Hanspeter Kraft, May 1, 2017

On the Lie Algebra of Vector Fields of Affine Varieties

There is some fundamental work of Grabowski and Siebert on the structure of the Lie algebra of vector fields of an affine variety X, or, more generally, on the Lie algebra of derivations of an algebra A, see [Gra79] and [Sie96]. A central question is how much information about X (or A) can be retrieved from this Lie algebra.

For example, it follows from the work of Seidenberg [Sei67] that the singular locus X_{sing} is invariant under all vector fields and that every strict invariant subvariety is contained in X_{sing}, and Siebert shows that X is smooth if and only if $\mathrm{Vec}(X)$ is a simple Lie algebra (one implication was proved earlier by Jordan in [Jor86]. The main result of [Sie96] is that two normal affine varieties X, Y are isomorphic if and only if the vector fields are isomorphic as Lie algebras. (The case of two smooth varieties X, Y goes back to Grabowski.)

The aim of these notes is to explain these notions and to prove some of these results in the case of an affine variety over an algebraically closed field of characteristic zero.

1.1. **Vector fields.** We start with some basic notions related to vector fields on an affine variety X. Our base field $\mathbb{K}$ is algebraically closed of characteristic zero. We denote by $\mathrm{Vec}(X)$ the Lie algebra of vector fields on X, i.e. $\mathrm{Vec}(X) = \mathrm{Der}(\mathcal{O}(X))$, the derivations of the coordinate ring $\mathcal{O}(X)$ of X. The vector fields $\mathrm{Vec}(X)$ form a Lie algebra, namely a Lie subalgebra of the linear operators $\mathrm{End}_k(\mathcal{O}(X))$: $[\xi, \eta] := \xi \circ \eta - \eta \circ \xi$. They also have a structure of an $\mathcal{O}(X)$-module where $f \cdot \xi$ is defined by $(f \cdot \xi)_x := f(x)\xi_x$ for $x \in X$. The two structures are interrelated by the formula

$$[\xi, f \cdot \eta] = \xi(f) \cdot \eta + f \cdot [\xi, \eta].$$

Note that the evaluation map $\varepsilon_x \colon \mathrm{Vec}(X) \to T_x X$, $\varepsilon_x(\xi) := \xi_x$ is an $\mathcal{O}(X)$-module homomorphism with the obvious $\mathcal{O}(X)$-module structure on $T_x X$.

In the following, $\mathcal{L} \subseteq \mathrm{Vec}(X)$ will be an $\mathcal{O}(X)$-Lie subalgebra, i.e. a Lie subalgebra which is also an $\mathcal{O}(X)$-submodule. The most interesting case for us is $\mathcal{L} = \mathrm{Vec}(X)$, but many results hold in the more general setting. We define some maps between subsets of $\mathcal{L}$, subsets of $\mathcal{O}(X)$ and subsets of X. We use the following notation for $\mathcal{D} \subseteq \mathcal{L}$ and $F \subseteq \mathcal{O}(X)$:

$$\mathcal{D}(F) := \langle \xi(f) \mid \xi \in \mathcal{D}, f \in F \rangle \subseteq \mathcal{O}(X), \text{ the linear span of the images } \xi(f).$$

Definition 1. (1) For a subset $\mathcal{D} \subseteq \mathcal{L}$ we define the *ideal* $\mathcal{I}_{\mathcal{D}} \subseteq \mathcal{O}(X)$ *of* $\mathcal{D}$ and the *zero set* $\mathcal{V}(\mathcal{D}) \subseteq X$ *of* $\mathcal{D}$:

$$\mathcal{I}_{\mathcal{D}} := \mathcal{O}(X)\mathcal{D}(\mathcal{O}(X)) \subseteq \mathcal{O}(X), \text{ the ideal generated by } \mathcal{D}(\mathcal{O}(X)),$$
$$\mathcal{V}(\mathcal{D}) := \{x \in X \mid \xi_x = 0 \text{ for all } \xi \in \mathcal{D}\} = \mathcal{V}_X(\mathcal{I}_{\mathcal{D}}) \subseteq X.$$

(2) For an ideal $I \subset \mathcal{O}(X)$ we set

$$\mathcal{L}_I := \{\xi \in \mathcal{L} \mid \xi(\mathcal{O}(X)) \subset I\} \subseteq \mathcal{L},$$

and for a subset $Y \subseteq X$

$$\mathcal{L}_Y := \{\xi \in \mathcal{L} \mid \xi_y = 0 \text{ for all } y \in Y\} = \mathcal{L}_{I(Y)} \subseteq \mathcal{L}$$

where $I(Y) \subset \mathcal{O}(X)$ is the vanishing ideal of Y.

VF.rem *Remark* 1. By definition, $\mathcal{L}_x := \mathcal{L}_{\{x\}}$ is the kernel of the evaluation map $\varepsilon_x\colon \mathcal{L} \to T_x X$ and therefore of finite codimension in $\mathcal{L}$. It is easy to see that $\mathcal{L}_x$ is an $\mathcal{O}(X)$-Lie subalgebra of $\mathcal{L}$. More generally, $\mathcal{L}_I$ is an $\mathcal{O}(X)$-Lie subalgebra for every ideal $I \subseteq \mathcal{O}(C)$, and $I \cdot \mathcal{L} \subseteq \mathcal{L}_I$. We also see that $\mathcal{V}(\mathcal{D}) \subseteq X$ is closed and that

$$\begin{aligned}\mathcal{V}(\mathrm{Vec}(X)) &= \{x \in X \mid \xi_x = 0 \text{ for all } \xi \in \mathrm{Vec}(X)\}\\ &= \{x \in X \mid \mathrm{Vec}(X)_x = \mathrm{Vec}(X)\}.\end{aligned}$$

Example 1. The vector fields $\mathrm{Vec}(\mathbb{A}^n)$ of affine n-space $\mathbb{A}^n = \mathbb{K}^n$ form a free $\mathcal{O}(\mathbb{A}^n) = \mathbb{K}[x_1, \ldots, x_n]$-module generated by $\partial_{x_1}, \ldots, \partial_{x_n}$ where $\partial_{x_i} = \frac{\partial}{\partial x_i}$:

$$\mathrm{Vec}(\mathbb{A}^n) = \bigoplus_{i=1}^{n} \mathbb{K}[x_1, \ldots, x_n] \cdot \partial_{x_i}.$$

It follows that the vector fields generate the tangent space at every point $a \in \mathbb{A}^n$, i.e. the evaluation map $\varepsilon_a\colon \mathrm{Vec}(X) \to T_a\mathbb{A}^n$ is surjective for all $a \in \mathbb{A}^n$. If $\mathfrak{m}_a \subset \mathcal{O}(\mathbb{A}^n)$ denotes the maximal ideal corresponding to $a \in \mathbb{A}^n$ we see that $\mathrm{Vec}(\mathbb{A}^n)_a = \mathfrak{m}_a \cdot \mathrm{Vec}(\mathbb{A}^n)$, and this it is a maximal Lie subalgebra of $\mathrm{Vec}(\mathbb{A}^n)$ of codimension n. More generally, we have $\mathrm{Vec}(\mathbb{A}^n)_I = I \cdot \mathrm{Vec}(\mathbb{A}^n)$. Moreover,

$$\mathrm{Vec}(\mathbb{A}^n)(\mathcal{O}(X)) = \langle \xi(f) \mid \xi \in \mathrm{Vec}(\mathbb{A}^n), f \in \mathcal{O}(\mathbb{A}^n)\rangle = \mathcal{O}(\mathbb{A}^n).$$

In fact, for every homogeneous $f \in \mathcal{O}(\mathbb{A}^n)$ we have $\eta(f) = \deg(f) f$ for the Euler-field $\eta := x_1 \cdot \partial_{x_1} + \cdots + x_n \cdot \partial_{x_n}$.

