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Perpetuants: A Lost Treasure

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To the memory of Gian-Carlo Rota.

The purpose of this paper is to discuss the classical, and forgotten, notion of *perpetuants*, see Definition 2.13, and in particular to exhibit a basis of these elements in Theorem 4.9, thus closing an old line of investigation started by J. J. Sylvester in 1882. In order to do this we also give a proof of the classical Theorem of Stroh computing their dimensions.

Introduction

Perpetuant (see Definition 2.13) is one of the several concepts invented by J. J. Sylvester in his investigations of *covariants* for binary forms.

One of the main goals of classical invariant theorists was to exhibit a *minimal set of generators or "Groundforms"* for the rings of invariants under consideration, in particular for covariants of binary forms. This proved soon to be a formidable task achieved only for forms of degree up to 6. Perpetuants are strictly connected to the quest of a minimal set of generators for a *limit algebra S of covariants*, defined below in Formula 1.

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The simplest description of S, but not very instructive, is as the subalgebra of the polynomial ring $R = \mathbb{C}[a_0, a_1, a_2, \ldots]$ in the infinitely many variables a_i, $i = 0, \ldots, \infty$ which is the kernel of the derivation $\boldsymbol{D} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i}$.

To the best of our knowledge such an explicit description was not achieved. With our method we shall in fact exhibit such a minimal set of generators that we call a *basis of perpetuants*. This is our main new result, Theorem 3.9. The term *perpetuant* appears in one of the 1st issues of the *American Journal of Mathematics* [21], which SYLVESTER had founded a few years before. This name will hardly appear in a mathematical paper of the past 70 years due to the complex history of invariant theory that was at some time declared dead only to resurrect several decades later.

We learned of this word from GIAN-CARLO ROTA who pronounced it with an enigmatic smile. In fact, in [9] he laments that "This area is in a particularly sorry state." We were surprised to find an entry in Wikipedia where one finds useful information, but the wrong paper of STROH is quoted.

In this entry it is mentioned that MACMAHON conjectured and STROH proved the following result.

Theorem 0.1 ([20]). The dimension of the space of perpetuants of degree $n > 2$ and weight g is the coefficient of x^g in

$$\frac{x^{2^{n-1}-1}}{(1-x^2)(1-x^3)\cdots(1-x^n)}.$$

For $n = 1$ there is just one perpetuant, of weight 0, and for $n = 2$ the number is given by the coefficient of x^g in $x^2/(1-x^2)$.

In order to prove our main Theorem 3.9 we need first to review in modern language STROH's proof that is quite remarkable and in a way already *very modern*, see Theorem 2.19. The basic new idea here is to understand STROH's mysterious "Potenziante" as a dualizing tensor.

For a history of these ideas and the contributions of CAYLEY and HAMMOND we refer to MACMAHON [13]. More about perpetuants can be found in [3, 6, 7, 10–12, 22–26].

Organization of the paper

The paper is divided into four sections. Section 1 establishes the basic notation and recalls some standard techniques from classical invariant theory. Sections 2 and 3 form the bulk of the paper. In Section 2 we give an explicit basis of S and a proof of STROH's

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Theorem. In Section 3 we prove the main theorem giving a basis of the perpetuants. Finally, Section 4 is an appendix, explaining the role of this material in the classical theory of binary forms. We also explain a direct approah which can be used to calculate a minimal set of generating covariants.

1 Back to 19th century

1.1 Semi-invariants and covariants

One has to start with the classical notion of *semi-invariant*. The name is probably due to Cayley (see [21], cf. [2]), but today, with this name, we understand a different notion, so that we will use the term *U-invariant*.

Consider the $n+1$-dimensional vector space $P_n = P_n(x) \subset \mathbb{C}[x]$ of polynomials of degree $\leq n$ in the variable x.

On this acts the additive group $\mathbb{C}^+$ by

$$p(x) \mapsto p(x-\lambda) \text{ for } \lambda \in \mathbb{C}^+ \text{ and } p(x) \in P_n.$$

As usual this action extends to an action of $\mathbb{C}^+$ as automorphisms of the algebra $\mathcal{O}(P_n)$ of polynomial functions on P_n.

Definition 1.2. The *algebra* $S(n)$ of *U-invariants* of polynomials of degree n is the subalgebra of the algebra of polynomial functions on P_n which are invariant under the action of the group $\mathbb{C}^+$:

$$S(n) := \mathcal{O}(P_n)^{\mathbb{C}^+}.$$

The symbol U is justified since, as we shall see, the space P_n can be identified with the space of *binary forms*, that is homogeneous polynomials of degree n in two variables, over which acts the group $\mathrm{SL}(2,\mathbb{C})$.

The action of $\mathbb{C}^+$ should be understood as the action of the unipotent subgroup U of $\mathrm{SL}(2,\mathbb{C})$,

$$U := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{C} \right\}.$$

Remark 1.3. For the invariant theorists of the 19th century $S(n)$ is an *avatar* of covariants of binary forms of degree n, a basic tool to compute invariants. We will explain later what this means.

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The operator of derivative $\frac{d}{dx}$ maps P_n surjectively to P_{n-1} commuting with the actions of $\mathbb{C}^+$.

This induces an inclusion of duals $P_{n-1}^* \subset P_n^* \subset P_{n+1}^* \dots$, hence an inclusion of the rings $\mathcal{O}(P_n)$ of polynomial functions on P_n, and finally an inclusion $S(n) \subset S(n+1)$ of U-invariants. We thus obtain a limit ring

$$S = \bigcup_{n=0}^{\infty} S(n), \quad \text{the algebra of } U\text{-invariants.} \tag{1}$$

In order to have a more concrete description of S one needs to keep the same coordinates for the duals. It is then necessary to write a polynomial $p(x)$ as a sum of *divided powers*, setting

$$x^{[i]} := \frac{x^i}{i!}, \quad p(x) := \sum_{j=0}^{n} a_j x^{[n-j]} = \sum_{j=0}^{n} b_j x^{n-j}, \quad b_j := \frac{a_j}{(n-j)!}. \tag{2}$$

Then the operator of derivative $\dfrac{d}{dx}$ acts as $\dfrac{d}{dx}x^{[i]} = x^{[i-1]}$; hence,

$$\frac{d}{dx}p(x) = \frac{d}{dx}\sum_{j=0}^{n} a_j x^{[n-j]} = \sum_{j=0}^{n-1} a_j x^{[n-1-j]}.$$

It follows that the coordinates $a_0, a_1, \dots, a_{n-1}$ that are a basis of P_{n-1}^* are mapped to the same coordinates in P_n^*.

Therefore, the algebra $R = \mathbb{C}[a_0, \dots, a_n, \dots]$ of polynomials in the infinitely many variables a_i, $i = 0, \dots, \infty$, is the union of the algebras $R_n = \mathcal{O}(P_n)$ of polynomials on the spaces P_n, and the algebra $S = R^U$ is the ring of invariants of this infinite polynomial algebra under the action of $U = \mathbb{C}^+$.

A basic feature of divided powers is that, in the *binomial formula*, the binomial coefficients disappear:

$$(a+b)^{[i]} = \frac{(a+b)^i}{i!} = \sum_{j=0}^{i} \frac{1}{i!}\binom{i}{j} a^{i-j} b^j = \sum_{j=0}^{i} a^{[i-j]} b^{[j]}. \tag{3}$$

We thus get, for $\lambda \in \mathbb{C}^+$,

$$\lambda \cdot x^{[i]} := \frac{(x-\lambda)^i}{i!} = \sum_{j=0}^{i} (-1)^j \lambda^{[j]} x^{[i-j]},$$

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and so

$$\begin{aligned}\lambda\cdot p(x) := \lambda\cdot\sum_{j=0}^{n} a_j x^{[n-j]} &= \sum_{j=0}^{n} a_j \sum_{h=0}^{n-j}(-1)^h\lambda^{[h]}x^{[n-j-h]}\\ &= \sum_{k=0}^{n}\left(\sum_{j+h=k}(-1)^h a_j\lambda^{[h]}\right)x^{[n-k]}.\end{aligned} \tag{4}$$

By duality the action of λ on the coefficients is that a_k is transformed into the kth coefficient of $(-\lambda)\cdot p(x)$, that is

$$\lambda\cdot a_k = \sum_{j+h=k} a_j\lambda^{[h]} = \sum_{j=0}^{k} a_j\lambda^{[k-j]}, \tag{5}$$

for instance

$$\lambda\cdot a_0 = a_0,\ \ \lambda\cdot a_1 = a_0\lambda + a_1,\ \ \lambda\cdot a_2 = a_0\lambda^{[2]} + a_1\lambda + a_2,$$
$$\lambda\cdot a_3 = a_0\lambda^{[3]} + a_1\lambda^{[2]} + a_2\lambda + a_3, \ldots.$$

We thus see in an explicit way that the dual action on polynomials in $a_0, a_1, \ldots$ is defined in a way independent of n.

The map $\lambda \mapsto \lambda\cdot f(a_0, a_1, \ldots) = f(\lambda\cdot a_0, \lambda\cdot a_1, \ldots)$ is an additive 1-parameter group of automorphisms with infinitesimal generator given by

$$\begin{aligned}\frac{d}{d\lambda} f(\lambda\cdot a_0, \lambda\cdot a_1, \ldots)|_{\lambda=0} &= \sum_i \frac{\partial}{\partial a_i} f(a_0, a_1, \ldots)\frac{d}{d\lambda}(\lambda\cdot a_i)|_{\lambda=0}\\ &= \sum_{i=1}^{\infty} a_{i-1}\frac{\partial}{\partial a_i} f(a_0, a_1, \ldots),\end{aligned}$$

because $\frac{d}{d\lambda}(\lambda\cdot a_i)|_{\lambda=0} = a_{i-1}$. One deduces the next theorem which—we believe—is due to Cayley.

Theorem 1.4. The algebra S of U-invariants is formed by the polynomials f in the infinitely many variables $a_0, a_1, a_2, \ldots$, satisfying

$$\boldsymbol{D}f := \sum_{i=1}^{\infty} a_{i-1}\frac{\partial}{\partial a_i} f(a_0, a_1, a_2, \ldots) = 0, \quad \boldsymbol{D} = \sum_{i=1}^{\infty} a_{i-1}\frac{\partial}{\partial a_i}.$$

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1.5 Weight

In classical literature, the *weight* is a way of counting in a monomial in the a_i the sum of the indices i appearing. That is, a_i has weight i and $\prod_j a_j^{h_j}$ has weight $\sum_j h_j j$. The use of this is in the following.

Definition 1.6. A polynomial $f(a_0, a_1, \ldots)$ is *isobaric of weight* g if all the monomials appearing in f have weight g.

Of course, every polynomial $f(a_0, \ldots, a_n)$ decomposes in a unique way into the sum of homogeneous and isobaric components.

Definition 1.7. The algebra $R = \mathbb{C}[a_0, a_1, \ldots]$ decomposes into the direct sum of its components $R_{n,g}$ formed by the homogeneous polynomials of degree n and weight g. This is a bigrading as algebra.

Of course, the polynomials in $R_{n,g}$ depend only on the variables $a_0, a_1, \ldots, a_g$.

Remark 1.8. If $n \geq g$, then one has $R_{n,g} = a_0^{n-g} R_{g,g}$.

This definition in modern language is that of characters of a torus. It applies to U-invariants due to the following considerations. The multiplicative group $\mathbb{C}^*$ of nonzero complex numbers acts by automorphisms on polynomials $f(a_0, a_1, \ldots)$ by $\mu \cdot a_i = \mu^i a_i$. Then a polynomial $f(a_0, a_1, \ldots)$ is isobaric of weight g if and only if

$$\mu \cdot f(a_0, a_1, \ldots) = \mu^g f(a_0, a_1, \ldots).$$

It is easy to see that this action by $\mathbb{C}^*$ has the following commutation with the operator $\boldsymbol{D} := \sum_{i=1}^{\infty} a_{i-1} \dfrac{\partial}{\partial a_i}$:

$$\mu \cdot \boldsymbol{D} f = \mu^{-1}(\boldsymbol{D}(\mu \cdot f)).$$

As a consequence, since $\boldsymbol{D}(a_i) = a_{i-1}$, we have the following result.

