colon \operatorname{Lie}\mathcal{G} \to \operatorname{Lie}\mathcal{H}$. Then $\varphi = \psi$.*

A final result which we will use can be found in [KRZ17]. Denote by $\operatorname{Aff}_n \subseteq \operatorname{Aut}(\mathbb{A}^n)$ *the subgroup of affine transformations*, i.e., $\operatorname{Aff}_n = \operatorname{GL}_n(\Bbbk) \ltimes (\Bbbk^n)^+$. Similarly, the subgroup $\operatorname{SAff}_n \subseteq \operatorname{Aff}_n$ consists of the affine transformations with determinant 1, i.e., $\operatorname{SAff}_n = \operatorname{SL}_n(\Bbbk) \ltimes (\Bbbk^n)^+$.

Proposition 2.8. *Let X be a connected affine variety with a faithful action of SAff_n. If $\dim X \leq n$, then X is SAff_n-isomorphic to $\mathbb{A}^n$.*

Remark 2.9. It is shown in [KRZ17] that the same holds if we replace SAff_n by Aff_n. Using Theorem 1.6 we see that we can replace SAff_n by $\operatorname{Aut}(\mathbb{A}^n)$ or $\operatorname{SAut}(\mathbb{A}^n)$ as well.

3. The adjoint representation

Following [Kum02, section 4.2] we define a *representation of an ind-group* $\mathcal{G}$ on a vector space V of countable dimension to be a homomorphism $\rho\colon \mathcal{G} \to \operatorname{GL}(V)$ of groups such that the induced map $\mathcal{G} \times V \to V$ is a morphism of ind-varieties. Note that $\operatorname{GL}(V)$ does not have the structure of an ind-variety if $\dim V = \infty$. However, if L is a finitely generated Lie algebra, then $\operatorname{Aut}_{\mathrm{Lie}}(L)$ has a natural structure of an ind-group which is defined in the following way (see [FK17, section 7] where we define an ind-group structure on $\operatorname{Aut}(R)$ for any finitely generated general algebra R, i.e., a $\Bbbk$-vector space R endowed with a bilinear map $R \times R \to R$).

Choose a finite-dimensional subspace $L_0 \subseteq L$ which generates L as a Lie algebra. Then the restriction map $\operatorname{End}_{\mathrm{Lie}}(L) \to \operatorname{Hom}(L_0, L)$ is injective and the image is a closed affine ind-subvariety. (To see this write L as the quotient of the free Lie algebra $F(L_0)$ over L_0 modulo an ideal I.) Choosing a filtration $L = \bigcup_{k\geq 0} L_k$ by finite-dimensional subspaces, we set $\operatorname{End}_{\mathrm{Lie}}(L)_k := \{\alpha \in \operatorname{End}_{\mathrm{Lie}}(L) \mid \alpha(L_0) \subseteq L_k\}$ which is a closed subvariety of $\operatorname{Hom}(L_0, L_k)$ (see Example 2.2). Then we define the ind-structure on $\operatorname{Aut}_{\mathrm{Lie}}(L)$ by identifying $\operatorname{Aut}_{\mathrm{Lie}}(L)$ with the closed subset

$$\bigl\{(\alpha,\beta) \in \operatorname{End}_{\mathrm{Lie}}(L) \times \operatorname{End}_{\mathrm{Lie}}(L) \mid \alpha\circ\beta = \beta\circ\alpha = \mathrm{id}_L\bigr\} \subseteq \operatorname{End}_{\mathrm{Lie}}(L) \times \operatorname{End}_{\mathrm{Lie}}(L),$$

i.e.,

$$\operatorname{Aut}_{\mathrm{Lie}}(L)_k := \bigl\{\alpha \in \operatorname{Aut}_{\mathrm{Lie}}(L) \mid \alpha, \alpha^{-1} \in \operatorname{End}_{\mathrm{Lie}}(L)_k\bigr\}.$$

It follows that $\operatorname{Aut}_{\mathrm{Lie}}(L)$ is an affine ind-group with the usual functorial properties. In particular, we have the following result.

Lemma 3.1. *Let $\mathcal{G}$ be an ind-group, and let $\rho\colon \mathcal{G} \to \operatorname{Aut}_{\mathrm{Lie}}(L)$ be an abstract homomorphism where L is a finitely generated Lie algebra. Then ρ is a homomorphism of ind-groups if and only if ρ is a representation, i.e., the map $\rho\colon \mathcal{G} \times L \to L$ is a morphism of ind-varieties.*

Proof. Assume that L is generated by the finite-dimensional subspace $L_0 \subseteq L$. If $\mathcal{G} = \bigcup_j \mathcal{G}_j$ and if $\rho\colon \mathcal{G} \times L \to L$ is a morphism, then, for any j, there is a $k = k(j)$ such that $\rho(\mathcal{G}_j \times L_0) \subseteq L_k$ and $\rho(\mathcal{G}_j^{-1} \times L_0) \subseteq L_k$. Hence, $\rho(\mathcal{G}_j) \subseteq \operatorname{Aut}_{\text{Lie}}(L)_k$, and the map $\mathcal{G}_j \to \operatorname{Hom}(L_0, L_k)$ is clearly a morphism.

Now assume that $\mathcal{G} \to \operatorname{Aut}_{\text{Lie}}(L)$ is a homomorphism of ind-groups. Then, for any j, there is a $k = k(j)$ such that $\rho(\mathcal{G}_j) \subseteq \operatorname{Aut}_{\text{Lie}}(L)_k \hookrightarrow \operatorname{Hom}(L_0, L_k)$. Hence, $\rho(\mathcal{G}_j \times L_0) \subseteq L_k$, and $\mathcal{G}_j \times L_0 \to L_k$ is a morphism. □

The *adjoint representation* $\operatorname{Ad}\colon \mathcal{G} \to \operatorname{Aut}_{\text{Lie}}(\operatorname{Lie}\mathcal{G})$ of an ind-group $\mathcal{G}$ is defined in the usual way: $\operatorname{Ad} g := (d \operatorname{Int} g)_e \colon \operatorname{Lie}\mathcal{G} \xrightarrow{\sim} \operatorname{Lie}\mathcal{G}$ where $\operatorname{Int} g$ is the inner automorphism $h \mapsto ghg^{-1}$.