One can show that every maximal strict Lie subalgebra $L \subset \mathrm{Vec}(\mathbb{A}^n)$ of finite codimension is equal to $\mathrm{Vec}(\mathbb{A}^n)_a$ for some $a \in \mathbb{A}^n$, so that we have a bijection

$$\mathbb{A}^n \xrightarrow[\simeq]{a \mapsto \mathrm{Vec}(\mathbb{A}^n)_a} \left\{\begin{matrix}\text{proper maximal Lie subalgebras}\\ L \subset \mathrm{Vec}(\mathbb{A}^n) \text{ of finite codimension}\end{matrix}\right\},$$

see Theorem main-theorem 1.

1.2. **Invariant subvarieties.** We shortly discuss the concept of invariant (or integral) subvarieties with respect to a given set of vector fields and prove some basic results.

Definition 2. Let $\mathcal{D} \subseteq \mathrm{Vec}(X)$ be a set of vector fields.

(1) A closed subvariety $Y \subseteq X$ is called *$\mathcal{D}$-invariant* if $\xi(y) \in T_y Y$ for all $y \in Y$ and all $\xi \in \mathcal{D}$. We also say that the vector fields $\xi \in \mathcal{D}$ are *parallel to* Y or that Y *is integral with respect to* $\mathcal{D}$.
(2) A subspace $W \subseteq \mathcal{O}(X)$ is called *$\mathcal{D}$-invariant* if $\xi(W) \subset W$ for all $\xi \in \mathcal{D}$.

Remark 2. If ξ is a vector field parallel to Y and f a rational function on X defined in a neighborhood U of $y \in Y$, then $\xi(y)f = \xi(y)(f|_{U\cap Y})$. In particular, if f is regular on X, then $(\xi f)|_Y = \xi|_Y(f|_Y)$.

Let $\mathcal{D} \subset \mathrm{Vec}(X)$ be a set of vector fields.

Lemma 1. *If $I(Y) \subseteq \mathcal{O}(X)$ denotes the vanishing ideal of Y, then Y is $\mathcal{D}$-invariant if and only if $I(Y)$ is $\mathcal{D}$-invariant.*

Proof. If $f \in I(Y)$, then, for $y \in Y$, $(\xi f)(y) = \xi(y)f = \xi(y)f|_Y = 0$, hence $\xi f \in I(Y)$. Conversely, if $\xi(I(Y)) \subseteq I(Y)$, then ξ induces a derivation of $\mathcal{O}(X)/I(Y) = \mathcal{O}(Y)$, and the claim follows. □

For a closed subvariety $Y \subseteq X$ we can define the Lie subalgebra of the vector fields on X parallel to Y:

$$\mathrm{Vec}_Y(X) := \{\xi \in \mathrm{Vec}(X) \mid \xi(y) \in T_y Y \text{ for all } y \in Y\} \subseteq \mathrm{Vec}(X).$$

We get a homomorphism of Lie algebras

$$\rho\colon \mathrm{Vec}_Y(X) \to \mathrm{Vec}(Y), \quad \xi \mapsto \xi|_Y,$$

whose kernel consists of the vector fields on X vanishing on Y. This map is surjective if X is a vector space, but not in general as one can see from Example 5 below.

embedding.lem

Lemma 2. *Let $X \subseteq \mathbb{A}^n$ be a closed subvariety. Denote by $\partial_i \in \mathrm{Vec}(X)$ the images of the ∂_{x_i} and by $\bar{x}_i \in \mathcal{O}(X)$ the images of the x_i. Then we have an embedding*

$$\mathrm{Vec}(X) \hookrightarrow \mathcal{O}(X)^n, \ \xi \mapsto (\xi(\bar{x}_1), \ldots, \xi(\bar{x}_n)).$$

In particular, every vector field $\xi \in \mathrm{Vec}(X)$ has a uniquely defined representation $\xi = f_1\partial_1 + \cdots + f_n\partial_n$ with $f_i \in \mathcal{O}(X)$. Moreover, $\mathrm{Vec}(X)$ is a torsion-free $\mathcal{O}(X)$-module.

Proof. We have mentioned above that $\mathrm{Vec}_X(\mathbb{A}^n) \to \mathrm{Vec}(X)$ is surjective, i.e. every vector field ξ of X has the form $\xi = f_1\partial_1 + \cdots + f_n\partial_n$ with $f_i \in \mathcal{O}(X)$. This representation is unique, because $f_i = \xi(\bar{x}_i)$. The last statement is clear. □

Remark 3. It is in general not true that for a closed subvariety $Y \subseteq X$ the induced surjective homomorphism $\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} \mathrm{Vec}_Y(X) \twoheadrightarrow \mathrm{Vec}(Y)$ is an isomorphism. An example will be given below, see Example 2.

The following lemma can be found in [GK17, Lemma 2.5]. It is essentially due to Seidenberg [Sei67].

D-invariant.lem

Lemma 3. *Let $\mathcal{D} \subseteq \mathrm{Vec}(X)$ be a set of vector fields.*

1. *Sums and intersections of $\mathcal{D}$-invariant ideals are $\mathcal{D}$-invariant.*
2. *If $I \subseteq \mathcal{O}(X)$ is a $\mathcal{D}$-invariant ideal, then so is $\sqrt{I}$.*
3. *If $Y_\lambda \subseteq X$, $\lambda \in \Lambda$, are $\mathcal{D}$-invariant closed subvarieties, then so is $\bigcap_{\lambda\in\Lambda} Y_\lambda$.*
4. *If the closed subvariety $Y \subseteq X$ is $\mathcal{D}$-invariant, then every irreducible component of Y is $\mathcal{D}$-invariant.*

Proof. (1) is clear, and (3) follows from (1) and (2).

(2) It suffices to show that if $f^n = 0$, then $(\xi f)^m = 0$ for some $m > 0$. Let $e_0 \geq 0$ be the minimal e such that there exists a $q \geq 0$ with $f^e \cdot (\xi f)^q = 0$. If $e_0 = 0$, we are done. So assume that $e_0 > 0$. Then

$$0 = \xi(f^{e_0} \cdot (\xi f)^q) \cdot \xi f = e_0 f^{e_0 - 1} \cdot (\xi f)^{q+1} + q f^{e_0} \cdot (\xi f)^q \cdot \xi^2 f = e_0 f^{e_0-1} \cdot (\xi f)^{q+1},$$

contradicting the minimality of e_0.