Lemma 1.9. The operator $\boldsymbol{D} := \sum_{i=1}^{\infty} a_{i-1} \dfrac{\partial}{\partial a_i}$ maps polynomials of degree n and weight g into polynomials of degree n and weight $g-1$.

By Theorem 2.4, this implies the next result.

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Corollary 1.10. The isobaric components of a U-invariant are also U-invariants.

Definition 1.11. We denote by S_n the subspace of U-invariants homogeneous of degree n and by $S_{n,g} \subset S_n$ the ones isobaric of weight g:

$$S_n = \bigoplus_{g=0}^{\infty} S_{n,g}, \quad S = \bigoplus_{n=0}^{\infty} S_n = \bigoplus_{n,g \in \mathbb{N}} S_{n,g}.$$

1.12 Reducible elements

In the old literature an invariant of some positive degree k is called *reducible* if it is equal to a polynomial in invariants of strictly lower degree.

This idea of course applies to an element of any commutative graded algebra, but today it is an unfortunate expression, since in commutative algebra reducible means something else. So we shall use the word *decomposable*.

Nevertheless, the invariant theorists of the 19th century used this idea in order to understand *a minimal set of generators* for a ring of invariants.

In modern terms, if $A = \oplus_{i=0}^{\infty} A_i$ is a commutative graded algebra, with $A_0 = F$ the base field and setting $I := \oplus_{i=1}^{\infty} A_i$, we know that a minimal set of generators for A is a basis of I modulo I^2. So in order to describe such minimal bases one has to describe I/I^2, or rather to describe a *complementary space to* I^2 *in* I. This, of course, is not canonical, and in fact it is interesting to read some disputes between Sylvester and Faà di Bruno about the best choice of representatives.

In our situation, we have for each n the algebra $S(n)$ and the corresponding maximal ideal I_n. For general n little is known about I_n/I_n^2. In fact, one of the high points of the theory was to prove that I_n/I_n^2 is finite dimensional. This is Gordan's famous finiteness theorem [4].

But what was quickly discovered is that an element of $S(n)$ that is indecomposable in $S(n)$ need not remain indecomposable in $S(n+1)$. In other words, the maps $I_n/I_n^2 \to I_{n+1}/I_{n+1}^2$ need not be injective, or also, a minimal set of generators for $S(n)$ cannot be completed to one for $S(n+1)$. As an example we will see in Section 4.2 that the generator D for $S(3)$ given by Remark 5.6 is decomposable in $S(4)$.

We are now ready to give the main definition of this paper, that of *perpetuant* ([21]).

Definition 1.13. A *perpetuant* is an indecomposable element of $S(n)$ that remains indecomposable in all $S(k)$, $k \geq n$.

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In other words it gives an element of I_n/I_n^2 that *lives forever*, that is, it remains nonzero in all I_k/I_k^2, $\forall\, k \geq n$. In this sense it is *perpetuant*.

Of course a perpetuant is just an indecomposable element of the limit algebra S. Thus, to describe perpetuants is strictly related to describe minimal sets of generators for the graded algebra S. In other words, denoting by $I \subset S$ the maximal homogeneous ideal of S we want to describe I/I^2.

This space decomposes into a direct sum

$$I/I^2 = \bigoplus_{n,g\in\mathbb{N}} P_{n,g}$$

with $P_{n,g}$ the image of $S_{n,g}$ (the elements in I of degree n and weight g). We may, for convenience and abuse, refer to this space as *the space of perpetuants*.

Definition 1.14. A bigraded subspace of I that is a complement of I^2 will be called *a space of perpetuants*.

Observe that, while the decomposable elements, that is, elements of I^2, as well as the perpetuants, that is, the elements of $I \setminus I^2$, are intrinsic objects, a *space of perpetuants* that is a complement of I^2 in I is not intrinsic, but depends on the choice of some basis of I, completing a basis of I^2.

The theorem of Stroh gives the generating function for the dimensions of the isobaric components of I/I^2 and can be stated as

$$\sum_{g=0}^{\infty} \dim(P_{n,g})\, x^g = \begin{cases} \dfrac{x^{2^{n-1}-1}}{(1-x^2)(1-x^3)\cdots(1-x^n)} & \text{for } n > 2, \\ x^2/(1-x^2) & \text{for } n = 2, \\ 1 & \text{for } n = 1. \end{cases}$$

In Theorem 2.19 we will construct a complement of I^2 in I, see Theorem 4.9, thus giving a possible solution to the problem posed by SYLVESTER of describing the spaces of perpetuants.

1.15 Umbral calculus

This is one of the forgotten parts of old invariant theory, but it really is an anticipation of some aspects of tensor calculus. The problem arises in the computation of invariants of some group G of linear transformations on a vector space W (cf. [18] and [19]).

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1st step: polarization and restitution

The 1st step is general. The homogeneous polynomial functions on W of degree k can be *fully polarized* giving rise, in characteristic 0, to a G-equivariant isomorphism between this space and the space of multilinear and symmetric functions in k copies of W. In formulas

$$\mathcal{O}(W)_k = \mathrm{Sym}^k(W^*) \simeq (W^{*\otimes k})^{\mathcal{S}_k},$$

where $\mathrm{Sym}^k(W^*)$ is the kth symmetric power of W^*, and $\mathcal{S}_k$ is the symmetric group in k letters acting in the obvious way on $W^{*\otimes k}$.

Given a homogeneous polynomial function $f(w)$ on W of degree k, its *polarizations* f_α are obtained from the expansion

$$f(t_1 w_1 + \cdots + t_k w_k) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_k) \\ \alpha_1+\cdots+\alpha_k=k}} t^\alpha f_\alpha(w_1, \ldots, w_k), \quad t^\alpha := t_1^{\alpha_1} \cdots t_k^{\alpha_k}. \tag{6}$$

The *full polarization* Pf is the coefficient of the product $t_1 \cdots t_k$:

$$Pf(w_1, \ldots, w_k) := f_{1,\ldots,1}(w_1, \ldots, w_k).$$

It is multilinear and symmetric. In order to obtain the inverse map, called *restitution*, one sets all the w_i equal to w, that is, $F(w_1, \ldots, w_n) \mapsto F(w, \ldots, w)$. Starting with $f = f(w)$ and setting $w_i = w$ in equation (6) we find

$$f((t_1 + \cdots + t_k)w) = (t_1 + \cdots + t_k)^k f(w) = \sum t^\alpha f_\alpha(w, \ldots, w).$$

In particular, $Pf(w, \ldots, w) = k!f(w)$ (the presence of the factor $k!$ explains why this procedure works well only in characteristic zero).

2nd step: umbrae of invariants

Next assume that W is some tensor representation of $\mathrm{GL}(V)$ for some space V. For instance, in the classical literature one finds $W = \mathcal{O}(V)_n = \mathrm{Sym}^n(V^*)$, the space of homogeneous polynomial functions of degree n on V. Inside W one then has some special vectors, usually the *decomposable vectors* that span W and are stable under $\mathrm{GL}(V)$. For instance, for $W = \mathrm{Sym}^n(V^*)$ we have the functions φ^n, the nth powers of the linear forms $\varphi \in V^*$, or their divided powers $\varphi^{[n]}$.

Then a linear function on W restricted to the elements $\varphi^{[n]}$, $\varphi \in V^*$, gives rise to a homogeneous polynomial of degree n on V^*, and this establishes an isomorphism,

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between W^* and the space of homogeneous polynomials of degree n on V^*, which is $\mathrm{GL}(V)$-equivariant. Similarly, a multilinear function on k copies of W restricted to $\varphi_1^{[n]}, \dots, \varphi_k^{[n]}$, $\varphi_j \in V^*$, gives rise to a homogeneous polynomial of degree n in each of the k variables $\varphi_i \in V^*$, and this also establishes a $\mathrm{GL}(V)$-equivariant isomorphism between $(W^{\otimes k})^*$ and the space of multihomogeneous polynomials of degree n on k vector variables in V^* (compatible with the two actions of S_k).

Thus, a multilinear invariant of k copies of W under some subgroup $G \subset \mathrm{GL}(V)$ is encoded in a polynomial in k linear forms $\varphi_1, \dots, \varphi_k \in V^*$, called *umbrae*, which is multihomogeneous of degree n, symmetric in the variables $\varphi_1, \dots, \varphi_k$, and invariant under G.

The symbolic method

Combining the two steps above we obtain a map, denoted by $\boldsymbol{E}$, that associates to a multihomogeneous symmetric invariant of degree n of k *umbral*–variables in V^*, a homogeneous invariant of degree k on W, after interpretation as multilinear polynomial in k variables in W and setting all variables equal.

This is the basis of the *symbolic method for binary forms* where $W := \mathrm{Sym}^g(\mathbb{C}^2) = \mathbb{C}[x, y]_g$ that we will describe now.

1.16 Symbolic method for binary forms

For binary forms an explicit algorithmic way (which extends of course to forms in any number of variables) is the following. A linear form is an expression of type $\alpha_1 x + \alpha_2 y$. With the notation of (2), we have

$$(\alpha_1 x + \alpha_2 y)^{[g]} = \sum_{i=0}^{g} \alpha_1^{[g-i]} \alpha_2^{[i]} x^{[n-i]} y^{[i]} := \sum_{i=0}^{g} a_i x^{[g-i]} y^{[i]}.$$

Let $\varphi_i := \alpha_{1,i} x + \alpha_{2,i} y$, $i = 1, \dots, n$, be n linear forms. A polynomial homogeneous of degree g in each of the φ_i is a linear combination of monomials of type $\prod_{i=1}^n \alpha_{1,i}^{g-r_i} \alpha_{2,i}^{r_i}$ where $0 \le r_i \le g$. Then the corresponding function on $W = \mathbb{C}[x, y]_g$ under the map $\boldsymbol{E}$ is given by

$$\boldsymbol{E} \colon \prod_{i=1}^{n} \alpha_{1,i}^{[g-r_i]} \alpha_{2,i}^{[r_i]} \mapsto \prod_{i=1}^{n} a_{r_i}. \tag{7}$$

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The $\mathrm{SL}(2,\mathbb{C})$-invariants of the forms φ_i, $i = 1,\ldots,k$ are generated by the quadratic invariants:

$$(i,j) = (\varphi_i,\varphi_j) := \det\begin{bmatrix}\alpha_{1,i} & \alpha_{2,i}\\ \alpha_{1,j} & \alpha_{2,j}\end{bmatrix} = \alpha_{1,i}\alpha_{2,j} - \alpha_{1,j}\alpha_{2,i}.$$

Therefore, the space of invariants of degree k of binary forms of degree n is spanned by the evaluation $\boldsymbol{E}$ of the symbols $\prod_t (i_t,j_t)$, $i_t,j_t \in \{1,\ldots,k\}$ in which each of the indices $1,\ldots,k$ appears exactly n times.

The 1st problem of this method is to exhibit a list of symbols that give a basis of the corresponding invariants. But the main difficulty is to understand which symbols correspond to decomposable invariants.

A large number of papers of 19th century invariant theory is devoted to these problems culminating with GORDAN's proof of finite generation; see [4], [5].

A complete answer is not known and probably too complex to be made explicit. For this reason Theorem 1.1 of STROH, and our main result, Theorem 4.9, are quite remarkable.

Remark 1.17.