Proposition 3.2. *For any ind-group $\mathcal{G}$ the canonical map $\operatorname{Ad}\colon \mathcal{G} \to \operatorname{Aut}_{\text{Lie}}(\operatorname{Lie}\mathcal{G})$ is a homomorphism of ind-groups.*

Proof. Let $\gamma\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denote the morphism $(g,h) \mapsto ghg^{-1}$. For any $g \in \mathcal{G}$, the map $\gamma_g\colon \mathcal{G} \to \mathcal{G}$, $h \mapsto ghg^{-1}$, is an isomorphism of ind-groups, and its differential $\operatorname{Ad}(g) = (d\gamma_g)_e \colon \operatorname{Lie}\mathcal{G} \to \operatorname{Lie}\mathcal{G}$ is an isomorphism of Lie algebras. If $\mathcal{G} = \bigcup_k \mathcal{G}_k$, then for any $p, q \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $\gamma\colon \mathcal{G}_p \times \mathcal{G}_p \to \mathcal{G}_m$. Clearly, for $g \in \mathcal{G}_p$, $\operatorname{Ad} g$ is given by $(d\gamma_g)_e \colon T_e\mathcal{G}_q \to T_e\mathcal{G}_m$, and the map $\mathcal{G}_k \to \operatorname{Hom}(T_e\mathcal{G}_q, T_e\mathcal{G}_m)$ is a morphism, by the following lemma. Now the claim follows from Lemma 3.1 above. □

Lemma 3.3. *Let $\Phi\colon X \times Y \to Z$ be a morphism of affine varieties and set $\Phi_x(y) := \Phi(x,y)$. Assume that there exist $y_0 \in Y$ and $z_0 \in Z$ such that $\Phi_x(y_0) = z_0$ for all $x \in X$. Then the induced map $X \to \operatorname{Hom}(T_{y_0}Y, T_{z_0}Z)$, $x \mapsto d_{y_0}\Phi_x$, is a morphism.*

Proof. We can assume that Y, Z are vector spaces, $Y = W$ and $Z = V$. Choose bases $(w_1, \ldots, w_m)$ of W and $(v_1, \ldots, v_n)$ of V. Then Φ is given by an element of the form

$$\sum_{i=1}^{n} \sum_j f_{ij} \otimes h_{ij} \otimes v_i, \quad \text{where} \quad f_{ij} \in \mathcal{O}(X) \text{ and } h_{ij} \in \mathcal{O}(Y) = \Bbbk[y_1, \ldots, y_m],$$

and so the differential $(d\Phi_x)_{y_0} \colon W \to V$ is given by the matrix

$$\left(\sum_j f_{ij}(x) \frac{\partial h_{ij}}{\partial y_k}(y_0) \right)_{(i,k)}$$

whose entries are regular functions on x. The claim follows. □

We have shown in [KR17] that the adjoint representation

$$\operatorname{Ad}_{\operatorname{Aut}(\mathbb{A}^n)} \colon \operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \operatorname{Aut}_{\text{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n))$$

and the induced homomorphism

$$\rho\colon \operatorname{Aut}_{\text{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)) \xrightarrow{\sim} \operatorname{Aut}_{\text{Lie}}(\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n))$$

are both bijective. They are also homomorphisms of ind-groups: for $\operatorname{Ad}_{\operatorname{Aut}(\mathbb{A}^n)}$ this is Proposition 3.2 above, and for ρ it is obvious. But this does not necessarily imply that the maps are isomorphisms of ind-groups; see Proposition 1.8. However, for $\operatorname{Aut}(\mathbb{A}^n)$ it is true, and we will need this for the proof of Theorem 1.4 in the following section.

Proposition 3.4. *The adjoint representation*

$$\operatorname{Ad}_{\operatorname{Aut}(\mathbb{A}^n)} \colon \operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \operatorname{Aut}_{\text{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n))$$

is an isomorphism of ind-groups.

Proof. We will use here the identification of $\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$ with $\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)$; see Remark 2.6. Put $\partial_{x_i} := \frac{\partial}{\partial x_i}$.

Let $\mathbf{f} = (f_1, \dots, f_n) \in \operatorname{Aut}(\mathbb{A}^n)$ and set $\theta := \operatorname{Ad}(\mathbf{f}^{-1}) \in \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n))$. Then the matrix $(\theta(\partial_{x_k})x_j)_{(j,k)}$ is invertible, and

$$(*) \qquad (\theta(\partial_{x_k})x_j)^{-1}_{(j,k)} = \operatorname{Jac}(\mathbf{f}) = \left(\frac{\partial f_j}{\partial x_i}\right)_{(i,j)};$$

see [KR17, Remark 4.2]. We now claim that the map

$$\theta \mapsto (\theta(\partial_{x_k})x_j)^{-1}_{(j,k)} \colon \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)) \to \mathrm{M}_n(\Bbbk[x_1, \dots, x_n])$$

is a well-defined morphism of ind-varieties. In fact, $\theta \mapsto \theta(\partial_{x_k})x_j$ is the composition of the orbit map $\theta \mapsto \theta(\partial_{x_k})\colon \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)) \to \operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)$ and the evaluation map $\delta \mapsto \delta(x_j)\colon \operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n) \to \Bbbk[x_1, \dots, x_n]$, hence $\theta \mapsto \Theta := (\theta(\partial_{x_k})x_j)_{(j,k)}$ is a morphism. Since $\operatorname{jac}(\Theta) \in \Bbbk^*$ the claim follows.

Now recall that the gradient

$$\Bbbk[x_1, \dots, x_n] \to \Bbbk[x_1, \dots, x_n]^n, \quad f \mapsto \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right),$$

defines an isomorphism

$$\gamma\colon \Bbbk[x_1, \dots, x_n]_{\geq 1} \xrightarrow{\sim} \Gamma := \left\{(h_1, \dots, h_n) \mid \frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i} \text{ for all } i < j\right\}.$$

It follows from $(*)$ that the rows of the matrix $(h_{ij})_{(i,j)} := (\theta(\partial_{x_k})x_j)^{-1}_{(j,k)}$ belong to Γ, so that we get a morphism

$$\psi\colon \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)) \to \Bbbk[x_1, \dots, x_n]^n, \quad \theta \mapsto (f_1, \dots, f_n),$$

where $f_i := \gamma^{-1}(h_{i1}, \dots, h_{in}) \in \Bbbk[x_1, \dots, x_n]_{\geq 1}$. By construction, we have

$$(**) \qquad \psi(\theta) = \psi(\operatorname{Ad}(\mathbf{f}^{-1})) = \mathbf{f}_0 := \big(f_1 - f_1(0), \dots, f_n - f_n(0)\big) = \mathbf{t}_{-\mathbf{f}(0)} \circ \mathbf{f},$$

where $\mathbf{t}_a$ is the translation $v \mapsto v + a$. Let $S \subseteq \operatorname{Aff}_n$ be the subgroup of translations, and set $\widetilde{S} := \operatorname{Ad}(S)$. Then $\widetilde{S} \subseteq \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n))$ is a closed algebraic subgroup and $\operatorname{Ad}\colon S \to \widetilde{S}$ is an isomorphism. It follows from $(**)$ that

$$\operatorname{Ad}(\psi(\theta)) \cdot \theta = \operatorname{Ad}(\mathbf{t}_{-\mathbf{f}(0)}) \in \widetilde{S},$$

and so

$$\widetilde{\psi}(\theta) := \psi(\theta)^{-1} \cdot (\operatorname{Ad}|_S)^{-1}(\operatorname{Ad}(\psi(\theta)) \cdot \theta)$$

is a well-defined morphism $\widetilde{\psi}\colon \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}^{\mathrm{c}}(\mathbb{A}^n)) \to \operatorname{Aut}(\mathbb{A}^n)$ with the property that

$$\operatorname{Ad}(\widetilde{\psi}(\theta)) = \operatorname{Ad}(\psi(\theta)^{-1}) \cdot \operatorname{Ad}(\psi(\theta)) \cdot \theta = \theta.$$

Thus $\operatorname{Ad}\colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n))$ is an isomorphism, with inverse $\widetilde{\psi}$. □

Remark 3.5. Clearly, the restriction

$$\rho\colon \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)) \to \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n))$$

is a homomorphism of ind-groups, and it is bijective; see [KR17]. It follows from (1) that the composition $\rho \circ \operatorname{Ad}\colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n))$ is a bijective homomorphism of ind-groups. Now we use Theorem 1.4(1) to conclude that $\rho \circ \operatorname{Ad}$ is an isomorphism, hence ρ is an isomorphism, too.