(4) It suffices to consider the case where $Y = X$, hence $(0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_k$ where the $\mathfrak{p}_i$ are the minimal primes of $\mathcal{O}(X)$. For every i choose an element $p_i \in \bigcap_{j\neq i} \mathfrak{p}_j \setminus \mathfrak{p}_i$. Then $\mathfrak{p}_i = \{p \in \mathcal{O}(X) \mid p_i p = 0\}$, and the same holds for every power of p_i. For every $p \in \mathfrak{p}_i$ we find

$$0 = p_i \xi(p_i p) = p_i(p_i \xi p + p \xi p_i) = p_i^2 \xi p,$$

hence $\xi(p) \in \mathfrak{p}_i$. □

cusp.exa **Example 2.** Consider the cuspidal curve $C := \mathcal{V}(y^2 - x^3) \subset \mathbb{K}^2$. Then one shows that $\operatorname{Vec}(C)$ is generated, as an $\mathcal{O}(C)$-module, by $2\bar{x}\partial_x + 3\bar{y}\partial_y$ and $2\bar{y}\partial_x + 3\bar{x}^2\partial_y$. Thus all vector fields vanish in the origin: $\mathcal{V}(\operatorname{Vec}(C)) = \{(0,0)\}$. One also sees that the canonical homomorphism $\mathcal{O}(C) \otimes_{\mathbb{K}[x,y]} \operatorname{Vec}_C(\mathbb{A}^2) \to \operatorname{Vec}(C)$ is surjective, but has a non-trivial kernel.
(In fact, the vector fields $\xi := 2x\partial_x + 3y\partial_y, \eta := 2y\partial_x + 3x^2\partial_y \in \operatorname{Vec}_C(\mathbb{A}^2)$ generate $\operatorname{Vec}_C(\mathbb{A}^2)$ as an $\mathbb{K}[x,y]$-module. Then $\bar{y} \otimes \xi - \bar{x} \otimes \eta \in \mathcal{O}(C) \otimes_{\mathbb{K}[x,y]} \operatorname{Vec}_C(\mathbb{A}^2)$ is a nonzero element from the kernel.)

no-zeroes.exa **Example 3.** Let X be an arbitrary affine variety. Then the constant vector field ∂_t parallel to $\mathbb{A}^1$ has no zeroes, hence $\mathcal{V}(\operatorname{Vec}(X \times \mathbb{A}^1)) = \emptyset$. More generally, if the vector fields on Y have no zeroes, then the same is true for the vector fields on $X \times Y$ for any X, because any vector field ξ on Y defines a vector field on $X \times Y$ by $\xi(f \otimes h) := f \otimes \xi(h)$.

surface.exa **Example 4.** For the normal surface $S := \mathcal{V}(x^2 + y^2 + z^2) \subset \mathbb{K}^3$ with an isolated singularity in 0 we find that $\operatorname{Vec}(S)$ is generated, as an $\mathcal{O}(S)$-module, by

$$\xi_1 := \bar{y}\partial_x - \bar{x}\partial_y, \quad \xi_2 := \bar{z}\partial_x - \bar{x}\partial_z, \quad \xi_3 := \bar{z}\partial_y - \bar{y}\partial_z,$$

with the relation $\bar{z}\xi_1 - \bar{y}\xi_2 + \bar{x}\xi_3 = 0$. It follows that all vector fields vanish in the singular point 0, and the $\operatorname{Vec}(S)$ becomes a free module of rank 2 over the open sets S_x, S_y and S_z.

Whitney.exa **Example 5.** Consider the Whitney umbrella $Y := \mathcal{V}(x^2 - y^2 z) \subset \mathbb{K}^3$. The singular locus $Y_{sing} = \mathcal{V}_Y(\bar{x}, \bar{y})$ is the z-axis, and the vector fields are generated, as an $\mathcal{O}(Y)$-module, by

$$\bar{y}\partial_y - 2\bar{z}\partial_z, \quad \bar{x}\partial_x + 2\bar{z}\partial_z, \quad \bar{y}\bar{z}\partial_x + \bar{x}\partial_y, \quad \bar{y}^2\partial_x + 2\bar{x}\partial_z.$$

They are all parallel to Y_{sing}, and they all vanish in the origin. Thus Y_{sing} is invariant (see Proposition 2 from the next section), and one finds $\mathcal{V}(\operatorname{Vec}(Y)) = \{0\}$. This shows that the vector field ∂_z of the singular locus $Y_{sing} \simeq \mathbb{A}^1$ cannot be lifted to a vector field on Y, i.e. the restriction map $\operatorname{Vec}_{Y_{sing}}(Y) \to \operatorname{Vec}(Y_{sing})$ is not surjective.

Sing.sec 1.3. **Vector fields and singularities.** A first important result due to Seidenberg says that the singular locus of an affine variety is invariant under all vector fields, and that every invariant subvariety is contained in the singular locus.

Localization.lem **Lemma 4.** *Let X be an affine variety.*

(1) *If $f \in \mathcal{O}(X)$ is a nonzero element, then we have an isomorphism*

$$\mathcal{O}(X_f) \otimes_{\mathcal{O}(X)} \operatorname{Vec}(X) \xrightarrow{\sim} \operatorname{Vec}(X_f).$$

(2) *If X is irreducible, then $\operatorname{Vec}(X)$ is a torsion-free $\mathcal{O}(X)$-module of rank equal to $\dim X$.*
(3) *If X is smooth, then $\operatorname{Vec}(X)$ is a projective $\mathcal{O}(X)$-module.*
(4) *If $x \in X$ is a smooth point, then $\varepsilon_x \colon \operatorname{Vec}(X) \to T_x X$ is surjective.*

Proof. (1) This follows from the universal properties of $\operatorname{Der}(R)$ and of the localization R_f. The details are well-known and left to the reader.

(2) Denote by K the field of fractions of $\mathcal{O}(X)$. Then $K \otimes_{\mathcal{O}(X)} \operatorname{Der}(\mathcal{O}(X)) \simeq \operatorname{Der}(K)$, and the latter is known to be a free K-module of rank $\operatorname{tdeg}_{\mathbb{K}} K = \dim X$.

(3) Denote by $p \colon TX \to X$ the tangent bundle. This is defined for any (affine) variety, and the sections of p are exactly the vector fields, see [Kra16, Appendix A.4.5].

If X is smooth, then $TX \to X$ is a vector bundle, and so the $\mathcal{O}(X)$-module of sections is projective.

(4) If $f \in \mathcal{O}(X)$ is nonzero in $x \in X$, then $\varepsilon_x\colon \operatorname{Vec}(X) \to T_xX$ has the same image as $\varepsilon_x\colon \operatorname{Vec}(X_f) \to T_xX$. Localizing with a suitable $f \in \mathcal{O}(X)$, $f(x) \neq 0$, we can therefore assume that X is smooth and that the vector bundle $TX \to X$ is trivial, i.e. the $\mathcal{O}(X)$-module $\operatorname{Vec}(X)$ is free. Then the claim is clear. ☐

This has the following consequence.

variant-is-singular.prop

Proposition 1. *If a strict closed subvariety $Y \subsetneqq X$ is invariant under all vector fields $\xi \in \operatorname{Vec}(X)$, then $Y \subseteq X_{sing}$.*

Proof. We can assume that Y is irreducible (Lemma 3(4)). If $Y \not\subseteq X_{sing}$, then there is an open dense subset $U \subseteq Y$ consisting of points which are smooth in Y and smooth in X. For these points $y \in U$ we have $T_yY \subsetneqq T_yX$, contradicting the fact that the vector fields $\xi \in \operatorname{Vec}(X)$ span the tangent space T_xX in every smooth point $x \in X$ (Lemma 4(4)). ☐

The following result is due to SEIDENBERG, see [Sei67].

Sing-is-invariant.prop

Proposition 2. *For an affine variety X, the singular locus X_{sing} is invariant under all vector fields.*

We give a proof following SIEBERT, see [Sie96, Lemma 4 and Remark 3]. We start with the following lemma.