(1) The map $\boldsymbol{E}$ is a well defined linear homomorphism from the space of polynomials in the n variables φ_i with coordinates $\alpha_{1,i},\alpha_{2,i}$ and homogeneous of degree g in each of these variables to the space of polynomials of degree n in the variables $a_0,\ldots,a_g$:

$$\boldsymbol{E}\colon \mathbb{C}[\alpha_{1,1},\alpha_{2,1},\ldots,\alpha_{1,n},\alpha_{2,n}]_{(g,g,\ldots,g)} \to \mathbb{C}[a_0,\ldots,a_g]_n.$$

(2) In the classical literature one may see a statement as

$$\varphi_1 \cong \varphi_2 \cong \ldots \cong \varphi_n$$

to mean that the map $\boldsymbol{E}$ takes the same values when permuting the k umbrae φ_i. More precisely, the map $\boldsymbol{E}$ is an isomorphism when restricted to symmetric polynomials in the k variables φ_i.

(3) Finally, $\boldsymbol{E}$ is equivariant with respect to the action of $\mathrm{GL}(2,\mathbb{C})$ induced by the action on the variables x,y.

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Formula (7) suggests, when considering U-invariants, to work only with the variables $\alpha_i := \alpha_{2,i}$ and to replace the map $\boldsymbol{E}$ with

$$\boldsymbol{E}\colon \prod_{i=1}^{n} \alpha_i^{[r_i]} \mapsto \prod_{i=1}^{n} a_{r_i}, \quad (r_i \geq 0). \tag{8}$$

Notice that, in this formula the α_i that do not appear, that is, for which $r_i = 0$, contribute each to a factor a_0.

It follows that $\boldsymbol{E}$ maps linearly the space of polynomials in $\alpha_1, \ldots, \alpha_n$ to the space of polynomials homogeneous of degree n in the variables $a_0, a_1, a_2, \ldots,$

$$\boldsymbol{E}\colon \mathbb{C}[\alpha_1, \ldots, \alpha_n] \to \mathbb{C}[a_0, a_1, a_2, \ldots], \ \alpha_1^{[r_1]} \cdots \alpha_n^{[r_n]} \mapsto a_{r_1} \cdots a_{r_n},$$

where a homogeneous polynomial of degree g is mapped to an isobaric polynomial of weight g and homogeneous of degree n. The map $\boldsymbol{E}$ commutes with the permutation action on the α_i, $i = 1, \ldots, n$, for example,

$$\boldsymbol{E}(\alpha_1^{[3]}\alpha_2^{[2]}) = \boldsymbol{E}(\alpha_3^{[3]}\alpha_1^{[2]}) = a_0^{n-2} a_2 a_3 \ \text{ and } \boldsymbol{E}(\alpha_i^{[2]}\alpha_j^{[2]}) = a_0^{n-2} a_2^2.$$

Remark 1.18. The map $\boldsymbol{E}$ is not a homomorphism of algebras. Nevertheless, if we decompose the umbrae in two disjoint subsets $\alpha_1, \ldots, \alpha_h$ and $\alpha_{h+1}, \ldots, \alpha_k$ and consider two polynomials $f(\alpha_1, \ldots, \alpha_h)$ and $g(\alpha_{h+1}, \ldots, \alpha_k)$ we have

$$\boldsymbol{E}\left(f(\alpha_1, \ldots, \alpha_h) \cdot g(\alpha_{h+1}, \ldots, \alpha_n)\right) = \boldsymbol{E}(f(\alpha_1, \ldots, \alpha_h)) \cdot \boldsymbol{E}(g(\alpha_{h+1}, \ldots, \alpha_n)).$$

In this formalism we loose the action of $\mathrm{GL}(2, \mathbb{C})$, but we still have the translation action by $\mathbb{C}^+$, which commutes with $\boldsymbol{E}$. In term of differential operators, we see that

$$\boldsymbol{E} \circ \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i} = \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \circ \boldsymbol{E}. \tag{9}$$

When working with U-invariants the previous umbral calculus corresponds to the further simplification of considering, instead of binary forms, polynomials in x, and the action is just translation $x \mapsto x - \lambda$. One may further restrict to monic polynomials, and so use the formula

$$(x + \alpha)^{[g]} = \sum_{i=0}^{g} \alpha^{[i]} x^{[g-i]} = x^{[n]} + \sum_{i=1}^{n} a_i x^{[g-i]}.$$

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Given any polynomial in the $a_1, a_2, \ldots$ which is invariant under translation and of given weight one can reconstruct a corresponding U-invariant by making it homogeneous by inserting powers of a_0.

2 Stroh's Potenziante and Duality

2.1 Potenziante

Following Stroh we define the *potenziante* by

$$\pi_{n,g} := \boldsymbol{E}\left(\left(\sum_{j=1}^{n} \lambda_j \alpha_j\right)^{[g]}\right) = \sum_{\substack{r_1,\ldots,r_n \in \mathbb{N} \\ r_1+\cdots+r_n=g}} \lambda_1^{r_1} \cdots \lambda_n^{r_n} a_{r_1} \cdots a_{r_n} \tag{10}$$

where the $\alpha_1 \ldots, \alpha_n$ are all umbrae, and one uses Formula (8). Moreover,

$$\sum_{g=0}^{\infty} \pi_{n,g} = \boldsymbol{E}\left(\exp \sum_{j=1}^{n} \lambda_j \alpha_j\right).$$

Notice that $\pi_{n,g}$ is a polynomial in the variables $\lambda_1, \ldots, \lambda_n$ and the variables $a_0, \ldots, a_g$. Sometimes we need to change the variables λ_j, so we shall stress this dependence by writing

$$\pi_{n,g} = \pi_{n,g}(\lambda; a) = \pi_{n,g}(\lambda_1, \ldots, \lambda_n; a_0, a_1, \ldots, a_g).$$

By construction, $\pi_{n,g}(\lambda; a)$ is homogeneous of degree n and isobaric of weight g in the a_i, and symmetric and of degree g in the λ_j. So it can be developed in term of the symmetric functions of degree g in the λ_j.

Definition 2.2. Denote by $\Sigma_{n,g} = \mathbb{C}[\lambda_1, \ldots, \lambda_n]_g^{\mathcal{S}_n} \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ the subspace of *symmetric polynomials* in $\lambda_1, \ldots, \lambda_n$ that are homogeneous of degree g.

The space $\Sigma_{n,g}$ has several useful bases, all indexed by partitions:

$$h_1 \geq h_2 \geq \cdots \geq h_n \geq 0, \; h_i \in \mathbb{N}, \quad h_1 + \cdots + h_n = g.$$

We first take as basis of $\Sigma_{n,g}$ the *total monomial sums* $m_{h_1,\ldots,h_n}$, that is, the sum over the $\mathcal{S}_n$-orbit of $\lambda_1^{h_1} \cdots \lambda_n^{h_n}$ ($\mathcal{S}_n$ the symmetric group on n elements) where $h_1 \geq h_2 \geq$

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$\cdots \geq h_n \geq 0$ and $h_1 + \cdots + h_n = g$:

$$m_{h_1,\ldots,h_n}(\lambda) := \sum_{\mathcal{S}_n\text{-orbit}} \sigma(\lambda_1^{h_1} \cdots \lambda_n^{h_n}).$$

It follows from the definition of $\pi_{n,g}(\lambda; a)$ (see formula (10)) that the monomial $a_{h_1} \cdots a_{h_n}$ appears in $\pi_{n,g}$ with coefficient $m_{h_1,\ldots,h_n}(\lambda)$. From this we easily get the following result.

Proposition 2.3.

(1) In $\pi_{n,g}(\lambda; a)$ the total monomial sum $m_{h_1,\ldots,h_n}(\lambda)$ has as coefficient the product $a_{h_1} a_{h_2} \cdots a_{h_n}$:

$$\pi_{n,g}(\lambda; a) = \boldsymbol{E}\left(\left(\sum_{r=1}^{n} \lambda_r \alpha_r\right)^{[g]}\right) = \sum_{\substack{h_1 \geq \ldots \geq h_n \geq 0 \\ h_1 + \cdots + h_n = g}} m_{h_1,\ldots,h_n} a_{h_1} a_{h_2} \cdots a_{h_n}.$$

(2) These coefficients form a basis of the space $\mathbb{C}[a]_{n,g}$ of homogeneous polynomials in $a_0, a_1, a_2, \ldots$ of degree n and weight g.

Example 2.4. With $n = g = 3$ we find

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3)^{[3]} = \lambda_1^3\alpha_1^{[3]} + \lambda_2^3\alpha_2^{[3]} + \lambda_3^3\alpha_3^{[3]} + \lambda_1^2\lambda_2\alpha_1^{[2]}\alpha_2 + \lambda_1^2\lambda_3\alpha_1^{[2]}\alpha_3$$
$$+ \lambda_2^2\lambda_1\alpha_2^{[2]}\alpha_1 + \lambda_2^2\lambda_3\alpha_2^{[2]}\alpha_3 + \lambda_3^2\lambda_1\alpha_3^{[2]}\alpha_1 + \lambda_3^2\lambda_2\alpha_3^{[2]}\alpha_2 + \lambda_1\lambda_2\lambda_3\alpha_1\alpha_2\alpha_3.$$

Applying $\boldsymbol{E}$ we get

$$(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)a_0^2a_3 + (\lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_2^2\lambda_1 + \lambda_2^2\lambda_3 + \lambda_3^2\lambda_1 + \lambda_3^2\lambda_2)a_0a_1a_2 + \lambda_1\lambda_2\lambda_3a_1^3,$$

which is equal to

$$m_{3,0,0}(\lambda)a_0^2a_3 + m_{2,1,0}(\lambda)a_0a_1a_2 + m_{1,1,1}(\lambda)a_1^3.$$

With $n = 2$ and $g = 4$ we find

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2)^{[4]} = \lambda_1^4\alpha_1^{[4]} + \lambda_2^4\alpha_2^{[4]} + (\lambda_1^3\lambda_2\alpha_1^{[3]}\alpha_2 + \lambda_2^3\lambda_1\alpha_2^{[3]}\alpha_1) + \lambda_1^2\lambda_2^2\alpha_1^{[2]}\alpha_2^{[2]}.$$

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Applying $\boldsymbol{E}$ this gives

$$(\lambda_1^4+\lambda_2^4)a_0^3a_4+(\lambda_1^3\lambda_2+\lambda_2^3\lambda_1)a_0^2a_1a_3+\lambda_1^2\lambda_2^2a_2^2,$$

which is equal to

$$m_{4,0}(\lambda)a_0^3a_4+m_{3,1}(\lambda)a_0^2a_1a_3+m_{2,2}(\lambda)a_2^2.$$

Finally, with $n=3$ and $g=2$ we have

$$(\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3\alpha_3)^{[2]}=\lambda_1^2\alpha_1^{[2]}+\lambda_2^3\alpha_2^{[2]}+\lambda_3^3\alpha_3^{[2]}+\lambda_1\lambda_2\alpha_1\alpha_2+\lambda_1\lambda_3\alpha_1\alpha_3+\lambda_2\lambda_3\alpha_2\alpha_3.$$

Applying $\boldsymbol{E}$ we get

$$(\lambda_1^2+\lambda_2^2+\lambda_3^2)a_0^2a_2+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)a_0a_1^2,$$

which is equal to

$$m_{2,0,0}(\lambda)a_0^2a_2+m_{1,1,0}(\lambda)a_0a_1^2.$$

Notice that all coefficients are divisible by a_0 as expected, because $n>g$.

2.5 Duality

Proposition 2.1 can be understood in the following way. The tensor

$$\pi_{n,g}(\lambda;a)\in\Sigma_{n,g}\otimes\mathbb{C}[a]_{n,g}\subset\mathbb{C}[\lambda_1,\ldots,\lambda_n]^{\mathcal{S}_n}\otimes_{\mathbb{C}}\mathbb{C}[a_0,\ldots,a_g]$$

defines a *duality* between the subspace $\Sigma_{n,g}\subseteq\mathbb{C}[\lambda_1,\ldots,\lambda_n]^{\mathcal{S}_n}$ of homogeneous symmetric polynomials of degree g in n variables, and the subspace $\mathbb{C}[a]_{n,g}\subseteq\mathbb{C}[a_0,\ldots,a_g]$ of polynomials, homogeneous of degree n, and with weight g.