4. Proof of the Theorems 1.4 and 1.6

Proof of Theorem 1.4. (1) Let $\varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups such that $d\varphi$ is injective. We can assume that $\mathcal{G} = \overline{\varphi(\mathrm{Aut}(\mathbb{A}^n))}$, and we will show that φ is an isomorphism. The basic idea is to construct a homomorphism $\psi\colon \mathcal{G} \to \mathrm{Aut}(\mathbb{A}^n)$ such that $\psi \circ \varphi = \mathrm{id}$. By Proposition 4.1 below this implies that φ is a closed immersion, hence an isomorphism.

Denote by $L \subseteq \mathrm{Lie}\mathcal{G}$ the image of $d\varphi$. For any $g \in \mathrm{Aut}(\mathbb{A}^n)$ we have

$$d\varphi \circ \mathrm{Ad}(g) = \mathrm{Ad}(\varphi(g)) \circ d\varphi.$$

In particular, L is stable under $\mathrm{Ad}(\varphi(g))$, hence stable under $\mathrm{Ad}(\mathcal{G})$, because $\varphi(\mathrm{Aut}(\mathbb{A}^n))$ is dense in $\mathcal{G}$. Thus we get the following commutative diagram of homomorphisms of ind-groups

$$\begin{array}{ccc}
\mathrm{Aut}(\mathbb{A}^n) & \xrightarrow{\varphi} & \mathcal{G} \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{Ad}_{\mathcal{G}}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))) & \xrightarrow{\simeq} & \mathrm{Aut}_{\mathrm{Lie}}(L),
\end{array}$$

where the first vertical map is an isomorphism, by Proposition 3.4. Thus, the composition $\mathrm{Ad}_{\mathcal{G}} \circ \varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(L) \simeq \mathrm{Aut}(\mathbb{A}^n)$ is an isomorphism, and so φ is also an isomorphism, by Proposition 4.1 below.

If $d\varphi$ is not injective, then $\ker d\varphi \supseteq \mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ (Remark 2.6) and so $d\varphi = f \circ d\,\mathrm{jac}$ where $f\colon \Bbbk \to \mathrm{Lie}\mathcal{G}$ is a Lie algebra homomorphism. If $\Bbbk^* \subseteq \mathrm{GL}_n(\Bbbk)$ denotes the center, then $\varphi|_{\Bbbk^*}\colon \Bbbk^* \to \mathcal{G}$ factor through $?^n\colon \Bbbk^* \to \Bbbk^*$, because $\mathrm{SL}_n(\Bbbk) \subseteq \ker\varphi$, i.e., $\varphi(z) = \rho(z^n)$ for any $z \in \Bbbk^*$ and a suitable homomorphism $\rho\colon \Bbbk^* \to \mathcal{G}$ of ind-groups. By construction, $d\rho_e = f\colon \Bbbk \to \mathrm{Lie}\mathcal{G}$, and so the two homomorphisms φ and $\rho \circ \mathrm{jac}$ have the same differential. Thus, by Proposition 2.7, we get $\varphi = \rho \circ \mathrm{jac}$, and we are done.

(2) Let $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups. If $d\varphi_e$ is not injective, then $d\varphi_e$ is the trivial map (Remark 2.6), hence $d\varphi_e = d\bar{\varphi}_e$ where $\bar{\varphi}\colon \mathbf{g} \mapsto e$ is the constant homomorphism. Again by Proposition 2.7 we get $\varphi = \bar{\varphi}$.

If $d\varphi_e$ is injective, set $L := d\varphi_e(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)) \subseteq \mathrm{Lie}\mathcal{G}$. As above we can assume that $\mathcal{G} = \overline{\varphi(\mathrm{SAut}(\mathbb{A}^n))}$. Since L is stable under $\mathrm{Ad}\,\varphi(\mathbf{g})$ for all $\mathbf{g} \in \mathrm{SAut}(\mathbb{A}^n)$ it is also stable under $\mathcal{G}$, and we get, as above, the following commutative diagram,

$$\begin{array}{ccccc}
\mathrm{Aut}(\mathbb{A}^n) & \xleftarrow{\supseteq} & \mathrm{SAut}(\mathbb{A}^n) & \xrightarrow{\varphi} & \mathcal{G} \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \mathrm{Ad}_{\mathrm{SAut}(\mathbb{A}^n)}}\downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \mathrm{Ad}_{\mathcal{G}}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))) & \xrightarrow[\text{bijective}]{\rho} & \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))) & \xrightarrow[\simeq]{\Phi} & \mathrm{Aut}_{\mathrm{Lie}}(L),
\end{array}$$

where $\mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}$ is an isomorphism, by (1). Since ρ is bijective ([KR17]) the composition $\rho \circ \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}$ is an isomorphism, again by (1). Therefore, the image $\mathfrak{A} := \mathrm{Ad}(\mathrm{SAut}(\mathbb{A}^n)) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ is a closed subgroup isomorphic to $\mathrm{SAut}(\mathbb{A}^n)$, and $\mathfrak{A} \xrightarrow{\sim} \Phi(\mathfrak{A}) = \mathrm{Ad}_{\mathcal{G}}(\varphi(\mathrm{SAut}(\mathbb{A}^n))$. But $\varphi(\mathrm{SAut}(\mathbb{A}^n)) \subseteq \mathcal{G}$ is dense, and so $\mathrm{Ad}_{\mathcal{G}}(\mathcal{G}) = \Phi(\mathfrak{A})$. Thus, the composition $\mathrm{Ad}_{\mathcal{G}} \circ \varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \Phi(\mathfrak{A})$ is an isomorphism, hence φ is an isomorphism, by Proposition 4.1 below. □

Proposition 4.1. *Let $\mathcal{H}, \mathcal{G}$ be two ind-groups, and let $\varphi\colon \mathcal{H} \to \mathcal{G}$, $\psi\colon \mathcal{G} \to \mathcal{H}$ be two homomorphisms. If $\psi \circ \varphi = \mathrm{id}_{\mathcal{H}}$, then φ is a closed immersion, i.e., $\varphi(\mathcal{H}) \subseteq \mathcal{G}$ is a closed subgroup and φ induces an isomorphism $\mathcal{H} \xrightarrow{\sim} \varphi(\mathcal{H})$.*