Lemma 5. *Let $I = (f_1, \ldots, f_m) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal, and let $\xi \in \operatorname{Vec}(\mathbb{A}^n)$ be a vector field such that $\xi(I) \subseteq I$. Let $J_r \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be the ideal generated by I and all r-minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$. Then J_r is ξ-invariant.*

Proof. If A, B are two $r \times r$-matrices we define by $d(A,B) := \sum_{i=1}^r \det A_i$ where A_i is obtained from A by replacing the ith row by the ith row of B. We have $d(A,B) = \sum_{j=1}^r \det A^j$ where A^j is obtained from A by replacing the jth column by the j-column of B. In fact, writing $\det A$ as an alternating some of monomials $m := a_{1i_1}a_{2j_2}\cdots a_{rj_r}$, then $d(A,B)$ is obtained by replacing each m with the sum of r terms obtained by replacing successively each a_{ij_i} by b_{ij_i}. Clearly, $d(A,B)$ is linear in B.

Now let $d \in \mathbb{K}[x_1, \ldots, x_n]$ be a r-minor of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$:

$$d = \det M \quad \text{where} \quad M = \begin{bmatrix} \frac{\partial f_{m_1}}{\partial x_{n_1}} & \cdots & \frac{\partial f_{m_1}}{\partial x_{n_r}} \\ \\ \frac{\partial f_{m_r}}{\partial x_{n_1}} & \cdots & \frac{\partial f_{m_r}}{\partial x_{n_r}} \end{bmatrix} = \left(\frac{\partial f_{m_i}}{\partial x_{n_j}}\right)_{i,j=1,\ldots,r}.$$

For the vector field $\xi = \sum_k g_k \partial_k$ we get $\xi(d) = d(M, \xi(M))$ where the entries of $\xi(M)$ are $\xi(\frac{\partial f_{m_i}}{\partial x_{n_j}}) = \xi(\partial_{n_j}(f_{m_i}))$. We have

$$\xi(\partial_j(f_i)) = \partial_j(\xi(f_i)) - \partial_j(\xi)(f_i) \text{ where } \partial_j(\xi) := \sum_k \partial_j(g_k)\partial_k,$$

so that $\xi(M) = N' - N''$ with $N' := \left(\partial_{n_j}(\xi(f_{m_i}))\right)_{i,j}$ and $N'' := \left(\partial_{n_j}(\xi)(f_{m_i})\right)_{i,j}$. Note that $\xi(f_i) \in I$, hence $\xi(f_i) = \sum h_{ik} f_k$, and so

$$\partial_j(\xi(f_i)) = \sum_{k=1}^{m} \partial_j(h_{ik}) f_k + \sum_{k=1}^{m} h_{ik} \partial_j(f_k).$$

Thus N' is of the form $N' = N_0 + \sum_k N'_k$ where the entries of N_0 are in I and the entries of N'_k are of the form $h_{m_i k}\partial_{n_j}(f_k)$. It follows that $d(M, N_0) \in I$ and $d(M, N_k) \in J_r$. In fact, the ith row of N_k is $h_{m_i,k}(\frac{\partial f_k}{\partial x_{n_1}}, \ldots, \frac{\partial f_k}{\partial x_{n_r}})$ and so $d(M, N'_k) = \sum_i h_{m_i k} d'_i$ where each d'_i is an r-minor of the jacobian. By the linearity this shows that $d(M, N') \in J_r$.

The argument for the second term is similar, but we have to replace the rows by columns. We have

$$\partial_j(\xi)(f_i) = \sum_k \partial_j(g_k)\partial_k(f_i),$$

and so $N'' = \sum_k N''_k$ where the entries of N''_k are $\partial_{n_j}(g_k)\partial_k(f_{m_i})$. Therefore, the jth column of N''_k is $\partial_{n_j}(g_k)(\frac{\partial f_{m_1}}{\partial x_k}, \ldots, \frac{\partial f_{m_r}}{\partial x_k})^t$, and so $d(M, N''_k) = \sum_j \partial_{n_j}(g_k) d''_j$ where each d''_j is an r-minor of the jacobian. Hence, by linearity, $d(M, N'') \in J_r$, and the claim follows. □

Proof of Proposition 2. (1) Let X be irreducible of dimension $d = \dim X$. Fix an embedding $X \subseteq \mathbb{A}^n$, and let $I(X) = (f_1, \ldots, f_m) \subset \mathbb{K}[x_1, \ldots, x_n]$. Then X_{sing} is the zero locus of the ideal J generated by $I(X)$ and the $n - d + 1$-minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$. The previous lemma shows that J is invariant under all vector fields $\xi \in \operatorname{Vec}(\mathbb{A}^n)$ such that $\xi(I(X)) \subseteq I(X)$. Hence, X_{sing} is invariant under all vector fields of X, because $\operatorname{Vec}_X(\mathbb{A}^n) \to \operatorname{Vec}(X)$ is surjective.

(2) If X is reducible and $X = \bigcup_k X^{(k)}$ is the decomposition into irreducible components, then $X_{sing} = \bigcup_k X^{(k)}_{sing} \cup \bigcup_{i \neq j} X^{(i)} \cap X^{(j)}$. It follows from (1) and Lemma 3 that each member of this union is invariant under $\operatorname{Vec}(X)$, hence the claim. □

Remark 4. The invariance of the singular locus implies that the Lie algebra $\operatorname{Vec}(X)$ of a singular variety X is not simple, as shown by Siebert. In fact, for $I := I(X_{sing}) \subset \mathcal{O}(X)$ and for any $k \geq 1$ the vector fields $\mathcal{L}_{I^k}$ are ideals, because I^k is $\operatorname{Vec}(X)$-invariant. Since $\bigcap_k \mathcal{L}_{I^k} = \{0\}$, we have $\mathcal{L}_{I^k} \neq \operatorname{Vec}(X)$ for a large enough k, hence the claim.

On the other hand, Jordan showed in [Jor86] that if $\operatorname{Vec}(X)$ is not a simple Lie algebra, then there exists an invariant subvariety $Y \subsetneq X$, hence X is singular by Proposition 1. We will prove this below, following an idea of Siebert.

Smooth-is-simple.prop **Proposition 3** (Siebert, [Sie96, Proposition 1])**.** *The Lie algebra* $\operatorname{Vec}(X)$ *is simple if and only if* X *is smooth.*

Proof. Let $M \subset \operatorname{Vec}(X)$ be a proper nonzero Lie ideal. Then we find a nonzero $\mathcal{O}(X)$-Lie ideal $M' \subseteq M$ of $\operatorname{Vec}(X)$ by Lemma 9 from section 1.5 below. It follows that the ideal $I := M'(\mathcal{O}(X)) = \langle \xi(f) \mid \xi \in M', f \in \mathcal{O}(X) \rangle \subseteq \mathcal{O}(X)$ is invariant. In fact, for $\delta \in \operatorname{Vec}(X)$, $\xi \in M'$ and $f \in \mathcal{O}(X)$ we get

$$\delta(\xi(f)) = [\delta, \xi](f) + \xi(\delta(f)) \in M'(\mathcal{O}(X)).$$

Clearly, $I \neq (0)$, and we claim that $I \neq \mathcal{O}(X)$. Otherwise, we have $1 = \sum_i \xi_i f_i$ for some $\xi_i \in M'$ and $f_i \in \mathcal{O}(X)$. But this implies for all $\delta \in \mathrm{Vec}(X)$ that

$$\delta = \sum_i \xi_i(f_i)\delta = \sum_i [\xi_i, f_i\delta] - \sum_i f_i[\xi_i, \delta] \in M',$$

contradicting the fact that $M' \subsetneqq \mathrm{Vec}(X)$. Hence, the zero set of I is a proper invariant subvariety of X, and the claim follows from Proposition 1. invariant-is-singular.prop ☐

1.4. **A Galois-correspondence.** We shortly describe a Galois-correspondence between ideals in the coordinate ring $\mathcal{O}(X)$ and $\mathcal{O}(X)$-Lie subalgebras of the vector fields. It turns out that this correspondence restricts to a correspondence between invariant ideals in $\mathcal{O}(X)$ and $\mathcal{O}(X)$-Lie ideals.