In general, given two finite dimensional vector spaces U,W and denoting by $U^\vee,W^\vee$ their duals, one has the canonical isomorphisms

$$U\otimes W\simeq\mathrm{Hom}(U^\vee,W)\simeq\mathrm{Hom}(W^\vee,U). \tag{11}$$

For example, if $\pi\in U\otimes W$ is a tensor, then the corresponding map $\pi:U^\vee\to W$ is given by $\varphi\mapsto(\varphi\otimes\mathrm{id}_W)(\pi)$.

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A *dualizing tensor* $\pi \in U \otimes W$ is an element that corresponds, under these isomorphisms, to an isomorphism $U^\vee \xrightarrow{\sim} W$ (or $W^\vee \xrightarrow{\sim} U$). Thus, a dualizing tensor π equals, for any basis $u_1, \dots, u_k$ of U, to $\pi = \sum_{i=1}^k u_i \otimes w_i$, where $w_1, \dots, w_k$ is a basis of W.

Definition 2.6. Let $\pi \in U \otimes W$ be a dualizing tensor. If $M \subseteq U$ is a subspace, then the *orthogonal subspace* $M^\perp \subseteq W$ is defined to be the image of $(U/M)^\vee$ in W under the isomorphism $U^\vee \xrightarrow{\sim} W$ corresponding to π.

Choosing a basis $(u_i)_{i=1}^n$ of U such that $(u_j)_{j=1}^m$ is a basis of M and writing $\pi = \sum_i u_i \otimes w_i$, then $(w_k)_{k=m+1}^n$ is a basis of $M^\perp$. Moreover, the image of π in $U/M \otimes W$ defines a dualizing tensor $\bar\pi \in U/M \otimes M^\perp$.

Remark 2.7. In general, a tensor $\pi \in U \otimes W$ gives, via the isomorphism (11), two maps

$$\pi_1 : W^\vee \to U, \quad \pi_2 : U^\vee \to W \tag{12}$$

and we have the following:

(1) $\pi_2 = \pi_1^\vee$, $\mathrm{im}(\pi_1) = \ker(\pi_2)^\perp$, $\mathrm{im}(\pi_2) = \ker(\pi_1)^\perp$.
(2) $\pi \in \mathrm{im}(\pi_1) \otimes \mathrm{im}(\pi_2)$ is a dualizing tensor for these two spaces.
(3) If $\pi = \sum_{i=1}^k u_i \otimes w_i$, and if the u_i are linearly independent, then the image $\mathrm{im}(\pi_2)$ of the corresponding map is the span of the elements w_i. Similarly, if the w_i are linearly independent.
(4) If $U' \subseteq U$ is a subspace and $\pi \in U' \otimes W$, then the associated subspace in W is the same when computed with U or U'.

Sketch of Proof. Let $\pi = \sum_{i=1}^k u_i \otimes w_i$ with k minimal. Then clearly both the u_i as well as the w_j are linearly independent, and they span the two spaces $\mathrm{im}(\pi_1)$ and $\mathrm{im}(\pi_2)$. Now the various claims follow easily. ■

Applying this to the dualizing tensor

$$\pi_{n,g} \in \Sigma_{n,g} \otimes_{\mathbb{C}} \mathbb{C}[a_0, \dots, a_g]_{n,g}$$

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we obtain, from Proposition 3.1, the following two isomorphisms:

$$\Sigma_{n,g}^{\vee} \xrightarrow{\sim} \mathbb{C}[a]_{n,g} \colon \varphi \mapsto (\varphi \otimes \mathrm{id})(\pi_{n,g}) = (\varphi \otimes \mathrm{id})(\textstyle\sum_h m_h \otimes a_h) = \sum_h \varphi(m_h) a_h,$$
$$\mathbb{C}[a]_{n,g}^{\vee} \xrightarrow{\sim} \Sigma_{n,g} \colon \psi \mapsto (\mathrm{id} \otimes \psi)(\pi_{n,g}) = (\mathrm{id} \otimes \psi)(\textstyle\sum_h m_h \otimes a_h) = \sum_h \psi(a_h) m_h.$$

As a consequence of the remark above this gives the next result.

Proposition 2.8. For every subspace $M \subset \Sigma_{n,g}$ we have the orthogonal subspace $M^{\perp} \subset \mathbb{C}[a]_{n,g}$, and the duality between $\Sigma_{n,g}/M$ and $M^{\perp}$ is given by the image of $\pi_{n,g}$ in $\Sigma_{n,g}/M \otimes \mathbb{C}[a_0, \ldots, a_g]_{n,g}$.

Given a basis $u_1, \ldots, u_N$ of $\Sigma_{n,g}$ such that $u_1, \ldots, u_m$ is a basis of M, and writing $\pi_{n,g} = \sum_{j=1}^{N} u_j \otimes b_j$, then the elements $b_{m+1}, \ldots, b_N$ form a basis of $M^{\perp}$.

Now consider some homogeneous ideal $J \subset \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$ and the corresponding quotient map $\mathbb{Q}[\lambda_1, \ldots, \lambda_n] \to \mathbb{Q}[\lambda_1, \ldots, \lambda_n]/J$. By restriction, we have a homomorphism

$$\mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{\mathcal{S}_n} \to \mathbb{Q}[\lambda_1, \ldots, \lambda_n]/J$$

with kernel $\mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{\mathcal{S}_n} \cap J$. Denoting the images of the λ_i by $\bar{\lambda}_i$ we get an image of the potenziante

$$\pi_{n,g}(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) = \boldsymbol{E}\left(\left(\sum_{j=1}^{n} \bar{\lambda}_j \alpha_j\right)^{[g]}\right) \in \mathbb{C}[\bar{\lambda}_1, \ldots, \bar{\lambda}_n] \otimes_{\mathbb{C}} \mathbb{C}[a_0, \ldots, a_n],$$

and we obtain the following result.

Proposition 2.9. If $(B_i)_{i \in I_g}$ is a basis of $\Sigma_{n,g}/\Sigma_{n,g} \cap J$ and if

$$\pi_{n,g}(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) = \sum_{i \in I_g} B_i \otimes A_i,$$

then $(A_i)_{i \in I_g}$ is a basis of the subspace of $\mathbb{C}[a]_{n,g}$ orthogonal to $\Sigma_{n,g} \cap J$.

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2.10 Description of the U-invariants

We have seen in Theorem 2.4 that the U-invariants $S = \mathbb{C}[a_0, a_1, \dots]^U$ form the kernel of the differential operator $\boldsymbol{D} := \sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i}$. Moreover, S is bigraded by degree and weight: $S = \bigoplus_{n,g \in \mathbb{N}} S_{n,g}$, see Definition 2.11.

A very remarkable formula

From

$$\left(\sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i}\right)\left(\sum_{r=1}^{n} \lambda_r \alpha_r\right)^{[g]} = \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{r=1}^{n} \lambda_r \alpha_r\right)^{[g-1]}$$

we obtain, by Formula (9),

$$\sum_{i=1}^{\infty} a_{i-1} \frac{\partial}{\partial a_i} \pi_{n,g} = \left(\sum_{i=1}^{n} \lambda_i\right) \pi_{g-1,n}, \quad \text{or}$$
$$\boldsymbol{D}\left(\boldsymbol{E}\left(\exp \sum_{j=1}^{n} \lambda_j \alpha_j\right)\right) = e_1(\lambda)\, \boldsymbol{E}\left(\exp \sum_{j=1}^{n} \lambda_j \alpha_j\right). \tag{13}$$

Remark 2.11. The meaning of this formula is that, using the duality between symmetric functions in n variables and polynomials in the a_i of degree n, the transpose of the operator $\boldsymbol{D}$ is the multiplication by $\sum_{i=1}^{n} \lambda_i$.

The remarkable formula (13) gives us a straightforward way of describing both, the operator $\boldsymbol{D}$, and also a basis of the U-invariants.

For this we change the basis of the space $\Sigma_{n,g}$ of symmetric functions from the total monomial sums $m_{h_1,\dots,h_n}$ to the monomials $e_1^{k_1} \dots e_n^{k_n}$, $\sum_j jk_j = g$, where $e_i = e_i(\lambda_1, \dots, \lambda_n)$ is the ith elementary symmetric function.

Expressing the total monomial sums $m_{h_1,\dots,h_n}$ in the e_i's we get, for some $\alpha_{h_1,\dots,h_n,k_1,\dots,k_n} \in \mathbb{Z}$:

$$m_{h_1,\dots,h_n} = \sum_{k_1,\dots,k_n} \alpha_{h_1,\dots,h_n,k_1,\dots,k_n} e_1^{k_1} \dots e_n^{k_n}. \tag{14}$$

The potenziante $\pi_{n,g}$ in this new basis appears as

$$\pi_{n,g} = \sum_{\substack{0 \le k_1,\dots,k_n \\ k_1+2k_2+\dots+nk_n=g}} e_1^{k_1} \dots e_n^{k_n} \tilde{U}_{k_1,\dots,k_n},$$

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where the new elements $\tilde{U}_{k_1,\dots,k_n}$ are given by

$$\tilde{U}_{k_1,\dots,k_n} = \sum_{h_1,\dots,h_n} \alpha_{h_1,\dots,h_n,k_1,\dots,k_n} \prod_{j=1}^{n} a_{h_j}, \tag{15}$$

and also form a basis of $\mathbb{C}[a]_{n,g}$. By formula (13) we get

$$\boldsymbol{D}\pi_{n,g} = \sum_{\substack{k_1,\dots,k_n\geq 0\\ \sum ik_i=g}} e_1^{k_1}\dots e_n^{k_n}\boldsymbol{D}\tilde{U}_{k_1,\dots,k_n} = \sum_{\substack{j_1,\dots,j_n\geq 0\\ \sum ij_i=g-1}} e_1^{j_1+1}\dots e_n^{j_n}\tilde{U}_{j_1,\dots,j_n} = e_1\pi_{n,g-1},$$

which implies the following result.

Corollary 2.12.

(1) We have

$$\boldsymbol{D}\tilde{U}_{k_1,\dots,k_n} = \begin{cases} 0 & \text{if } k_1 = 0 \\ \tilde{U}_{k_1-1,\dots,k_n} & \text{if } k_1 > 0. \end{cases} \tag{16}$$

(2) The elements $U_{k_2,\dots,k_n} := \tilde{U}_{0,k_2,\dots,k_n}$ form a basis of the space $S_{n,g}$ of the U-invariants of degree n and weight g.

It is interesting to remark that these results hold over $\mathbb{Z}$ and not just over $\mathbb{C}$.

In an alternative way we can impose the relation $\sum_{r=1}^{n}\lambda_r = 0$ and denote by $\bar{\lambda}_r$ the class of λ_r modulo $\sum_{r=1}^{n}\lambda_r = 0$. Denote by $\bar{\Sigma}_n$ the algebra of symmetric functions in the $\bar{\lambda}_r$, which is a polynomial algebra over the elements $e_2(\bar{\lambda}),\dots,e_n(\bar{\lambda})$. The space of symmetric functions of degree g in $\bar{\lambda}_1,\dots,\bar{\lambda}_n$,

$$\bar{\Sigma}_{n,g} = (\mathbb{C}[\lambda_1,\dots,\lambda_n]^{\mathcal{S}_n}/(\lambda_1+\cdots+\lambda_n))_g = \mathbb{C}[\bar{\lambda}_1,\dots,\bar{\lambda}_n]_g^{\mathcal{S}_n}$$

has as basis the monomials of weight g in the elements $\bar{e}_2 := e_2(\bar{\lambda}),\dots,\bar{e}_n := e_n(\bar{\lambda})$. This implies the following result.

Proposition 2.13.