Proof. By base change we can assume that the base field $\Bbbk$ is uncountable. Let $\mathcal{H} = \bigcup_i \mathcal{H}_i$ and $\mathcal{G} = \bigcup_j \mathcal{G}_j$, where we can assume that $\mathcal{H}_i \subseteq \mathcal{G}_i$ for all i. Moreover, for every i there is a $k = k(i)$ such that $\psi(\mathcal{G}_i) \subseteq \mathcal{H}_k$. By assumption, the composition $\psi \circ \varphi\colon \mathcal{H}_i \to \mathcal{G}_i \to \mathcal{H}_k$

is the closed embedding $\mathcal{H}_i \hookrightarrow \mathcal{H}_k$, hence the first map is a closed embedding. Thus $H_i := \varphi(\mathcal{H}_i)$ is a closed subset of $\mathcal{G}_i$ and $H := \varphi(\mathcal{H}) = \bigcup_i H_i$. Now the claim follows from Lemma 4.2 below by setting $S := \ker \psi$. □

Recall that a subset $S \subseteq \mathfrak{V}$ of an ind-variety $\mathfrak{V}$ is called *ind-constructible* if $S = \bigcup_i S_i$ where $S_i \subseteq S_{i+1}$ are constructible subsets of $\mathfrak{V}$.

Lemma 4.2. *Let $\mathcal{G}$ be an ind-group, $H \subseteq \mathcal{G}$ a subgroup and $S \subseteq \mathcal{G}$ an ind-constructible subset. Assume that $\Bbbk$ is uncountable and that*
(1) *$H = \bigcup_i H_i$ where $H_i \subseteq H_{i+1} \subseteq \mathcal{G}$ are closed algebraic subsets,*
(2) *the multiplication map $S \times H \to \mathcal{G}$ is bijective.*
Then H is a closed subgroup of $\mathcal{G}$.

Proof. Let $\mathcal{G} = \bigcup_k \mathcal{G}_k$. We have to show that for every k there exists an $i = i(k)$ such that $H \cap \mathcal{G}_k = H_i \cap \mathcal{G}_k$. We can assume that $e \in S = \bigcup_i S_i$. Then, by assumption, $\mathcal{G} = \bigcup_j S_j H_j$. Since $S_j H_j \cap \mathcal{G}_k$ is a constructible subset of $\mathcal{G}_k$ it follows that there exists a $j = j(k)$ such that $\mathcal{G}_k \subseteq S_j H_j$ ([FK17, Lemma 1.6.4]). Setting $\dot{S} := S \setminus \{e\}$ we get $\dot{S}H \cap H = \emptyset$. Thus, $\mathcal{G}_k = (\dot{S}_i H_i \cap \mathcal{G}_k) \cup (H_i \cap \mathcal{G}_k)$ and $H \cap \dot{S}_i H_i = \emptyset$, hence $H \cap \mathcal{G}_k = H_i \cap \mathcal{G}_k$. □

Finally, we can prove Theorem 1.6.

Proof of Theorem 1.6. (1) We already know from Theorem 1.4 that an injective homomorphism $\varphi\colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}(\mathbb{A}^n)$ is a closed immersion. We claim that $d\varphi_e\colon \operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$ is an isomorphism. To show this, consider the linear action of $\operatorname{GL}_n(\Bbbk)$ on $\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$. We then have

$$\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n) \subseteq \operatorname{Vec}(\mathbb{A}^n) \simeq \Bbbk^n \otimes \Bbbk[x_1, \ldots, x_n] = \bigoplus_d \Bbbk^n \otimes \Bbbk[x_1, \ldots, x_n]_d$$

and the latter is multiplicity-free as a $\operatorname{GL}_n(\Bbbk)$-module as well as an $\operatorname{SL}_n(\Bbbk)$-module.

Now $\varphi(\operatorname{GL}_n(\Bbbk)) \subseteq \operatorname{Aut}(\mathbb{A}^n)$ is a closed subgroup isomorphic to $\operatorname{GL}_n(\Bbbk)$. Moreover, $d\varphi_e\colon \operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$ is an injective linear map which is equivariant with respect to $\varphi\colon \operatorname{GL}_n(\Bbbk) \xrightarrow{\sim} \varphi(\operatorname{GL}_n(\Bbbk))$. Since $\varphi(\operatorname{GL}_n(\Bbbk))$ is conjugate to the standard $\operatorname{GL}_n(\Bbbk) \subseteq \operatorname{Aut}(\mathbb{A}^n)$ and since the representation of $\operatorname{GL}_n(\Bbbk)$ on $\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$ is multiplicity-free, it follows that $d\varphi_e$ is an isomorphism. Thus $\mathcal{G} := \varphi(\operatorname{Aut}(\mathbb{A}^n)) \subseteq \operatorname{Aut}(\mathbb{A}^n)$ is a closed subgroup with the same Lie algebra as $\operatorname{Aut}(\mathbb{A}^n)$, and we get the following commutative diagram (see proof of Theorem 1.4):

$$\begin{array}{ccccc}
\operatorname{Aut}(\mathbb{A}^n) & \xrightarrow{\varphi} & \mathcal{G} & \xrightarrow{\subseteq} & \operatorname{Aut}(\mathbb{A}^n) \\
{\scriptstyle \operatorname{Ad}_{\operatorname{Aut}(\mathbb{A}^n)}}\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle \operatorname{Ad}_{\mathcal{G}}} & & \downarrow{\scriptstyle \operatorname{Ad}_{\operatorname{Aut}(\mathbb{A}^n)}} \\
\operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)) & \xrightarrow{\simeq} & \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\mathcal{G}) & = & \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)).
\end{array}$$

As a consequence, all maps are isomorphisms, and so $\mathcal{G} = \operatorname{Aut}(\mathbb{A}^n)$ and φ is an isomorphism.

It remains to see that every automorphism $\varphi \in \operatorname{Aut}(\mathbb{A}^n)$ is inner. Since Ad is bijective (see [KR17]) and $d\varphi_e \in \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n))$ we get $d\varphi_e = \operatorname{Ad}(\mathbf{g})$ for some $\mathbf{g} \in \operatorname{Aut}(\mathbb{A}^n)$. This means that $d\varphi_e = (d\operatorname{Int}\mathbf{g})_e$ and so $\varphi = \operatorname{Int}\mathbf{g}$, by Proposition 2.7.

(2) The same argument as above shows that every nontrivial homomorphism $\operatorname{SAut}(\mathbb{A}^n) \to \operatorname{SAut}(\mathbb{A}^n)$ is an isomorphism where we use the fact that the action of $\operatorname{SL}_n(\Bbbk)$ on $\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n)$ is multiplicity-free.