Assume again that $\mathcal{L} \subseteq \mathrm{Vec}(X)$ an $\mathcal{O}(X)$-Lie subalgebra. Recall that $\mathcal{L}_I := \{\xi \in \mathcal{L} \mid \xi(\mathcal{O}(X)) \subseteq I\} \subseteq \mathcal{L}$ for an ideal $I \subseteq \mathcal{O}(X)$. We thus get a map

$$\Phi\colon \{\text{ideals in } \mathcal{O}(X)\} \xrightarrow{I \mapsto \mathcal{L}_I} \{\mathcal{O}(X)\text{-Lie subalgebras of } \mathcal{L}\}.$$

Note that $\Phi(0) = \mathcal{L}_{(0)} = (0)$ and $\Phi(\mathcal{O}(X)) = \mathcal{L}_{\mathcal{O}(X)} = \mathcal{L}$. There is also a map in the other direction, namely

$$\Psi\colon \{\mathcal{O}(X)\text{-Lie subalgebras of } \mathcal{L}\} \xrightarrow{L \mapsto \mathcal{I}_L} \{\text{ideals in } \mathcal{O}(X)\}$$

where $\mathcal{I}_L := \mathcal{O}(X)L(\mathcal{O}(X)) \subseteq \mathcal{O}(X)$. Here we get $\Psi(0) = \mathcal{I}_{(0)} = (0)$, whereas $\Psi(\mathcal{O}(X)) = \mathcal{I}_{\mathcal{O}(X)} \subseteq \mathcal{O}(X)$ can be a proper ideal.

The following lemma is easy. It shows that Φ and Ψ define a *Galois correspondence* between the ideals in $\mathcal{O}(X)$ and the $\mathcal{O}(X)$-Lie subalgebras of $\mathcal{L}$.

Lemma 6. (1) *If $I \subseteq J$ are ideals in $\mathcal{O}(X)$, then $\mathcal{L}_I \subseteq \mathcal{L}_J$. If $Z \subseteq Y \subseteq X$ are closed subvarieties, then $\mathcal{L}_Z \supseteq \mathcal{L}_Y$.*
(2) *If $L \subseteq M$ are $\mathcal{O}(X)$-Lie subalgebras of $\mathcal{L}$, then $\mathcal{I}_L \subseteq \mathcal{I}_M$.*
(3) *For an ideal $I \subseteq \mathcal{O}(X)$ we have $\mathcal{I}_{\mathcal{L}_I} \subseteq I$, with equality if $I = \mathcal{I}_L$.*
(4) *For an $\mathcal{O}(X)$-Lie subalgebra L we have $L \subseteq \mathcal{L}_{\mathcal{I}_L}$, with equality if $L = \mathcal{L}_I$.*

Thus we obtain a Galois correspondence

$$\{\textit{ideals of } \mathcal{O}(X)\} \underset{\mathcal{I}_L \leftarrow L\,:\Psi}{\overset{\Phi\colon I \mapsto \mathcal{L}_I}{\rightleftarrows}} \{\mathcal{O}(X)\textit{-Lie subalgebras of } \mathcal{L}\}$$

where Φ and Ψ induce bijections between the images of Φ and of Ψ.

Proof. (1) and (2) are clear.

The first part of (3) follows from $L(\mathcal{O}(X)) \subseteq \mathcal{I}_L$, and the first part of (4) from $\mathcal{L}_I(\mathcal{O}(X)) \subseteq I$.

For $L = \mathcal{L}_I$ we have that $L \subseteq \mathcal{L}_{\mathcal{I}_L}$ by (3), and $\mathcal{I}_L \subseteq I$ by (4), and so $\mathcal{L}_{\mathcal{I}_L} \subseteq \mathcal{L}_I = L$ by (1), and the second claim of (3) follows. The second claim of (4) is obtained in a similar way. ☐

The next lemma shows that the Galois correspondence above restricts to a Galois correspondence

$$\{\mathcal{L}\text{-invariant ideals of } \mathcal{O}(X)\} \underset{\mathcal{I}_L \leftarrow L\,:\Psi}{\overset{\Phi\colon I \mapsto \mathcal{L}_I}{\rightleftarrows}} \{\mathcal{O}(X)\text{-Lie ideals of } \mathcal{L}\}$$

Lemma 7. (1) *If $L \subseteq \mathcal{L}$ is an $\mathcal{O}(X)$-Lie ideal, then $\mathcal{I}_L \subseteq \mathcal{O}(X)$ is an $\mathcal{L}$-invariant ideal.*

(2) *If $I \subseteq \mathcal{O}(X)$ is an $\mathcal{L}$-invariant ideal, then $\mathcal{L}_I \subseteq \mathcal{L}$ is an $\mathcal{O}(X)$-Lie ideal.*

Proof. (1) Let $f \in \mathcal{I}_L$ and $\delta \in \mathcal{L}$. We have to show that $\delta(f) \in \mathcal{I}_L$. By definition, $f = \sum_i \xi_i(f_i)$ for some $\xi_i \in L$ and $f_i \in \mathcal{O}(X)$. Hence

$$\delta(f) = \sum_i \delta(\xi_i(f_i)) = \sum_i [\delta, \xi_i](f_i) + \sum_i \xi_i(\delta(f_i)) \in L(\mathcal{O}(X)) = \mathcal{I}_L.$$

(2) Let $\delta \in \mathcal{L}$ and $\xi \in \mathcal{L}_I$. Then

$$[\delta, \xi](f) = \delta(\xi(f)) - \xi(\delta(f)) \in I$$

for every $f \in \mathcal{O}(X)$, hence $[\delta, \xi] \in \mathcal{L}_I$. □

fundamental.sec

1.5. **The Fundamental Lemma.** In this section we prove a kind of "Nullstellensatz", namely a relation between maximal Lie subalgebras of the vector fields and the points of the variety X. A central result is the following lemma which is a variation of results due to Grabowski (see [Gra79]).

max-LA.lem

Fundamental Lemma. *Let $\mathcal{L} \subseteq \operatorname{Vec}(X)$ be $\mathcal{O}(X)$-Lie subalgebra, and let $L \subset \mathcal{L}$ be a maximal proper Lie subalgebra of finite codimension. Set $I := \{f \in \mathcal{O}(X) \mid f \cdot \mathcal{L} \subseteq L\}$. Then we have the following.*