(1) The potenziante $\bar{\pi}_{n,g}(\bar{\lambda};a) \in \bar{\Sigma}_{n,g}\otimes\mathbb{C}[a]_g$ has the form

$$\bar{\pi}_{n,g}(\bar{\lambda}_1,\dots,\bar{\lambda}_n;a_0,a_1,\dots,a_g) = \sum_{\substack{k_2,\dots,k_n\geq 0\\ \sum ik_i=g}} \bar{e}_2^{k_2}\dots\bar{e}_n^{k_n}U_{k_2,\dots,k_n}. \tag{17}$$

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(2) In particular, $\bar{\pi}_{n,g}(\bar{\lambda}, a)$ is a dualizing tensor between the space $\bar{\Sigma}_{n,g}$ of symmetric functions of degree g in $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$ and the space $S_{n,g}$ of U-invariants of degree n and weight g.

(3) The elements $U_{k_2,\ldots,k_n} = \tilde{U}_{0,k_2,\ldots,k_n}$ of formula (15) form a basis of the space $S_{n,g} \subset \mathbb{C}[a_0, \ldots, a_n]$ of U-invariants of degree n and weight g, dual to the basis $\bar{e}_2^{k_2} \ldots \bar{e}_n^{k_n}$ of $\bar{\Sigma}_{n,g}$.

As a corollary we have

$$\sum_{g=0}^{\infty} \dim(S_{n,g}) x^g = \frac{1}{(1-x^2)(1-x^3)\cdots(1-x^n)}.$$

Remark 2.14. Cayley and MacMahon use the word *non unitariants* for various symmetric functions in the differences of the roots, due to the fact that they are indexed by partitions with no part of size 1.

As a consequence of the proposition above and of Proposition 3.8 we see that the quotients of $\bar{\Sigma}_{n,g}$ correspond to subspaces of $S_{n,g}$. So our final task is to identify the subspace $O_{n,g}$ of $\bar{\Sigma}_{n,g}$ orthogonal to the subspace of decomposable elements of $S_{n,g}$ and from that a choice of perpetuants.

2.15 Decomposable U-invariants

For a given $h \in \mathbb{N}$, $1 \leq h < n$ we have (3):

$$(\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n)^{[g]} = \sum_{j=0}^{g} (\lambda_1\alpha_1 + \cdots + \lambda_h\alpha_h)^{[j]} (\lambda_{h+1}\alpha_{h+1} + \cdots + \lambda_n\alpha_n)^{[g-j]},$$

which implies, by Remark 2.18, the following decomposition of the potenziante:

$$\pi_{n,g}(\lambda_1, \ldots, \lambda_n; a) = \sum_{j=0}^{g} \pi_{h,j}(\lambda_1, \ldots, \lambda_h; a) \cdot \pi_{n-h,g-j}(\lambda_{h+1}, \ldots, \lambda_n; a). \tag{18}$$

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Consider the ideal $J_h \subset \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ generated by the two linear elements $\lambda_1 + \cdots + \lambda_h$ and $\lambda_{h+1} + \cdots + \lambda_n$ (or the ideal $\bar{J}_h \subset \mathbb{C}[\bar{\lambda}_1, \ldots, \bar{\lambda}_n]$ generated by $\bar{\lambda}_1 + \cdots + \bar{\lambda}_h$). Then

$$\mathbb{C}[\lambda_1, \ldots, \lambda_n]/J_h = \mathbb{C}[\bar{\lambda}_1, \ldots, \bar{\lambda}_n]/\bar{J}_h = \mathbb{C}[\lambda_1, \ldots, \lambda_h]/(\lambda_1 + \cdots + \lambda_h) \otimes_{\mathbb{C}} \mathbb{C}[\lambda_{h+1}, \ldots, \lambda_n]/(\lambda_{h+1} + \cdots + \lambda_n),$$

and the image of $\mathbb{C}[\lambda_1, \ldots, \lambda_n]^{\mathcal{S}_n}$ is contained in

$$\mathbb{C}[\lambda_1, \ldots, \lambda_h]^{\mathcal{S}_k}/(\lambda_1 + \cdots + \lambda_h) \otimes_{\mathbb{C}} \mathbb{C}[\lambda_{h+1}, \ldots, \lambda_n]^{\mathcal{S}_{n-k}}/(\lambda_{h+1} + \cdots + \lambda_n).$$

Denote by $T_{n,g,h}$ the subspace of this tensor product, formed by homogeneous elements of degree g, and let $\bar{\Sigma}_{n,g,h} \subseteq T_{n,g,h}$ be the image of $\bar{\Sigma}_{n,g} \subset \mathbb{C}[\bar{\lambda}_1, \ldots, \bar{\lambda}_n]$. Let us write $\tilde{\lambda}_i$ for the class of $\bar{\lambda}_i$ modulo $\bar{J}_h$, and consider the image $\bar{\pi}_{n,g,h}$ of $\bar{\pi}_{n,g}$ modulo $\bar{J}_h$. We get from Formula (18):

$$\bar{\pi}_{n,g,h} = \sum_{j=0}^{g} \pi_{h,j}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_h; a) \cdot \pi_{n-h,g-j}(\tilde{\lambda}_{h+1}, \ldots, \tilde{\lambda}_n; a) \tag{19}$$

as an element from $\bar{\Sigma}_{n,g,h} \otimes S_{n,g} \subset T_{n,g,h} \otimes S_{n,g}$.

Lemma 2.16. With the notation above we have the following results.

(a) If $(B_i)_{i \in I}$ is a basis of $\bar{\Sigma}_{n,g,h}$ and

$$\bar{\pi}_{n,g,h} = \sum_{i \in I} B_i \otimes A_i,$$

then the A_i form a basis of the U-invariants decomposable as

$$S_{n,g,h} := \sum_{j}^{g} S_{h,j} \cdot S_{n-h,g-j}.$$

In particular, the potenziante $\bar{\pi}_{n,g,h}(\tilde{\lambda}; a) \in \bar{\Sigma}_{n,g,h} \otimes S_{n,g}$ is a dualizing tensor between $\bar{\Sigma}_{n,g,h}$ and the space $S_{n,g,h} \subset S_{n,g}$.

(b) In the correspondence between subspaces of $S_{n,g}$ and of $\bar{\Sigma}_{n,g}$ given by the potenziante $\bar{\pi}_{n,g}$ (see Proposition 2.13(2)) the subspace $S_{n,g,h}$ is the orthogonal to $\bar{\Sigma}_{n,g} \cap \bar{J}_h$.

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Proof. (a) Developing Formula (19) for $\bar{\pi}_{n,g,h}$ by using Formula (17) for all terms we get

$$\bar{\pi}_{n,g,h} = \sum_{j=0}^{g} \sum_{\substack{k'_2,\dots,k'_h,k''_2,\dots,k''_{n-h} \\ \sum_i ik'_i=j,\ \sum_i ik''_i=g-j}} ({e'_2}^{k'_2} \cdots {e'_h}^{k'_h} \otimes {e''_2}^{k''_2} \cdots {e''_{n-h}}^{k''_{n-h}}) U_{k'_2\dots,k'_h} U_{k''_2,\dots,k''_{n-h}},$$

where $e'_i := e_i(\tilde{\lambda}_1,\dots,\tilde{\lambda}_h)$ and $e''_i := e_i(\tilde{\lambda}_{h+1},\dots,\tilde{\lambda}_n)$. Now the elements $U_{k_2,\dots,k_h}$ span the space $S_{h,j}$ for $j := \sum_i ik_i$, it follows that the elements $U_{k'_2\dots,k'_h} U_{k''_2,\dots,k''_{n-h}}$ span $S_{n,g,h} = \sum_{j=0}^{g} S_{h,j} \cdot S_{n-h,g-j}$. Since the tensor products ${e'_2}^{k'_2} \cdots {e'_h}^{k'_h} \otimes {e''_2}^{k''_2} \cdots {e''_{n-h}}^{k''_{n-h}}$ are linearly independent it follows that the A_i also span $S_{n,g,h}$ using parts (2) and (3) of Remark 2.7.

It follows from Proposition 2.13(2) and Remark 2.7 that $\bar{\pi}_{n,g,h}$ is a dualizing tensor between $\bar{\Sigma}_{n,g,h}$, the image of $\bar{\Sigma}_{n,g}$, and a subspace $S'_{n,g,h} \subseteq S_{n,g}$. By the 1st part $S'_{n,g,h} = S_{n,g,h}$ and the A_i are also linearly independent.

(b) By Remark 2.7 (1) or Proposition 2.9 this is clear, since $\bar{\Sigma}_{n,g} \cap \bar{J}_h$ is the kernel of the surjective map $\bar{\Sigma}_{n,g} \to \bar{\Sigma}_{n,g,h}$. ■

2.17 The symmetric functions p_h and q_n

The space $\bar{\Sigma}_{n,g} \cap \bar{J}_h$ consists of the symmetric functions in $\bar{\lambda}_1,\dots,\bar{\lambda}_n$ of degree n, which are divisible by $\bar{\lambda}_1 + \cdots + \bar{\lambda}_h$. Such a symmetric function is also divisible by the elements $\bar{\lambda}_T := \sum_{i\in T} \bar{\lambda}_i$ for all subsets $T \subset \{1,\dots,n\}$ of cardinality $|T| = h$.

For $h < \frac{n}{2}$ consider the symmetric function

$$p_h := \prod_{1\le j_1<j_2<\dots<j_h\le n} (\bar{\lambda}_{j_1} + \bar{\lambda}_{j_2} + \cdots + \bar{\lambda}_{j_h}) = \prod_{\substack{T\subset\{1,2,\dots,n\} \\ |T|=h}} \bar{\lambda}_T$$

of degree $\binom{n}{h}$. It follows that p_h is an irreducible element of $\bar{\Sigma}_n$ and that $\bar{\Sigma}_{n,g} \cap \bar{J}_h$ consists of the multiples of p_h of degree g. When $n = 2h$ is even and $h > 1$, then $\bar{\lambda}_T = -\bar{\lambda}_{\{1,2,\dots,n\}\setminus T}$. Therefore, we define

$$p_h := \prod_{1<j_2<\dots<j_h\le 2h} (\bar{\lambda}_1 + \bar{\lambda}_{j_2} + \cdots + \bar{\lambda}_{j_h}) = \prod_{\substack{T\subset\{1,2,\dots,n\} \\ |T|=h,\ 1\in T}} \bar{\lambda}_T.$$

We claim that p_h is irreducible, of degree $\frac{1}{2}\binom{2h}{h}$ and still symmetric for $h > 1$. In fact it is clearly symmetric with respect to the permutations which fix 1 so it is enough to see the symmetry under the transposition $(1,2)$. This fixes all factors for which $\lambda_2 = 2$; as

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for the product Π of the remaining factors $\bar{\lambda}_T,\ 1 \in T,\ 2 \notin T$ it replaces 1 with 2 and maps these set of factors bijectively to the set of factors associated to sets T with $1 \notin T,\ 2 \in T$.

For these sets the map $T \mapsto \{1, 2, \dots, n\} \setminus T$ is a bijection with the factors of Π. By formula $\bar{\lambda}_T = -\bar{\lambda}_{\{1,2,\dots,n\}\setminus T}$ the product of Π is thus equal to $\epsilon\Pi$ with $\epsilon = (-1)^{|\Pi|}$. Now clearly $|\Pi| = \binom{2h-2}{h-1} = 2\binom{2h-3}{h-2}$ is even. This proves the following lemma.

Lemma 2.18. For $n > 2$ and $h \le \frac{n}{2}$ the space $\bar{\Sigma}_{n,g} \cap \bar{J}_n$ consists of the elements of $\bar{\Sigma}_{n,g}$ that are multiples of the symmetric function p_h.