Moreover, $\operatorname{Ad}\colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n))$ is a bijective homomorphism of ind-groups; see [KR17]. Hence, for every $\varphi \in \operatorname{SAut}(\mathbb{A}^n)$ there is a $\mathbf{g} \in \operatorname{Aut}(\mathbb{A}^n)$ such that $d\varphi_e = \operatorname{Ad}\mathbf{g}$ which implies that $\varphi = \operatorname{Int}\mathbf{g}$. □

5. A special subgroup of $\mathrm{Aut}(X)$, proof of Theorem 1.1

Our Theorem 1.1 will follow from a more general result which we will describe now. For any affine variety X consider the normal subgroup $\mathcal{U}(X)$ of $\mathrm{Aut}(X)$ generated by the unipotent elements of $\mathrm{Aut}(X)$ or, equivalently, by the closed algebraic subgroups of $\mathrm{Aut}(X)$ isomorphic to the additive group $\Bbbk^+$. This is an instance of a so-called *connected group of automorphisms* defined by Ramanujam in [Ram64]. The group $\mathcal{U}(X)$ defined above was introduced and studied in [AFK$^+$13] where it is called the *group of special automorphisms*[1] *of* X. In particular, they give a very interesting connection between transitivity properties of the group $\mathcal{U}(X)$ and the flexibility of the variety X.

We do not know if $\mathcal{U}(X) \subseteq \mathrm{Aut}(X)$ is closed, but we still have the notion of an *algebraic subgroup* $G \subseteq \mathcal{U}(X)$, namely a subgroup which is algebraic as a subgroup of $\mathrm{Aut}(X)$; see Remark 2.3. We will also need the notion of an "algebraic" homomorphism between these groups.

Definition 5.1. A homomorphism $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ is *algebraic*, if for any algebraic subgroup $U \subseteq \mathcal{U}(X)$ isomorphic to $\Bbbk^+$ the image $\varphi(U) \subseteq \mathcal{U}(Y)$ is an algebraic subgroup and $\varphi|_U\colon U \to \varphi(U)$ is a homomorphism of algebraic groups. We say that $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ are *algebraically isomorphic*, $\mathcal{U}(X) \simeq \mathcal{U}(Y)$, if there exists a bijective homomorphism $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ such that φ and φ^{-1} are both algebraic.

Lemma 5.2. *Let $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ be an algebraic homomorphism. Then, for any algebraic subgroup $G \subseteq \mathcal{U}(X)$ generated by unipotent elements the image $\varphi(G) \subseteq \mathcal{U}(Y)$ is an algebraic subgroup and $\varphi|_G\colon G \to \varphi(G)$ is a homomorphism of algebraic groups.*

Proof. There exist closed subgroups $U_1, \ldots, U_m \subseteq G$ isomorphic to $\Bbbk^+$ such that the multiplication map $\mu\colon U_1 \times U_2 \times \cdots \times U_m \to G$ is surjective. This gives the following commutative diagram,

$$\begin{array}{ccc} U_1 \times U_2 \times \cdots \times U_m & \xrightarrow{\ \mu\ } & G \\ \Big\downarrow{\scriptstyle \widetilde{\varphi}:=\varphi|_{U_1}\times\cdots\times\varphi|_{U_m}} & & \Big\downarrow{\scriptstyle \varphi|_G} \\ \varphi(U_1) \times \varphi(U_2) \times \cdots \times \varphi(U_m) & \xrightarrow{\ \bar{\mu}\ } & \varphi(G), \end{array}$$

where all maps are surjective. It follows that $\overline{\varphi(G)} \subseteq \mathrm{Aut}(Y)$ is a (closed) algebraic subgroup, and thus $\varphi(G) = \overline{\varphi(G)}$, because $\varphi(G)$ is constructible. It remains to show that $\varphi|_G$ is a morphism. This follows from the next lemma, because G is normal, and μ and the composition $\varphi|_G \circ \mu = \bar{\mu} \circ \widetilde{\varphi}$ are both morphisms. □

Lemma 5.3. *Let X, Y, Z be irreducible affine varieties where Y is normal. Let $\mu\colon X \to Y$ be a surjective morphism and $\varphi\colon Y \to Z$ an arbitrary map. If the composition $\varphi \circ \mu$ is a morphism, then φ is a morphism.*

Proof. We have the following commutative diagram of maps,

$$\begin{array}{ccc} \Gamma_{\varphi\circ\mu} & \xrightarrow{\ \subseteq\ } & X \times Z \\ \Big\downarrow{\scriptstyle \bar{\mu}} & & \Big\downarrow{\scriptstyle \mu\times\mathrm{id}} \\ \Gamma_{\varphi} & \xrightarrow{\ \subseteq\ } & Y \times Z \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \mathrm{pr}_Y} \\ Y & = & Y, \end{array}$$

[1]They denote this group by $\mathrm{SAut}(X)$ which should not be confused with our definition of $\mathrm{SAut}(\mathbb{A}^n)$ and of $\mathrm{SAut}^{alg}(X)$ below.

where $\Gamma_{\varphi\circ\mu}$ and Γ_φ denote the graphs of the corresponding maps. We have to show that $\Gamma_\varphi \subseteq Y \times Z$ is closed and that p is an isomorphism. The diagram shows that $\bar{\mu}$ is surjective, hence Γ_φ is constructible, and p is bijective. Thus, the induced morphism $\bar{p}\colon \bar{\Gamma}_\varphi \to Y$ is birational and surjective, hence an isomorphism since Y is normal (see [Igu73, Lemma 4, page 379]). Since p is bijective, we finally get $\Gamma_\varphi = \bar{\Gamma}_\varphi$. □

Remark 5.4. If $\varphi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ is a homomorphism of ind-groups, then the induced homomorphism $\varphi_{\mathcal{U}}\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ is algebraic. If $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic as ind-groups, then $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ are algebraically isomorphic.

The remark shows that the following result generalizes Theorem 1.1. The proof will be given in the next section.

Theorem 5.5. *Let X be a connected affine variety. If $\mathcal{U}(X)$ is algebraically isomorphic to $\mathcal{U}(\mathbb{A}^n)$, then X is isomorphic to $\mathbb{A}^n$.*

Finally, we define the following closed subgroups of $\mathrm{Aut}(X)$:

$$\mathrm{Aut}^{alg}(X) := \overline{\langle G \mid G \subseteq \mathrm{Aut}(X) \text{ connected algebraic}\rangle},$$
$$\mathrm{SAut}^{alg}(X) := \overline{\langle U \mid U \subseteq \mathrm{Aut}(X) \text{ unipotent algebraic}\rangle}.$$

We have $\mathrm{SAut}^{alg}(X) = \overline{\mathcal{U}(X)} \subseteq \mathrm{Aut}^{alg}(X) \subseteq \mathrm{Aut}(X)$. A similar argument as above gives the next result, again as a consequence of Theorem 5.5 above.

Corollary 5.6. *Let X be a connected affine variety. If $\mathrm{SAut}^{alg}(X)$ is isomorphic to $\mathrm{SAut}^{alg}(\mathbb{A}^n)$ as ind-groups, then X is isomorphic to $\mathbb{A}^n$, and the same holds if we replace SAut^{alg} by Aut^{alg}.*

A special case of Theorem 1.1. Going back to our original Theorem 1.1 there is the following rather short proof in case X is irreducible which was suggested by a referee. We first remark that the subgroup of translations $\mathcal{T} \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is self-centralizing, i.e., $\mathrm{Cent}_{\mathrm{Aut}(\mathbb{A}^n)} \mathcal{T} = \mathcal{T}$. Denote by $\mathcal{T}' \subseteq \mathrm{Aut}(X)$ the image of $\mathcal{T}$. We claim that $\mathcal{T}'$ has a dense orbit. Since $\mathcal{T}'$ is a unipotent group, this implies that X is an orbit, hence isomorphic to $\mathbb{A}^m$ for some $m \leq n$. Since an n-dimensional torus acts faithfully on X, we have $n = m$, and we are done.