(1) *$I \neq (0)$, and I is L-invariant.*
(2) *If $\mathcal{V}(I) \subseteq X$ is not $\mathcal{L}$-invariant, then I is a maximal ideal $\mathfrak{m}$, and $L = \mathcal{L}_\mathfrak{m}$.*
(3) *If $\mathcal{L}_x \neq \mathcal{L}$ for some $x \in X$, then $\mathcal{L}_x$ is a proper maximal Lie subalgebra.*

Proof. (1) Setting $L' := \{\delta \in L \mid [\delta, \mathcal{L}] \subseteq L\}$, the following Lemma 8 shows that for $\delta, f \cdot \delta \in L'$ we have $\delta(f)^2 \in I$. Since $L' \subseteq \mathcal{L}$ has finite codimension, it follows that for every $\xi \in L$ the subspace $\{f \in \mathcal{O}(X) \mid f \cdot \xi \in L\} \subseteq \mathcal{O}(X)$ has finite codimension. On the other hand, $\ker \xi \subset \mathcal{O}(X)$ has infinite codimension if $\xi \neq 0$, because the image contains with $\xi(f)$ the span $\langle f^n \xi(f) \mid n \in \mathbb{N} \rangle$. Therefore, for every $\xi \in L'$, $\xi \neq 0$, there is an $f \in \mathcal{O}(X)$ such that $\xi, f \cdot \xi \in L'$ and $\xi(f) \neq 0$. Hence, $I \neq (0)$.

For $\xi \in \mathcal{L}$, $\delta \in L$ and $f \in I$ we have

$$[\delta, f \cdot \xi] = \delta(f) \cdot \xi + f \cdot [\delta, \xi] \in L,$$

showing that $\delta(f) \cdot \mathcal{L} \subseteq L$, hence $\delta(f) \in I$ for all $\delta \in L$ and $f \in I$. This means that I is an L-invariant ideal.

(2) It follows from (1) and Lemma 3 that $\sqrt{I}$ is L-invariant, as well as every minimal prime $\mathfrak{p} \supseteq I$. Hence

$$L \subseteq \mathcal{L}(I) := \{\xi \in \mathcal{L} \mid \xi(I) \subseteq I\} \subseteq \mathcal{L}(\sqrt{I}) \subseteq \mathcal{L}(\mathfrak{p}).$$

Since $\sqrt{I}$ is not $\mathcal{L}$-invariant, we have $\mathcal{L}(\mathfrak{p}) \neq \mathcal{L}$ for at least one of the minimal primes $\mathfrak{p} \supset I$, and so $L = \mathcal{L}(\mathfrak{p})$, because L is maximal. This implies that I is the annihilator of the $\mathcal{O}(X)$-module $\mathcal{L}/\mathcal{L}(\mathfrak{p})$. Since the latter is finite dimensional, we see that $I \subset \mathcal{O}(X)$ has finite codimension, and so $\mathfrak{p}$ is a maximal ideal, $\mathfrak{p} = \mathfrak{m}_x$. But then $\mathcal{L}(\mathfrak{m}_x) = \mathcal{L}_x$, because $\mathcal{O}(X) = \mathbb{K} \oplus \mathfrak{m}_x$, and so $L = \mathcal{L}_x$ as claimed.

(3) Assume that $\mathcal{L}_x \subseteq L \subsetneqq \mathcal{L}$. Then the ideal I is equal to $\mathfrak{m}_x$, because $\mathfrak{m}_x \mathcal{L} \subseteq \mathcal{L}_x$, and so $L = \mathcal{L}_x$. □

The next lemma is formulated in [Sie96, Lemma 1] and is contributed to [Ame75]. We will give a short proof.

transporter.lem

Lemma 8. *Let $\mathcal{L} \subseteq \mathrm{Vec}(X)$ be $\mathcal{O}(X)$-Lie subalgebra, and let $L \subseteq \mathcal{L}$ be a Lie subalgebra. Set $L' := \{\delta \in L \mid [\delta, \mathcal{L}] \subseteq L\}$. If $\delta \in L'$ and $f \cdot \delta \in L'$ for some $f \in \mathcal{O}(X)$, then $\delta(f)^2 \cdot \mathcal{L} \subseteq L$.*

Proof. Let $\delta, f \cdot \delta \in L'$. Then, for any $\xi \in \mathcal{L}$, we get

$$[f \cdot \delta, \xi] = f \cdot [\delta, \xi] - \xi(f) \cdot \delta \in L \quad \text{and}$$
$$[\delta, f \cdot \xi] = f \cdot [\delta, \xi] + \delta(f) \cdot \xi,$$

hence

$$(*) \qquad \delta(f) \cdot \xi + \xi(f) \cdot \delta \in L.$$

Substituting in $(*)$ ξ by $\xi(f) \cdot \delta$ we get $2\delta(f)\xi(f) \cdot \delta \in L$, and substituting ξ by $\delta(f) \cdot \xi$ we find $\delta(f)^2 \cdot \xi + \xi(f)\delta(f) \cdot \delta \in L$. Thus $\delta(f)^2 \cdot \xi \in L$. □

The lemma has another consequence which we used earlier in the proof of Proposition 3. smooth-is-simple.prop

O(X)-ideal.lem

Lemma 9. *A nonzero Lie ideal of $\mathcal{L}$ contains a nonzero $\mathcal{O}(X)$-Lie ideal.*

Proof. Let $M \subset \mathcal{L}$ be a Lie ideal.

(a) We first claim that the maximal $\mathcal{O}(X)$-submodule $M' \subset M$ is a Lie ideal. In fact, if $f \in \mathcal{O}(X)$, $\delta \in M'$ and $\xi \in \mathcal{L}$, we get

$$f \cdot [\xi, \delta] = [\xi, f \cdot \delta] - \xi(f) \cdot \delta \in M,$$

hence $[\xi, \delta] \in M'$.

(b) Now consider the ideal $I := \{f \in \mathcal{O}(X) \mid f \cdot \mathcal{L} \subseteq M\}$. If $I \neq 0$, then $I \cdot \mathcal{L} \subset M$ is a nontrivial $\mathcal{O}(X)$-submodul, and the claim follows by (a). For $\delta \in M$ and $f \in \mathcal{O}(X)$ we have $\delta(f) \cdot \delta = [\delta, f \cdot \delta] \in M$, hence $\delta(\delta(f))^2 \in I$ by Lemma 8. transporter.lem Choosing an embedding $X \subseteq \mathbb{A}^n$ and setting $\delta = \sum_i g_i \partial_i$, we find $\delta(\bar{x}_i) = g_i$. If $\delta(g_i) \neq 0$, we are done. If $\delta(g_i) = 0$, then $\delta(\bar{x}_i^2) = g_i^2$, and we are also done. □

max-LA.sec

1.6. **Reconstructing points from** $\mathrm{Vec}(X)$**.** Here we will show that one can reconstruct the smooth points of a variety X from the vector fields using only the Lie algebra structure of $\mathrm{Vec}(X)$.

If $L \subseteq \mathcal{L}$ is a Lie subalgebra we denote by $L^{[\infty]} \subseteq L$ the maximal Lie ideal of $\mathcal{L}$ contained in L. This notion has the following geometric interpretation.