Let us define the following symmetric function:

$$q_n := p_1 p_2 \cdots p_m \text{ where } m := \left\lfloor \frac{n}{2} \right\rfloor.$$

We claim that $\deg q_n = 2^{n-1} - 1$. In fact,

$$\sum_{j=1}^{\frac{n-1}{2}} \binom{n}{j} = \frac{1}{2} \sum_{j=1}^{n-1} \binom{n}{j} = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1 \qquad \text{if } n \text{ is odd,}$$

$$\sum_{j=1}^{h-1} \binom{2h}{j} + \frac{1}{2}\binom{2h}{h} = \frac{1}{2} 2^{2h} - 1 = 2^{n-1} - 1 \qquad \text{if } n = 2h \text{ is even.}$$

Theorem 2.19. Let us now assume $n > 2$. With respect to the potenziante $\bar{\pi}_{n,g} \in \bar{\Sigma}_{n,g} \otimes S_{n,g}$ the space of decomposable U-invariants of degree n and weight g is the orthogonal to $O_{n,g} := \bar{\Sigma}_{n,g} \cap (q_n)$. It has as basis the coefficients of the potenziante in the quotient algebra $\mathbb{C}[\bar{\lambda}_1, \dots, \bar{\lambda}_n]/(q_n)$.

Proof. This follows from the following argument. In the correspondence between subspaces of $\bar{\Sigma}_{n,g}$ and of $S_{n,g}$ given by the potenziante $\pi_{n,g}$ (Proposition 3.13(2)) we have seen in Lemma 3.16 that the space of U-invariants decomposed as a sum of products of U-invariants of degree h and $n - h$ is the orthogonal of the subspace of multiples of p_h. Hence, the entire space of decomposable U-invariants is the orthogonal of the intersection of all the subspaces of multiples of the various p_h. But these symmetric functions are all irreducible in the algebra of symmetric functions in $\bar{\lambda}_1, \dots, \bar{\lambda}_n$, and distinct, so that this intersection is exactly the space of multiples of q_n. ∎

This we believe is the main step in Stroh's proof.

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Corollary 2.20. If $M_{n,g} \subset \bar{\Sigma}_{n,g}$ is a complement to $O_{n,g} := \bar{\Sigma}_{n,g} \cap (q_n)$, then the orthogonal of $M_{n,g}$ in $S_{n,g}$ is a space of perpetuants (Definition 2.14) of degree n and weight g.

Proof. This follows by duality. The orthogonal of $M_{n,g}$ is a complement of the orthogonal of $O_{n,g}$, which is the space of decomposable elements. ∎

2.21 Generating functions and proof of Stroh's theorem

Define

$$N_{n,g} := \#\{2\mu_2 + 3\mu_3 + \cdots + n\mu_n = g\},$$

the number of ways of partitioning g with numbers between 2 and n. We have

$$\sum_{g=0}^{\infty} N_{n,g}\, x^g = \frac{1}{(1-x^2)\cdots(1-x^n)}.$$

The dimension of the space $\bar{\Sigma}_{n,g}$ is clearly $N_{n,g}$, and the subspace of those divisible by an element of degree i has dimension $N_{n,g-i}$ if $i \leq g$ and 0 otherwise. It follows that the space $O_{n,g} = (q_n) \cap \bar{\Sigma}_{n,g}$ of multiples of q_n has dimension $N_{n,g-2^{n-1}+1}$ if $g \geq 2^{n-1}-1$ and 0 otherwise. Now Corollary 2.20 shows that this is the dimension of the perpetuants of degree $n > 2$ and weight g, and we thus get for the generating function, in degree n:

$$\sum_{g=2^{n-1}-1}^{\infty} N_{n,g-2^{n-1}+1}\, x^g = \left(\sum_{k=0}^{\infty} N_{n,k}\, x^k\right) x^{2^{n-1}-1} = \frac{x^{2^{n-1}-1}}{(1-x^2)\cdots(1-x^n)}.$$

This proves the main theorem of Stroh provided we do the cases $n = 1$ and 2. For $n = 1$ we have $S_1 = \mathbb{C}a_0$, and so the only homogeneous perpetuant is clearly a_0. For $n = 2$ the only decomposable elements are the multiples of a_0^2. We have

$$E(\bar{\lambda}_1\alpha_1 + \bar{\lambda}_2\alpha_2)^{[g]} = \bar{\lambda}_1^g E\left((\alpha_1 - \alpha_2)^{[g]}\right) = \bar{\lambda}_1^g E\left(\sum_{j=0}^{g} \alpha_1^{[j]}(-\alpha_2)^{[g-j]}\right), \text{and}$$

$$E\left(\sum_{j=0}^{g} \alpha_1^{[j]}(-\alpha_2)^{[g-j]}\right) = \sum_{j=0}^{g}(-1)^{g-j} a_j a_{g-j} =$$

$$= \begin{cases} 0 & \text{if } g \text{ is odd,} \\ \sum_{j=0}^{h-1} 2(-1)^j a_j a_{g-j} + (-1)^h a_h^2 & \text{if } g = 2h \text{ is even.} \end{cases}$$

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This shows that there is exactly one perpetuant of degree 2 in every even weight > 0, and so the generating function is $x^2/(1-x^2)$ as claimed.

3 A Basis of the Perpetuants

In the next paragraph we construct an explicit basis for a space of perpetuants (Definition 2.14).

3.1 Leading exponents

Using Corollary 2.20, we will now define a special basis in order to obtain a basis of the perpetuants, see Theorem 3.9 below. As before, we will work in the polynomial algebra

$$\mathbb{C}[\lambda_1, \dots, \lambda_n]/(\lambda_1 + \cdots + \lambda_n) = \mathbb{C}[\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}],$$

where $\bar{\lambda}_i$ is the image of λ_i. For $\boldsymbol{r} = (r_1, \dots, r_{n-1}) \in \mathbb{N}^{n-1}$ we set $\bar{\lambda}^{\boldsymbol{r}} := \bar{\lambda}_1^{r_1} \cdots \bar{\lambda}_{n-1}^{r_{n-1}}$, so that any $f \in \mathbb{C}[\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}]$ can be written in the form $f = \sum_{finite} c_{\boldsymbol{r}} \bar{\lambda}^{\boldsymbol{r}}$.

We use the usual lexicographic order $\leq$ on the exponents:

$$(r_1, \dots, r_{n-1}) < (s_1, \dots, s_{n-1}) \iff r_k < s_k \text{ for } k := \min\{i \mid r_i \neq s_i\}.$$

Definition 3.2. For a nonzero polynomial $f \in \mathbb{C}[\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}]$, $f = \sum_i c_{\boldsymbol{r}} \bar{\lambda}^{\boldsymbol{r}}$, the maximum $\boldsymbol{r}_0 := \max\{\boldsymbol{r} \mid c_{\boldsymbol{r}} \neq 0\}$ is called the *leading exponent* of f and is denoted by $\ell_{exp}(f)$. Furthermore, $\ell_{mon}(f) := c_{\boldsymbol{r}_0} \bar{\lambda}^{\boldsymbol{r}_0}$ is called the *leading monomial* of f.

Remark 3.3. For two polynomials f, g we have $\ell_{exp}(f \cdot g) = \ell_{exp}(f) + \ell_{exp}(g)$.

As before, we denote by $\bar{e}_2, \dots, \bar{e}_n \in \mathbb{C}[\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}]$ the images of the elementary symmetric functions $e_2, \dots, e_n \in \mathbb{C}[\lambda_1, \dots, \lambda_n]$.

Lemma 3.4.

(1) The leading exponent of $\bar{e}^{\boldsymbol{h}} := \bar{e}_2^{h_2} \cdots \bar{e}_n^{h_n}$ is given by

$$\ell_{exp}(\bar{e}^{\boldsymbol{h}}) = (2(h_2 + \cdots + h_n), h_3 + \cdots + h_n, \dots, h_{n-1} + h_n, h_n) \in \mathbb{N}^{n-1}.$$

(2) The leading exponents of the monomials $\bar{e}_2^{h_2} \cdots \bar{e}_n^{h_n}$ are distinct and are formed by all sequences $(r_1, \dots, r_{n-1})$ with $r_1 - 2r_2 \in 2\mathbb{N}$ and $r_i \geq r_{i+1}$.

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Proof. (1) The leading monomial of $\bar{e}_j$ comes from the term

$$\begin{aligned}\bar{\lambda}_1\bar{\lambda}_2\cdots\bar{\lambda}_{j-1}\bar{\lambda}_n &= -\bar{\lambda}_1\bar{\lambda}_2\cdots\bar{\lambda}_{j-1}(\bar{\lambda}_1+\cdots+\bar{\lambda}_{n-1})\\ &= -\bar{\lambda}_1^2\bar{\lambda}_2\cdots\bar{\lambda}_{j-1} + \text{lower terms.}\end{aligned}$$

Therefore, we have

$$\ell_{exp}(\bar{e}_2) = (2,0,\ldots,0),\ \ell_{exp}(\bar{e}_3) = (2,1,0,\ldots,0),\cdots,\ell_{exp}(\bar{e}_n) = (2,1,\ldots,1),$$

and the claim follows from Remark 3.3.

(2) This follows immediately from (1) by setting $2h_2 := r_1 - 2r_2$, $h_j := r_{j-1} - r_j$, $n > j \geq 3$, $h_n = r_{n-1}$. ■

Recall the definition of the symmetric function $q_n \in \mathbb{C}[\bar{\lambda}_1,\ldots,\bar{\lambda}_n]$ from Section 2.17:

$$p_h := \prod_{1\leq j_1<j_2<\ldots<j_h\leq n} (\bar{\lambda}_{j_1}+\bar{\lambda}_{j_2}+\cdots+\bar{\lambda}_{j_h}) = \prod_{\substack{T\subset\{1,2,\ldots,n\}\\ |T|=h}} \bar{\lambda}_T \quad \text{for } 2h<n,$$

$$p_m := \prod_{1<j_2<\ldots<j_m\leq 2m} (\bar{\lambda}_1+\bar{\lambda}_{j_2}+\cdots+\bar{\lambda}_{j_m}) = \prod_{\substack{T\subset\{1,2,\ldots,n\}\\ |T|=m,\ 1\in T}} \bar{\lambda}_T \quad \text{for } n=2m,$$

and

$$q_n := p_1\cdots p_m \text{ where } m := \left\lfloor \frac{n}{2} \right\rfloor.$$

Lemma 3.5. The leading exponent of q_n is $\ell_{exp}(q_n) = (2^{n-2}, 2^{n-3}, \ldots, 2, 1)$.