It remains to see that $\mathcal{T}'$ has a dense orbit in X, or equivalently, that every $\mathcal{T}'$-invariant function on X is a constant. Assume that this is not the case, and let $f \in \mathcal{O}(X)^{\mathcal{T}'} \setminus \Bbbk$. Then we can "modify" every automorphism $\mathbf{t} \in \mathcal{T}$ by f (see the following §6) to obtain new unipotent automorphism $f \cdot \mathbf{t}$ in $\mathrm{Aut}(X)$ which do not belong to $\mathcal{T}'$, but commute with $\mathcal{T}'$, contradicting the fact that $\mathcal{T}'$ is self-centralizing. (It is here where we use the irreducibility of X. Otherwise it is not clear why these modified automorphisms do not belong to $\mathcal{T}'$.)

6. Modifications and root subgroups, proof of Theorem 5.5

Let X be an affine variety and consider a nontrivial action of $\Bbbk^+$ on X, given by $\lambda\colon \Bbbk^+ \to \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\Bbbk^+$-invariant, then we define the *modification* $f \cdot \lambda$ of the action λ in the following way (see [AFK+13] where a modified action is called a replica; cf. [FK17, section 12.4]):

$$(f \cdot \lambda)(s)x := \lambda(f(x)s)x \quad \text{for } s \in \Bbbk \text{ and } x \in X.$$

It is easy to see that this is again a $\Bbbk^+$-action. In fact, the action λ corresponds to a locally nilpotent vector field $\delta_\lambda \in \mathrm{Vec}(X)$. Since f is λ-invariant, it follows that $f\delta_\lambda \in \mathrm{Vec}(X)$ is again locally nilpotent, and defines the modified $\Bbbk^+$-action $f \cdot \lambda$. Note that if $U_\lambda \subseteq \mathrm{Aut}(X)$ denotes the image of λ, then $\mathrm{Lie}(U_\lambda) \xrightarrow{\sim} \Bbbk\delta_\lambda$ under the canonical homomorphism $\mathrm{Lie}\,\mathrm{Aut}(X) \hookrightarrow \mathrm{Vec}(X)$.

This modified action $f \cdot \lambda$ is trivial if and only if f vanishes on every irreducible component X_i of X, where the action λ is nontrivial. It is clear that the orbits of $f \cdot \lambda$ are contained in the orbits of λ, and that they are equal on the open subset $X_f := \{x \in X \mid f(x) \neq 0\}$ of X. In particular, if X is irreducible and $f \neq 0$, then λ and $f \cdot \lambda$ have the same invariants.

If $U \subseteq \mathrm{Aut}(X)$ is a closed subgroup isomorphic to $\Bbbk^+$ and if $f \in \mathcal{O}(X)^U$ is a U-invariant, then we can define the *modification* $f \cdot U$ *of* U by choosing an isomorphism $\lambda\colon \Bbbk^+ \xrightarrow{\sim} U$ and setting $f \cdot U := (f \cdot \lambda)(\Bbbk^+)$, the image of the modified action.

Let $\mathcal{G}$ be an ind-group, and let $T \subseteq \mathcal{G}$ be a torus.

Definition 6.1. An algebraic subgroup $U \subseteq \mathcal{G}$ isomorphic to $\Bbbk^+$ and normalized by T is called a *root subgroup* with respect to T. The character of T on $\mathrm{Lie}\,U \simeq \Bbbk$ is called the *weight of* U.

If $U = U_\lambda$ is the image of a nontrivial $\Bbbk^+$-action λ, then U is a root subgroup if and only if $\Bbbk\delta_\lambda \subseteq \mathrm{Vec}(X)$ is stable under T. If α is the weight of U_λ, we have

$$t \cdot \lambda(s) \cdot t^{-1} = \lambda(\alpha(t)s) \quad \text{for } t \in T, s \in \Bbbk.$$

If a torus T acts on an affine variety X, then we get a locally finite and rational representation of T on the coordinate ring $\mathcal{O}(X)$, and thus a decomposition of $\mathcal{O}(X)$ into weight spaces. A locally finite and rational representation of T is called *multiplicity-free* if the dimensions of the weight spaces are ≤ 1. The following lemma is crucial.

Lemma 6.2. *Let X be an irreducible affine variety, and let $T \subseteq \mathrm{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subseteq \mathrm{Aut}(X)$ with respect to T such that $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.*

Proof. The first inequality $\dim T \leq \dim X$ is clear, because T acts faithfully on X. It follows from [DK08, Propositions 2.7 and 2.9] that there exists a T–semi-invariant $f \in \mathcal{O}(X)^U$ such that the localization $\mathcal{O}(X)^U_f = \mathcal{O}(X_f)^U$ is finitely generated. Clearly, $\mathcal{O}(X)^U_f$ is T-stable and multiplicity-free, and $\mathcal{O}(X)^U_f$ is the coordinate ring of the algebraic quotient $Z := X_f/\!/U$ on which T acts. It follows from [Kra84, II.3.4 Satz 5]) that T has a dense orbit in Z, and so $\dim Z \leq \dim T$. Since $\dim Z = \dim X_f/\!/U = \dim X_f - 1 = \dim X - 1$, we get the second inequality. □

Lemma 6.3. *We have $\mathcal{U}(\mathbb{A}^n) \subseteq \mathrm{SAut}(\mathbb{A}^n)$, and its closure $\overline{\mathcal{U}(\mathbb{A}^n)}$ is connected. Moreover, $\mathrm{Lie}\overline{\mathcal{U}(\mathbb{A}^n)} = \mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$, hence it is a simple Lie algebra.*

Proof. The first statement is obvious, since every unipotent algebraic subgroup is contained in $\mathrm{SAut}(\mathbb{A}^n)$. The second claim follows from $\mathcal{U}(\mathbb{A}^n) \subseteq \overline{\mathcal{U}(\mathbb{A}^n)}^\circ$ (see Lemma 7.3 in the next section). For the last statement we remark that $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ is generated by the Lie algebras of the algebraic subgroups of $\mathrm{SAut}(\mathbb{A}^n)$ (Remark 2.6) which are all contained in $\mathcal{U}(\mathbb{A}^n)$ (Lemma 5.2). □

Denote by $T_n \subseteq \mathrm{GL}_n(\Bbbk) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ the diagonal torus and set $T_n' := T_n \cap \mathrm{SL}_n(\Bbbk)$. The next result can be found in [Lie11, Theorem 1].

Lemma 6.4. *Root subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ with respect to T_n' exist, and their weights are all different.*

Now we can give the proof of Theorem 5.5.