Proposition 4. (1) *If L is an $\mathcal{O}(X)$-Lie subalgebra, then $L^{[\infty]}$ is an $\mathcal{O}(X)$-Lie ideal of $\mathcal{L}$.*

(2) *Let $Y \subseteq X$ be an irreducible closed subvariety, and let $Z \subseteq X$ be the smallest closed $\mathcal{L}$-invariant subvariety containing Y. Then $\mathcal{L}_Y^{[\infty]} = \mathcal{L}_Z$.*

Proof. (1) If $M \subseteq \mathcal{L}$ is an ideal, then $\mathcal{O}(X) \cdot M$ is also an ideal. In fact, for $\delta \in \mathcal{L}$, $\mu \in M$ and $f \in \mathcal{O}(X)$ we get

$$[\delta, f \cdot \mu] = \delta(f) \cdot \mu + f \cdot [\delta, \mu] \in \mathcal{O}(X) \cdot M.$$

(2) Since $\mathcal{I}_{\mathcal{L}_Y} = I(Y)$ is a prime ideal, we have $\mathcal{I}_{\mathcal{L}_Y^{[\infty]}} \subseteq \sqrt{\mathcal{I}_{\mathcal{L}_Y^{[\infty]}}} \subseteq I(Y)$. Moreover, $\sqrt{\mathcal{I}_{\mathcal{L}_Y^{[\infty]}}} = I(Z')$ where $Z' \supset Y$ is an $\mathcal{L}$-invariant closed subvariety of X, hence $Z \subseteq Z'$. It follows that

$$\mathcal{L}_Y^{[\infty]} \subseteq \mathcal{L}_{\mathcal{I}_{\mathcal{L}_Y^{[\infty]}}} \subseteq \mathcal{L}_{Z'} \subseteq \mathcal{L}_Z \subseteq \mathcal{L}_Y,$$

and so $\mathcal{L}_Y^{[\infty]} = \mathcal{L}_Z$, because of the maximality of $\mathcal{L}_Y^{[\infty]}$. $\square$

max.cor **Corollary 1.** *Assume that X is irreducible, and let $x \in X$.*

(1) *If $\mathcal{L}_x \neq \mathcal{L}$, then $\mathcal{L}_x^{[\infty]}$ has infinite codimension.*
(2) *If x is a smooth point of X, then $\mathcal{L}_x^{[\infty]} = (0)$.*
(3) *If $x \in X_{sing}$ and $\mathcal{L}_x \neq \mathcal{L}$, then $\mathcal{L}_x^{[\infty]} \neq (0)$.*
(4) *Let $L \subset \mathcal{L}$ be a proper maximal Lie subalgebra different from $\mathcal{L}_x$ for any $x \in X$. Then $L^{[\infty]} \neq (0)$.*

Proof. Denote by $Z \subseteq X$ the smallest $\mathcal{L}$-invariant subvariety containing x. Then $\mathcal{L}_x^{[\infty]} = \mathcal{L}_Z$ by the proposition above.

(1) Since there are vector fields in $\mathcal{L}$ which do not vanish in x, we see that $\dim Z \geq 1$. Since $\mathcal{L}/\mathcal{L}_Z$ is an $\mathcal{O}(Z)$-Lie subalgebra of $\operatorname{Vec}(Z)$, it has infinite dimension.

(2) If $x \in X$ is a smooth point, then $Z = X$ by Proposition 1 from the next section. Hence $\mathcal{L}_x^{[\infty]} = \mathcal{L}_X = (0)$.

(3) If $x \in X_{sing}$, then $Z \subseteq X_{sing} \subsetneq X$ and so $\mathcal{L}_x^{[\infty]} = \mathcal{L}_Z \supseteq \mathcal{L}_{X_{sing}} \neq (0)$.

(4) Define $I := \{f \in \mathcal{O}(X) \mid f \cdot \mathcal{L} \subseteq L\}$. It follows from Lemma 1.5 that I is a nonzero ideal of $\mathcal{O}(X)$ and that $\sqrt{I}$ is $\mathcal{L}$-invariant. Then there is an $m \geq 1$ such that $J := (\sqrt{I})^m \subseteq I$. Since J is $\mathcal{L}$-invariant it follows that $J \cdot \mathcal{L} \subseteq L$ is a nonzero $\mathcal{O}(X)$-Lie ideal, hence $L^{[\infty]} \neq (0)$. In fact, for $\delta, \xi \in \mathcal{L}$ and $f \in J$ we have

$$[\delta, f \cdot \xi] = \delta(f) \cdot \xi + f \cdot [\delta, \xi] \in J \cdot \mathcal{L},$$

hence the claim. $\square$

We now show how one can reconstruct the (smooth) points of X from the Lie algebra $\operatorname{Vec}(X)$.

main-theorem **Theorem 1.** (1) *Assume that X does not contain a proper $\mathcal{L}$-invariant subvariety. Then the map $x \mapsto \mathcal{L}_x$ gives a bijection*

$$X \xrightarrow{\simeq} \operatorname{Max}(\mathcal{L}) := \left\{ \begin{array}{l} \textit{proper maximal Lie subalgebras} \\ L \subset \mathcal{L} \textit{ of finite codimension} \end{array} \right\},$$

(2) *If X is irreducible, then the map $x \mapsto \operatorname{Vec}(X)_x$ defines a bijection*

$$X \setminus X_{sing} \xrightarrow{\sim} \operatorname{Max}_0(\operatorname{Vec}(X)) := \left\{ \begin{array}{l} \textit{proper maximal Lie subalgebras } L \subset \operatorname{Vec}(X) \\ \textit{of finite codimension such that } L^\infty = (0) \end{array} \right\}.$$

Proof. (1) Since $\mathcal{L}(\mathcal{O}(X)) = \mathcal{O}(X)$ we see that $\mathcal{L}_x$ is a proper Lie subalgebra of finite codimension. The Fundamental Lemma implies that $\mathcal{L}_x$ is maximal for any $x \in X$, and that every maximal proper Lie subalgebra is of this form.

(2) Set $\mathcal{L} := \operatorname{Vec}(X)$. If $x \in X$ is a smooth point, then $\mathcal{L}_x \neq \mathcal{L}$ (Lemma 4(4)), hence is a maximal proper Lie subalgebra by the Fundamental Lemma, and $\mathcal{L}_x^{[\infty]} = (0)$ by Corollary 1(2). Hence, the map $x \mapsto \mathcal{L}_x$ sends the smooth point into $\operatorname{Max}_0(\operatorname{Vec}(X))$.

Now let $L \in \operatorname{Max}_0(\operatorname{Vec}(X))$. Then part (3) and (4) of Corollary 1 imply that $L = \mathcal{L}_x$ for a smooth point $x \in X$. $\square$

In [Sie96] there are several variants of the theorem above. For example, it was already shown by GRABOWSKI that (1) holds under the more general assumption that $\mathcal{L}$ has no zeroes. An example of such a variety is $X := C \times \mathbb{A}^1$ where $C =$

$\mathcal{V}(y^2-x^3) \subset \mathbb{K}^2$, see Example 3. In fact, the vector field ∂_z does not vanish anywhere on the singular locus $X_{sing} = \{(0,0)\} \times \mathbb{A}^1$.

1.7. **Regular functions.** In this last section we define regular functions on the set of maximal proper Lie subalgebras of $\operatorname{Vec}(X)$ using only the Lie algebra structure of $\operatorname{Vec}(X)$. With the relation between points of X and maximal proper Lie subalgebras proved in the previous section, we then show that for a normal variety X, these functions coincide with the regular functions on X. This implies that a normal affine variety X is determined, up to isomorphism, by the Lie algebra $\operatorname{Vec}(X)$.