Proof. For $T := \{j_1, j_2, \ldots, j_h\}$, $1 \leq j_1 < j_2 < \ldots < j_h \leq n$ we have

$$\bar{\lambda}_T = \begin{cases} \bar{\lambda}_{j_1}+\bar{\lambda}_{j_2}+\cdots+\bar{\lambda}_{j_h} & \text{if } j_h<n \\ \bar{\lambda}_{j_1}+\bar{\lambda}_{j_2}+\cdots+\bar{\lambda}_{j_{h-1}} - \sum_{i=1}^{n-1}\bar{\lambda}_i & \text{if } j_h=n \end{cases}$$

Thus, if $n \in T$, then $\bar{\lambda}_T = -\bar{\lambda}_{T'}$ where $T' := \{1,\ldots,n\} \setminus T$. The map $T \mapsto T'$ is a bijection between the subsets T of $\{1,2,\ldots,n\}$ containing n and of cardinality h with the subsets

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of $\{1, 2, \ldots, n-1\}$ of cardinality $n-h$. This implies, for $2h < n$:

$$p_h = \pm \prod_{\substack{T \subset \{1,\ldots,n-1\} \\ |T|=h}} \bar{\lambda}_T \prod_{\substack{T \subset \{1,\ldots,n-1\} \\ |T|=n-h}} \bar{\lambda}_T = \pm f_h f_{n-h}$$

$$\text{where } f_k := \prod_{\substack{T \subset \{1,\ldots,n-1\} \\ |T|=k}} \bar{\lambda}_T.$$

The leading term of $\bar{\lambda}_T$, $T \subset \{1, 2, \ldots, n-1\}$ is $\bar{\lambda}_j$ with $j := \min T$, and the number of subsets $T \subset \{1, 2, \ldots, n-1\}$ with $|T| = h$ and $j = \min T$ equals the number of subsets $T \subseteq \{j+1, \ldots, n-1\}$ with $|T| = h-1$. This number is equal to $\binom{n-1-j}{h-1}$ if $h \leq n-j$, and 0 otherwise. Setting $\binom{m}{k} = 0$ if $m < k$, we see that the leading exponent of f_k is given by

$$\ell_{exp}(f_k) = (\tbinom{n-2}{k-1}, \tbinom{n-3}{k-1}, \ldots, \tbinom{n-i-1}{k-1}, \ldots, \tbinom{1}{k-1}, \tbinom{0}{k-1}).$$

(Recall that $\binom{0}{0} = 1$.) The leading exponent of p_h is thus

$$\ell_{exp}(p_h) = (\tbinom{n-2}{h-1} + \tbinom{n-2}{n-h-1}, \ldots, \tbinom{n-i-1}{h-1} + \tbinom{n-i-1}{n-h-1}, \ldots, \\ \tbinom{1}{h-1} + \tbinom{1}{n-h-1}, \tbinom{0}{h-1} + \tbinom{0}{n-h-1}).$$

If $n = 2m+1$, then $q_n = p_1 \cdots p_m$, and we find for the leading exponent of q_n, $\ell_{exp}(q_n) = (r_1, \ldots, r_{n-1})$ where

$$r_i = \sum_{h=1}^{m} \left(\binom{n-i-1}{h-1} + \binom{n-i-1}{n-h-1} \right) = \sum_{h=0}^{2m-1} \binom{2m-i}{h} = 2^{2m-i} = 2^{n-i-1},$$

as claimed.

If $n = 2m$, we have $q_n = p_1 \cdots p_{m-1} p_m$ where $p_m = \prod_{\substack{T \subset \{1,2,\ldots,n\} \\ |T|=h,\ 1 \in T}} \bar{\lambda}_T$. In this case, the map $T \mapsto T' := \{1, \ldots, 2m\} \setminus T$ is a bijection between the subsets containing 1 and $2m$ and of cardinality m and the subsets of $\{2, 3, \ldots, 2m-1\}$ containing m elements. Hence,

$$p_m = \pm \prod_{\substack{T \subset \{1,\ldots,2m-1\} \\ 1 \in T,\, |T|=m}} \bar{\lambda}_T \prod_{\substack{T \subset \{2,\ldots,2m-1\} \\ |T|=m}} \bar{\lambda}_T.$$

The leading monomial of the 1st product is $\bar{\lambda}_1^{\binom{2m-2}{m-1}}$. For the 2nd product, we see as above that the number of subsets of $\{2, \ldots, 2m-1\}$ of cardinality m with minimum $j \geq 2$ is

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equal to $\binom{2m-j-1}{m-1}$. Hence,

$$\ell_{exp}(p_m) = (\binom{2m-2}{m-1}, (\binom{2m-3}{m-1}, \dots, \binom{2m-i-1}{m-1}, \dots, \binom{1}{m-1}, 0),$$

and thus we get for the leading exponent $\ell_{exp}(q_n) = (r_1, \dots, r_{n-1})$

$$\begin{aligned} r_i &= \sum_{h=1}^{m-1} \left(\binom{2m-i-1}{h-1} + \binom{2m-i-1}{2m-h-1} \right) + \binom{2m-i-1}{m-1} \\ &= \sum_{h=0}^{2m-2} \binom{2m-i-1}{h} = 2^{2m-i-1} = 2^{n-i-1}. \end{aligned}$$

This proves the lemma. ∎

Remark 3.6. For $n \geq 4$ we have

$$\ell_{exp}(q_n) = \ell_{exp}(e^{\boldsymbol{n}}) \text{ where } \boldsymbol{n} := (0, 2^{n-4}, 2^{n-5}, \dots, 2, 1, 1).$$

Moreover, $\ell_{exp}(q_3) = (2, 1) = \ell_{exp}(\bar{e}_3)$.

3.7 A basis for the perpetuants

Recall that $\bar{\Sigma}_{n,g}$ is the space of symmetric functions of degree g in $\bar{\lambda}_1, \dots, \bar{\lambda}_n$,

$$\bar{\Sigma}_{n,g} = (\mathbb{C}[\lambda_1, \dots, \lambda_n]^{\mathcal{S}_n} / (\lambda_1 + \dots + \lambda_n))_g = \mathbb{C}[\bar{\lambda}_1, \dots, \bar{\lambda}_n]_g^{\mathcal{S}_n}.$$

In the next lemma we use the partial order

$$(t_2, \dots, t_n) \succeq (s_2, \dots, s_n) \iff t_i \geq s_i \text{ for all } i.$$

Lemma 3.8. For $n \geq 3$ a basis $\mathcal{B}_n$ of a complement of $O_{n,g} := \bar{\Sigma}_{n,g} \cap (q_n)$, in the space of symmetric functions in $\bar{\Sigma}_{n,g}$, is formed by the monomials $e^{\boldsymbol{h}} := \prod_{k=2}^{n} e_k^{h_k}$ with $\sum_k k h_k = g$, satisfying

$$\boldsymbol{h} = (h_2, \dots, h_n) \not\succeq \boldsymbol{n} := (0, 2^{n-4}, 2^{n-5}, \dots, 2, 1, 1) \text{ for } n > 3,$$

respectively, $\boldsymbol{h} = (h_2, h_3) \not\succeq (0, 1)$ for $n = 3$.

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Proof. A basis of $O_{n,g}$ is formed by the symmetric functions $q_n\bar{e}^{\boldsymbol{k}} = q_n\prod_{j=2}^{n}\bar{e}_j^{k_j}$ with $\sum_j jk_j = g - 2^{n-1} + 1$. We have seen, in Remark 4.6, that the leading exponent of $q_n\bar{e}^{\boldsymbol{k}}$ equals the leading exponent of $\bar{e}^{\boldsymbol{n}+\boldsymbol{k}}$. It follows that the set

$$\mathbb{X}_n := \left\{q_n\bar{e}^{\boldsymbol{k}} \mid \sum_j jk_j = g - 2^{n-1} + 1\right\} \cup \left\{\bar{e}^{\boldsymbol{h}} \mid \sum_i ih_i = g,\ \boldsymbol{h} \not\succeq \boldsymbol{n}\right\} \subset \bar{\Sigma}_{n,g}$$

has the same leading exponents as the basis $\{\bar{e}^{\boldsymbol{h}} \mid \sum_i ih_i = g\}$ of $\bar{\Sigma}_{n,g}$. Since these leading exponents are distinct, by Lemma 3.1(2), it follows that $\mathbb{X}_n$ is a basis of $\bar{\Sigma}_{n,g}$; hence, $\mathcal{B}_n$ is a basis of a complement, in $\bar{\Sigma}_{n,g}$, of $O_{n,g}$, hence the claim. ∎

We have seen in Proposition 2.13 that the potenziante $\pi_{n,g}(\bar{\lambda}; a) \in \bar{\Sigma}_{n,g} \otimes S_{n,g}$ has the form

$$\pi_{n,g}(\bar{\lambda}_1, \ldots, \bar{\lambda}_n; a_0, a_1, \ldots, a_g) = \sum_{\substack{k_2,\ldots,k_n,\\ \sum ik_i = g}} e_2^{k_2} \ldots e_n^{k_n} U_{k_2,\ldots,k_n}, \tag{20}$$

where the $U_{k_2,\ldots,k_n}$ form a basis of the space $S_{n,g} \subset \mathbb{C}[a_0, \ldots, a_n]$ of U-invariants of degree n and weight g.

Using Corollary 2.20 with the basis $\mathcal{B}_n$ of a complement $M_{n,g} \subset \bar{\Sigma}_{n,g}$ to $O_{n,g}$ constructed above we get as consequence our main result.

Theorem 3.9. The elements $U_{k_2,\ldots,k_n}$ from Formula (20) with

$$\boldsymbol{k} \succeq \boldsymbol{n} = (0, 2^{n-4}, \ldots, 2, 1, 1)$$

(resp. $\boldsymbol{n} = (0, 1)$) form a basis of a space of perpetuants of degree $n > 3$ (resp. $n = 3$) and weight g.

Observe that the decomposable elements do not have a basis extracted from the elements $U_{k_2,\ldots,k_n}$.

Remark 3.10. Finally, in order to compute explicitly the perpetuants of Theorem 4.9 one needs to compute the numbers $\alpha_{h_1,\ldots,h_n,k_1,\ldots,k_n}$ of Formula (14). One possible algorithm is to compute first

$$e_1^{k_1} \ldots e_n^{k_n} = \sum_{h_1,\ldots,h_n} \beta_{h_1,\ldots,h_n,k_1,\ldots,k_n}\, m_{h_1,\ldots,h_n}.$$

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The numbers $\beta_{h_1,\dots,h_n,k_1,\dots,k_n}$ form an upper triangular matrix $E + A$ of nonnegative integers with 1 on the diagonal, and its inverse $E - A + A^2 - \dots$ has as entries the integers $\alpha_{h_1,\dots,h_n,k_1,\dots,k_n}$.

The integer $\beta_{h_1,\dots,h_n,k_1,\dots,k_n}$ is computed as the coefficient of the monomial $\prod_{i=1}^n \lambda_i^{h_i}$ in the development of $e_1^{k_1} \dots e_n^{k_n}$.

4 Binary Forms

This is a complement to set into the 19th century context the theory developed. The $q+1$-dimensional vector space $P_q = P_q(x) \subset \mathbb{C}[x]$ of polynomials of degree $\leq q$ in the variable x, introduced in Section 1.1, can be thought of as a *non-homogeneous form* of the space, still denoted by $P_q = P_q(x,y) \subset \mathbb{C}[x,y]$, of homogeneous polynomials of degree q in the variables x, y. These are the classical *binary forms* or *binary q-antics.* On this space acts the group $\mathrm{GL}(2,\mathbb{C})$, and, in fact, these spaces form the list of irreducible representations of $\mathrm{SL}(2,\mathbb{C})$. Of course this is the 1st case of the more general theory of *n-ary q-antics*, that is, of homogeneous polynomials of degree q in the n variables $x_1, \dots, x_n$.

One of the themes of Algebra of the 19th century was to study the algebra R_q of polynomial functions on P_q that are invariant under $\mathrm{SL}(2,\mathbb{C})$, and then try the general case of invariants of n-ary q-antics. In particular, to determine a minimal set of generators for such an algebra. The question whether such a minimal set of generators is finite was one of the main problems of this period, and proved by Gordan [4] for binary forms by a difficult combinatorial method.

The problem of finite generation of invariants for a general linear group action, also known as Hilbert*'s 14th problem*, has now a very long and complex history (cf. [14, 15]) with still several open questions.

In fact, R_q is a graded algebra, and if I_q denotes the ideal of R_q formed by elements with no constant term, the question is to study I_q/I_q^2. A partial question is to understand the graded dimension of I_q/I_q^2, which by Gordan's theorem is a polynomial. There are in fact various formulas for the graded dimension of R_q, but for I_q/I_q^2, to our knowledge, the only known cases are those in which one can exhibit generators for I_q/I_q^2. Thus, for binary forms only a few cases are explicitly known. It is therefore quite remarkable that for perpetuants such a formula exists.

The reason to introduce U-invariants comes from the theory of *covariants* of binary forms, a notion introduced as a tool to compute invariants of binary forms. Covariants appear in three different forms. For more details, we refer to the literature, e.g. [17, Chap. 15.1, Proposition 2, and Theorem 1].