Proof of Theorem 5.5. The algebraic subgroups $\mathrm{SL}_n(\Bbbk)$ and $\mathrm{SAff}_n(\Bbbk)$ of $\mathrm{Aut}(\mathbb{A}^n)$ both belong to $\mathcal{U}(\mathbb{A}^n)$ as well as all root subgroups U. Fix an algebraic isomorphism $\varphi\colon \mathcal{U}(\mathbb{A}^n) \xrightarrow{\sim} \mathcal{U}(X)$ and set by $T' := \varphi(T'_n) \subseteq \mathcal{U}(X)$.

Let $X = \bigcup_i X_i$ be the decomposition into irreducible components. Since $\overline{\mathcal{U}(X)}$ is connected by Lemma 6.3, the components X_i are stable under $\overline{\mathcal{U}(X)}$. Denote by $A \subseteq \mathcal{U}(X)$ the image of $\mathrm{Aff}_n(\Bbbk)$ under φ. Since every nontrivial closed normal subgroup of $\mathrm{Aff}_n(\Bbbk)$ contains the translations, one of the restriction maps $\rho_i\colon \mathcal{U}(X) \to \mathcal{U}(X_i)$, say ρ_1, is injective on A.

Let $T_1 := \rho_1(T') \subseteq \mathcal{U}(X_1)$ be the image of T'. We will show that there is a root subgroup $U_1 \subseteq \mathcal{U}(X_1)$ such that $\mathcal{O}(X_1)^{U_1}$ is multiplicity-free. Then Lemma 6.2 implies that $\dim X_1 \leq n$ and so, by Proposition 2.8, X_1 is isomorphic to $\mathbb{A}^n$ with a transitive action of A. Since X is connected, this implies that $X = X_1 \simeq \mathbb{A}^n$.

In order to construct U_1 we choose a root subgroup $U \subseteq \varphi(\mathrm{SL}_n(\Bbbk)) \subseteq \mathcal{U}(X)$, and set $U_1 := \rho_1(U) \subseteq \mathcal{U}(X_1)$. Since U is a maximal unipotent subgroup of a closed subgroup $S \subseteq \mathcal{U}(X)$ isomorphic to $\mathrm{SL}_2(\Bbbk)$ and since the restriction map $\mathrm{res}\colon \mathcal{O}(X) \to \mathcal{O}(X_1)$ is a surjective homomorphism of S-modules, it follows that $\mathrm{res}\colon \mathcal{O}(X)^U \to \mathcal{O}(X_1)^{U_1}$ is also surjective (see [Kra84, III.3.1, Bemerkung 2]). If α is the weight of U and U_1 and if $f \in \mathcal{O}(X_1)^{U_1}$ is an invariant of weight β, then $f = \widetilde{f}|_{X_1}$ for an invariant $\widetilde{f} \in \mathcal{O}(X)^U$ of weight β, and so $\widetilde{f} \cdot U$ is a root subgroup of weight $\alpha + \beta$ with $\rho_1(\widetilde{f} \cdot U) = f \cdot U_1$. Since the root subgroups of $\mathrm{Aut}(X)$ have different weights, it finally follows that $\mathcal{O}(X_1)^{U_1}$ is multiplicity-free. □

7. Finite-dimensional automorphism groups

It is well known that for a smooth affine curve C the automorphism group $\mathrm{Aut}(C)$ is finite except for $C \simeq \Bbbk, \Bbbk^*$. Theorem 1.2 implies that every finite group appears as automorphism group of a smooth affine curve. There also exist examples of smooth affine surfaces with a discrete nonfinite automorphism group; see [FK17, Proposition 12.7.1]. Recall that an ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *discrete* if $\mathcal{G}_k$ is finite for all k, or equivalently, if $\mathrm{Lie}\mathcal{G} = \{0\}$.

Definition 7.1. An ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *finite-dimensional*, $\dim \mathcal{G} < \infty$, if $\dim \mathcal{G}_k$ is bounded above. In this case we put $\dim \mathcal{G} := \max_k \dim \mathcal{G}_k$.

Definition 7.2. For an ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ we define $\mathcal{G}^\circ := \bigcup_k \mathcal{G}_k^\circ$, where $\mathcal{G}_k^\circ$ denotes the connected component of $\mathcal{G}_k$ which contains $e \in \mathcal{G}$.

An ind-variety $\mathcal{V}$ is called *curve-connected* if for every $v, w \in \mathcal{V}$ there is an irreducible curve D and a morphism $D \to \mathcal{V}$ whose image contains v and w. This is equivalent to the condition that $\mathcal{V}$ admits a filtration with irreducible varieties (see [FK17, Lemma 1.6.3]). The following result can be found in [FK17, Lemma 2.2.2]).

Lemma 7.3. *Let $\mathcal{G} = \bigcup_k \mathcal{G}_k$ be an ind-group.*

(1) *$\mathcal{G}^\circ \subseteq \mathcal{G}$ is a curve-connected open (and thus closed) normal subgroup of countable index. In particular,* $\mathrm{Lie}\mathcal{G} = \mathrm{Lie}\mathcal{G}^\circ$.

(2) *We have $\dim \mathcal{G} < \infty$ if and only if $\mathcal{G}^\circ \subseteq \mathcal{G}$ is an algebraic group.*

(3) *We have $\dim \mathcal{G} < \infty$ if and only if* $\dim \mathrm{Lie}\mathcal{G} < \infty$.

Example 7.4. (1) We have $\mathrm{Aut}(\Bbbk^*) \simeq \mathbb{Z}/2 \ltimes \Bbbk^*$, hence $\mathrm{Aut}(\Bbbk^*)^\circ \simeq \Bbbk^*$. Similarly, $\mathrm{Aut}(\Bbbk^{*n}) \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes \Bbbk^{*n}$, and so $\mathrm{Aut}(\Bbbk^{*n})^\circ \simeq \Bbbk^{*n}$.

(2) Let $C := V(y^2 - x^3) \subseteq \Bbbk^2$ be Neile's parabola. Then $\mathrm{Aut}(C) = \Bbbk^*$. In fact, every automorphism of C defines an automorphism of the normalization $\mathbb{A}^1$ of C fixing the origin. From this the claim follows immediately.

(3) Let C be a smooth curve with trivial automorphism group, and consider the one dimensional variety $Y_C = \mathbb{A}^1 \cup C$ where the two irreducible components meet in $\{0\} \in \mathbb{A}^1$. Then $\mathrm{Aut}(Y_C) \simeq \Bbbk^*$. Moreover, the disjoint union $Y_{C_1} \cup Y_{C_2} \cup \cdots \cup Y_{C_m}$ with pairwise nonisomorphic curves C_i has automorphism group ${\Bbbk^*}^m$. We will show in §8 that for every n there is even an irreducible affine variety X whose automorphism group $\mathrm{Aut}(X)$ is an n-dimensional torus.

Theorem 1.3 claims that if $\dim \mathrm{Aut}(X)$ is finite, then either $X \simeq \mathbb{A}^1$ or $\mathrm{Aut}(X)^\circ$ is a torus. This follows immediately from the next result.