From now on we assume that X is irreducible. Consider the open dense set $X' := X \setminus X_{sing}$ of smooth points of X. We have seen in Theorem 1(2) that there is a bijection $X' \xrightarrow{\sim} \mathcal{M}_X := \operatorname{Max}_0(\operatorname{Vec}(X))$, given by $x \mapsto \mathcal{L}_x$. Note that $\bigcap_{L \in \mathcal{M}_X} L = (0)$, because any ξ from the intersection vanishes on X', hence on X.

Definition 3. A $\mathbb{K}$-valued function $f\colon \mathcal{M}_X \to \mathbb{K}$ is called *regular* if the following holds:

For every $\delta \in \operatorname{Vec}(X)$ there is a $\mu \in \operatorname{Vec}(X)$ such that $f(L)\delta - \mu \in L$ for all $L \in \mathcal{M}_X$.

We denote by $\mathcal{R}(\mathcal{M}_X)$ the set of regular functions on $\mathcal{M}_X$.

reg-functions.prop

Proposition 5. (1) *The set $\mathcal{R}(\mathcal{M}_X)$ is $\mathbb{K}$-algebra.*

(2) *$\mathcal{L}$ is stable under $\mathcal{R}(\mathcal{M}_X)$.*

(3) *$\mathcal{R}(\mathcal{M}_X)$ contains $\mathcal{O}(X)$ and consists of rational functions defined on X'.*

(4) *If $\operatorname{codim}_X X_{sing} \geq 2$, then $\mathcal{R}(\mathcal{M}_X)$ is a finite extension of $\mathcal{O}(X)$.*

Proof. Set $\mathcal{L} := \operatorname{Vec}(X)$.

(1) Let $f_1, f_2 \in \mathcal{R}(\mathcal{M}_X)$ and $\delta \in \mathcal{L}$. Then there exist μ_1, μ_2 such that $f_i(L)\delta - \mu_i \in L$ for all $L \in \mathcal{M}_X$, $i = 1, 2$. Therefore, $(c_1 f_1 + c_2 f_2)(L)\delta - (c_1\mu_1 + c_2\mu_2) \in L$, hence $c_1 f_1 + c_2 f_2 \in \mathcal{R}(\mathcal{M}_X)$. There exists also $\mu_3 \in \mathcal{L}$ such that $f_2(L)\mu_1 - \mu_3 \in L$, hence $(f_1 f_2)(L)\delta - \mu_3 = f_2(L)(f_1(L)\delta - \mu_1) + (f_2(L)\mu_1 - \mu_3) \in L$, and so $f_1 f_2 \in \mathcal{R}(\mathcal{M}_X)$.

(2) For $f \in \mathcal{R}(\mathcal{M}_X)$ define $f \cdot \delta := \mu$ if $f(L)\delta - \mu \in L$ for all $L \in \mathcal{M}_X$. This is well-defined, because $\bigcap_{L \in \mathcal{M}_X} L = \{0\}$.

(3) For $f \in \mathcal{O}(X)$, $x \in X'$ and $L = \mathcal{L}_x \in \mathcal{M}_X$ we define $f(L) := f(x)$. This is a regular function on $\mathcal{M}_X$. In fact, if $\delta \in \mathcal{L}$ and $\mu := f \cdot \delta$, then $f(x)\delta_x = \mu_x$ for all $x \in X$, and so $f(\mathcal{L}_x)\delta - \mu \in \mathcal{L}_x$ for $x \in X'$.

We fix an embedding $X \subseteq \mathbb{K}^n$, so that every vector field $\xi \in \operatorname{Vec}(X)$ can be written as $\xi = \sum_{i=1}^n g_i \partial_i$ with uniquely defined $f_i \in \mathcal{O}(X)$ (see Lemma 2). If $f(L)\delta - \mu \in L$ for all $L \in \mathcal{M}_X$ and if we write $\delta = \sum g_i \partial_i$ and $\mu = \sum h_i \partial_i$, then

$$f(\mathcal{L}_x) g_i - h_i = (f(\mathcal{L}_x)\delta - \mu)(\bar{x}_i) \in \mathfrak{m}_x \text{ for } x \in X',$$

i.e. $f = \frac{h_i}{g_i}$ is a rational function on X which has no poles on X'.

(4) If $\operatorname{codim}_X X \setminus X' \geq 2$ and if $\tilde{X} \to X$ is the normalization, then the pullback $\tilde{r}$ of a rational function r on X with no poles on X' is regular. Hence $\mathcal{R}(\mathcal{M}_X) \subseteq \mathcal{O}(\tilde{X})$, and thus is a finite extension of $\mathcal{O}(X)$. □

As a consequence we get the following result due to Siebert [Sie96, Corollary 3].

Theorem 2. *Let X, Y be normal affine varieties. If there is an isomorphism $\operatorname{Vec}(X) \xrightarrow{\sim} \operatorname{Vec}(Y)$ of Lie algebras, then $X \simeq Y$ as varieties.*

Proof. The isomorphism $\mathrm{Vec}(X) \xrightarrow{\sim} \mathrm{Vec}(Y)$ of Lie algebras induces a bijection $\varphi\colon \mathcal{M}_X = \mathrm{Max}_0(\mathrm{Vec}(X)) \xrightarrow{\sim} \mathcal{M}_Y = \mathrm{Max}_0(\mathrm{Vec}(Y))$. The definition of regular functions above shows that φ induces an isomorphism $\varphi^*\colon \mathcal{R}(\mathcal{M}_Y) \xrightarrow{\sim} \mathcal{R}(\mathcal{M}_X)$ of $\mathbb{K}$-algebras. Now it follows from part (3) and (4) of Proposition 5 that $\mathcal{R}(\mathcal{M}_X) = \mathcal{O}(X)$ and $\mathcal{R}(\mathcal{M}_Y) = \mathcal{O}(Y)$, and the claim follows. □

Remark 5. It follows from the construction and the proof above that for normal varieties X, Y every isomorphism $\mathrm{Vec}(X) \xrightarrow{\sim} \mathrm{Vec}(Y)$ of Lie algebras is induced by an isomorphism $X \xrightarrow{\sim} Y$.

References

[Ame75] Ichiro Amemiya, *Lie algebra of vector fields and complex structure*, J. Math. Soc. Japan **27** (1975), no. 4, 545–549. MR 0400252

[GK17] Frank D. Grosshans and Hanspeter Kraft, *Endomorphisms and first integrals*, arXiv:1703.01890 [math.RT] (2017).

[Gra79] J. Grabowski, *Isomorphisms and ideals of the Lie algebras of vector fields*, Invent. Math. **50** (1978/79), no. 1, 13–33. MR 516602

[Jor86] D. A. Jordan, *On the ideals of a Lie algebra of derivations*, J. London Math. Soc. (2) **33** (1986), no. 1, 33–39. MR 829385

[Kra16] Hanspeter Kraft, *Algebraic Transformation Groups: An Introduction*, Mathematisches Institut, Universität Basel, `http://kraftadmin.wixsite.com/hpkraft`, 2016.

[Sei67] A. Seidenberg, *Differential ideals in rings of finitely generated type*, Amer. J. Math. **89** (1967), 22–42. MR 0212027

[Sie96] Thomas Siebert, *Lie algebras of derivations and affine algebraic geometry over fields of characteristic* 0, Math. Ann. **305** (1996), no. 2, 271–286. MR 1391215 (97e:17033)

Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, CH-4051 Basel

E-mail address: `Hanspeter.Kraft@unibas.ch`