Proposition 7.5. *Let X be a connected affine variety. If X admits a nontrivial action of the additive group $\Bbbk^+$, then either $X \simeq \mathbb{A}^1$ or $\dim \mathrm{Aut}(X) = \infty$.*

Proof. If X contains a one-dimensional irreducible component X_i with a nontrivial action of $\Bbbk^+$, then X_i is an orbit under $\Bbbk^+$, hence $X = X_i \simeq \mathbb{A}^1$. Otherwise, $\Bbbk^+$ acts nontrivially on an irreducible component X_j of dimension ≥ 2. Denote by $U \subseteq \mathrm{Aut}(X)$ the image of $\Bbbk^+$. We claim that the modifications $f \cdot U$ for $f \in \mathcal{O}(X)^U$ form an infinite-dimensional subgroup $\mathcal{O}(X)^U \cdot U \subseteq \mathrm{Aut}(X)$. This follows if we show that the image of $\mathcal{O}(X)^U$ in $\mathcal{O}(X_j)$ is infinite-dimensional. For that we first remark that there is a nonzero U-invariant f which vanishes on all X_k for $k \neq j$, because the vanishing ideal is U-stable. This implies that $X_f \subseteq X_j$, and so

$$\mathcal{O}(X)^U_f = \mathcal{O}(X_f)^U = \mathcal{O}(X_j)^U_f = (\mathcal{O}(X)^U|_{X_j})_f.$$

Thus the image $\mathcal{O}(X)^U|_{X_j} \subseteq \mathcal{O}(X_j)$ is infinite-dimensional. □

The following result—a partial converse of the proposition above— is due to Arzhantsev–Gaĭfullin.

Proposition 7.6 ([AG17]). *Let X be an affine variety which does not admit a nontrivial $\Bbbk^+$-action. Then $\mathrm{Aut}(X)$ contains a unique maximal torus T. If the action of T on X is one fix pointed, then $\mathrm{Aut}(X)^\circ = T$ and $\mathrm{Aut}(X)/T$ is a finite group.*

(A T-action on X is called *one fix pointed* if there is a unique fixed point $x_0 \in X$ and no other closed orbit.) The paper [AG17] contains many examples of such varieties, e.g., cones over projective varieties with a finite automorphism group, or the so-called trinomial hypersurfaces.

8. An example with a torus as automorphism group

In Example 7.4 we have mentioned that Neile's parabola $C := V(y^2 - x^3) \subseteq \Bbbk^2$ has an automorphism group isomorphic to $\Bbbk^*$, and we have given an example of a reducible curve with automorphism group isomorphic to ${\Bbbk^*}^m$. We now construct an irreducible variety X of dimension d with $\mathrm{Aut}(X) \xrightarrow{\sim} {\Bbbk^*}^d$.

Definition 8.1. A plane curve $C \subseteq \Bbbk^2$ given by an equation of the form $y^m - x^n = 0$, where $n > m \geq 2$ and m, n are relatively prime, is called a *cuspidal curve*. It has an isolated singularity in the origin 0.

For the cuspidal curve $C_{m,n}$ with equation $y^m = x^n$ we have a canonical isomorphism $\Bbbk^* \xrightarrow{\sim} \mathrm{Aut}(C_{m,n})$ given by the action $t(x, y) := (t^m x, t^n y)$. The induced representation on the tangent space $T_0 C_{m,n} = \Bbbk^2$ has weights m, n. In particular, $C_{m,n}$ is isomorphic to $C_{m',n'}$ if and only if $(m, n) = (m', n')$. Moreover, the normalization is given by the bijective morphism $\mu_{C_{m,n}} \colon \mathbb{A}^1 \to C_{m,n}$, $s \mapsto (s^m, s^n)$.

Proposition 8.2. *Let X be a product of d cuspidal curves which are pairwise nonisomorphic. Then $\mathrm{Aut}(X) \simeq {\Bbbk^*}^m$.*

Proof. (a) Let $X = C_1 \times C_2 \times \cdots \times C_d$ be such a product. We have a canonical injective homomorphism $\rho_X : {\Bbbk^*}^d \hookrightarrow \operatorname{Aut}(X)$. The normalization of X is given by the bijective morphism $\eta := \eta_1 \times \cdots \times \eta_d : \mathbb{A}^d \to X$ where $\eta_i : \mathbb{A}^1 \to C_i$ is the normalization of C_i. For $j = 1, \ldots, d$ define

$$\widetilde{C}_j := \{(0, \ldots, c_j, \ldots, 0) \mid c_j \in C_j\} \subseteq X,$$

i.e., $\widetilde{C}_j$ is the image of the jth coordinate line $L_j \subseteq \mathbb{A}^d$ under the normalization η. Then we have

$$A_X := \bigcup_j \widetilde{C}_j = \{x \in X \mid \dim T_x X \geq 2d - 1\}.$$

Now let $Y = D_1 \times D_2 \times \cdots \times D_d$ be another product of nonisomorphic cuspidal curves, and define $\widetilde{D}_j$ and A_Y as above. It follows from the description of A_X and A_Y that every isomorphism $\mu : X \xrightarrow{\sim} Y$ induces an isomorphism $A_X \xrightarrow{\sim} A_Y$. Hence there is a permutation σ of $\{1, \ldots, d\}$ such that $C_i \simeq \widetilde{C}_i \simeq \mu(\widetilde{C}_i) = \widetilde{D}_{\sigma(i)} \simeq D_{\sigma(i)}$.

(b) Now define $X_j := \{c = (c_1, \ldots, c_d) \in X \mid c_j = 0\} \simeq \prod_{i \neq j} C_i$. Clearly, X_j is the image of the jth coordinate hyperplane $H_j \subseteq \mathbb{A}^d$ given by $x_j = 0$ under the normalization $\eta : \mathbb{A}^d \to X$. Since the singular points of X are given by $X_{\text{sing}} = \bigcup_j X_j$, it follows that every automorphism $\varphi : X \xrightarrow{\sim} X$ permutes the irreducible components X_j of X_{sing}. Now (a) implies that the X_j are pairwise nonisomorphic, hence $\varphi(X_j) = X_j$.

(c) By induction, we can assume that

$$\rho_{X_j} : {\Bbbk^*}^{d-1} \to \operatorname{Aut}(X_j)$$

is an isomorphism, and so φ_{X_j} is given by an element $t_j \in {\Bbbk^*}^{d-1}$. Looking at the intersections $X_j \cap X_k$ we see that there is a $t \in {\Bbbk^*}^d$ such that $\varphi|_{X_{\text{sing}}}$ is given by t. Therefore, the automorphism $\psi := t^{-1} \circ \varphi \in \operatorname{Aut}(X)$ induces the identity on X_{sing}. It follows that the normalization $\widetilde{\psi} : \mathbb{A}^d \xrightarrow{\sim} \mathbb{A}^d$ fixes the coordinate hyperplanes H_j pointwise which implies that $\widetilde{\psi}$ is the identity. In fact, if $\widetilde{\psi} = (f_1, \ldots, f_d)$, $f_i \in \Bbbk[x_1, \ldots, x_d]$, then we get $x_i | f_i$, and the claim follows because all f_i are irreducible (see e.g., [Jel91] for a more general result). □

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