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Proposition 4.1. There are canonical bijections between the following objects, called covariants of P_n of degree k and order p:

(i) $\mathrm{SL}(2,\mathbb{C})$-equivariant polynomial maps $P_n \to P_p$ of degree k;
(ii) $\mathrm{SL}(2,\mathbb{C})$-invariant polynomials on $P_n \oplus \mathbb{C}^2$ of bidegree k, p;
(iii) U-invariants of P_n of degree k and isobaric of weight $\frac{nk-p}{2}$.

In particular an $\mathrm{SL}(2,\mathbb{C})$-invariant on P_n of degree k is a U-invariant of degree k and weight $\frac{nk}{2}$.

(The reader experienced in algebraic geometry may see that the geometric reason behind these statements is the fact that $\mathrm{SL}(2,\mathbb{C})/B \simeq \mathbb{P}^1$ is compact.)

Proof. (i) $\iff$ (ii): Given such a polynomial map $F\colon P_n \to P_p$ we can evaluate the form $F(f)$ in a point $(x,y) \in \mathbb{C}^2$, $\tilde{F}(f,(x,y)) := F(f)(x,y)$ obtaining an $\mathrm{SL}(2,\mathbb{C})$-invariant of the desired form. The opposite construction is essentially tautological by the definition of the actions.

(ii) $\iff$ (iii): Observe that a regular function on $P_n \oplus (\mathbb{C}^2 \setminus \{0\})$ extends as polynomial on $P_n \oplus \mathbb{C}^2$. Under $\mathrm{SL}(2,\mathbb{C})$, the space $\mathbb{C}^2 \setminus \{0\}$ is the orbit of e_1 with stabilizer U. This implies that the polynomials on $P_n \oplus \mathbb{C}^2$ invariant under $\mathrm{SL}(2,\mathbb{C})$ are in bijection with the polynomials on $P_n \times \{e_1\}$ invariant under U.

Now consider the torus elements $D_t := \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C})$. They act on the space $\mathbb{C}^2$ transforming $x \mapsto t^{-1}x$, $y \mapsto ty$. The action on the forms $f \in P_n$ is

$$(D_t f)(x,y) = f(tx, t^{-1}y) = \sum_{i=0}^{n} a_i (tx)^{[n-i]} (t^{-1}y)^{[i]} = \sum_{i=0}^{n} a_i t^{n-2i} x^{[n-i]} y^{[i]}.$$

In other words, D_t transforms $a_i \mapsto t^{n-2i} a_i$. A covariant F of degree k and order p must be an invariant function of this transformation on $P_n \oplus \mathbb{C}^2$, or

$$F(t^n a_0, \ldots, t^{n-2i} a_i, \ldots, t^{-n} a_n, t^{-1}x, ty) = F(a_0, \ldots, a_n, x, y).$$

By assumption, $F = \sum_{i=0}^{p} F_i(a_0, \ldots, a_n) x^{p-i} y^i$; hence,

$$F_0(t^n a_0, \ldots, t^{n-2i} a_i, \ldots, t^{-n} a_n)(t^{-1}x)^p = F_0(a_0, \ldots, a_n) x^p.$$

A monomial in F_0 in the a_i is of weight g; hence, it is multiplied by t^{nk-2g}. We deduce that for every monomial we have $nk - 2g - p = 0$, as required. ■

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The U-invariant F_0 associated to a covariant F is called its *source* (or *Quelle* in German). There is a simple formula to write down the covariant starting from its source, see [8].

4.2 U-invariants for binary forms

For the algebra $S(n)$ of U-invariants for P_n the results are not as precise as for the limit algebra S.

In classical literature explicit computations were done correctly only up to degree 6, and degree 8, with partial results in degree 7. With the help of computers now one has computations up to degree 12. Here we want to give a simple method that we believe is due to CAYLEY and that works very well up to degree 4.

Let us take a polynomial $f = \sum_{i=0}^{n} a_i x^{[n-i]}$ with $a_0 \neq 0$. Under the transformation $x \mapsto x - \frac{a_1}{a_0}$ it is transformed into a polynomial with $a_1 = 0$ (cf. Formula 5):

$$\begin{aligned} f\left(x - \frac{a_1}{a_2}\right) &= a_0\left(x - \frac{a_1}{a_0}\right)^{[n]} + a_1\left(x - \frac{a_1}{a_0}\right)^{[n-1]} + \cdots \\ &= a_0 x^{[n]} - a_0 \frac{a_1}{a_0} x^{[n-1]} + \cdots + a_1 x^{[n-1]} + \cdots \\ &= a_0 x^{[n]} + \left(-\frac{a_1^2}{2a_0} + a_2\right) x^{[n-2]} + \cdots \end{aligned}$$

More formally, let $P_n^0 \subset P_n$ be the set of polynomials of degree n with $a_0 \neq 0$, and let $P_n' \subset P_n^0$ be the set of polynomials of degree n with $a_0 \neq 0$, $a_1 = 0$. The previous remark shows that acting with U we have an isomorphism $U \times P_n' \xrightarrow{\sim} P_n^0$. Thus, we have an identification of the U-invariant functions on P_n^0 with the functions on P_n'. More precisely, the map (notation from Formula (4))

$$\pi : P_n^0 \to P_n', \quad f \mapsto \frac{a_1}{a_0} \cdot f,$$

is U-invariant, and so the pull-backs of the coordinate functions of P_n' together with a_0^{-1} generate the U-invariants on P_n^0. By Formula (5), these pull-backs are given by

$$\left(-\frac{a_1}{a_0}\right) \cdot a_k = \sum_{j=0}^{k} a_j \left(-\frac{a_1}{a_0}\right)^{[k-j]} = a_0^{-k+1} c_k,$$

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where

$$
\begin{aligned}
c_k &= (-a_1)^{[k]} + \sum_{j=1}^{k} a_0^{j-1} a_j (-a_1)^{[k-j]} \\
&= (-a_1)^{[k]} + a_1(-a_1)^{[k-1]} + \sum_{j=2}^{k} a_0^{j-1} a_j (-a_1)^{[k-j]} \\
&= (-1)^k (1-k) a_1^{[k]} + \sum_{j=2}^{k} (-1)^{k-j} a_0^{j-1} a_j a_1^{[k-j]}.
\end{aligned}
$$

Thus, we get the following result.

Theorem 4.3. We have $S(n)[a_0^{-1}] = \mathbb{C}[c_2, \ldots, c_n][a_0, a_0^{-1}]$ where $a_0, c_2, \ldots, c_n$ are algebraically independent. In particular, $\dim S(n) = n$.

Let us explicit some of these elements:

$$
\begin{aligned}
c_2 &= -a_1^{[2]} + a_0 a_2, \\
c_3 &= 2a_1^{[3]} - a_0 a_1 a_2 + a_0^2 a_3, \\
c_4 &= -3a_1^{[4]} + a_0 a_1^{[2]} a_2 - a_0^2 a_1 a_3 + a_0^3 a_4, \\
c_5 &= 4a_1^{[5]} - a_0 a_1^{[3]} a_2 + a_0^2 a_1^{[2]} a_3 - a_0^3 a_1 a_4 + a_0^4 a_5, \\
c_6 &= -5a_1^{[6]} + a_0 a_1^{[4]} a_2 - a_0^2 a_1^{[3]} a_3 + a_0^3 a_1^{[2]} a_4 - a_0^4 a_1 a_5 + a_0^5 a_6.
\end{aligned}
\tag{21}
$$

By construction, c_k is a U-invariant of degree k and weight k (cf. Definition 1.5).

Corollary 4.4. The subalgebra of $S(n)$ generated by the U-invariants with weight equal to the degree is the polynomial ring $\mathbb{C}[c_2, \ldots, c_n]$.

4.5 An algorithm

If we want to understand U-invariants from these formulas it is necessary to compute the intersection

$$
S(n) = \mathbb{C}[c_2, \ldots, c_n][a_0, a_0^{-1}] \cap \mathbb{C}[a_0, \ldots, a_n]. \tag{22}
$$

A general algorithm for these types of problems has been in fact developed by Bigatti–Robbiano in a recent preprint [1]. It gives by a computer program the U-invariants as explicit polynomials up to degree 6. The complexity of the algorithm,

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which is general, is much higher than that given by the symbolic method in the special case of U-invariants of binary forms with which those invariants were classically computed.

Roughly speaking the algorithm consists in finding polynomials in the c_i that are divisible by higher and higher powers of a_0.

For $n \leq 4$ the algorithm is quite simple and quickly gives the following.

4.5.1 *The case n = 2*

$S_2 = \mathbb{C}[a_0, c_2]$.

4.5.2 *The case n = 3*

$$8c_2^3 + 9c_3^2 = a_0^2(9a_0^2a_3^2 - 18a_0a_1a_2a_3 + 8a_0a_2^3 + 6a_1^3a_3 - 3a_1^2a_2^2) = a_0^2D,$$

with D of degree 4 and weight 6, thus an $\mathrm{SL}(2,\mathbb{C})$-invariant (Proposition 4.1), the *discriminant*. The algorithm stops after this point and S_3 is generated by the elements a_0, c_2, c_3, D modulo the relation $a_0^2D - 8c_2^3 - 9c_3^2$:

$$S_3 = \mathbb{C}[a_0, c_2, c_3, D], \quad a_0^2D - 8c_2^3 - 9c_3^2 = 0.$$

4.5.3 *The case n=4*

$$2c_4 + c_2^2 = a_0^2(2a_0a_4 - 2a_1a_3 + a_2^2) := a_0^2B,$$

with B of degree 2 and weight 4, hence an $\mathrm{SL}(2,\mathbb{C})$-invariant.

$$6c_2B - D = -a_0C \text{ with } C := 2a_2^3 - 6a_1a_2a_3 + 9a_0a_3^2 + 6a_1^2a_4 - 12a_0a_2a_4,$$

where C has degree 3 and weight 6, hence is an $\mathrm{SL}(2,\mathbb{C})$-invariant. Again, the algorithm stops here, the algebra S_4 is generated by the U-invariants a_0, c_2, c_3, B, C modulo the relation $6a_0^2c_2B + a_0^3C - 8c_2^3 - 9c_3^2$, and the subalgebra of $\mathrm{SL}(2,\mathbb{C})$-invariants is generated by B and C.

$$S_4 = \mathbb{C}[a_0, c_2, c_3, B, C], \quad 6a_0^2c_2B + a_0^3C - 8c_2^3 - 9c_3^2 = 0.$$

Remark 4.6. The computations above show that the indecomposable U-invariant $D \in S_3$ becomes decomposable in S_4.

A modern approach to computations of invariants and covariants for binary forms can be found in the thesis of Mihaela Popoviciu Draisma [16]. There, one can find also references to classical computations.

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References

[1] Bigatti, A. M. and L. Robbiano. "Saturations of subalgebras, SAGBI bases, and U-invariants." arXiv: 1909.10901v2, 2019.

[2] Faà de Bruno, F. *Théorie des Formes Binaires*. Turin: Librairie Brero, 1876.

[3] Gilham, C. W. "The minimum weight of irreducible q-ary perpetuant types." *J. London Math. Soc.* 2, no. 4 (1927): 209–10.

[4] Gordan, P. "Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist." *J. Reine Angew. Math.*, no. 69 (1868): 323–54.

[5] Gordan, P. *Vorlesungen über Invariantentheorie*, edited by G. Kerschensteiner, second ed. New York: Chelsea Publishing Co, 1987. Erster Band: Determinanten. [Vol. I: Determinants], Zweiter Band: Binäre Formen. [Vol. II: Binary forms].

[6] Grace, J. H. "On perpetuants." *Proc. London Math. Soc.* 35 (1903): 319–31.

[7] Grace, J. H. "Types of perpetuants." *Proc. London Math. Soc.* 35 (1903): 107–11.

[8] Hilbert, D. "Theory of algebraic invariants." *Translated from the German and with a preface by Reinhard C. Laubenbacher, Edited and with an introduction by Bernd Sturmfels*. Cambridge: Cambridge University Press, 1993.

[9] Kung, J. P. S. and G.-C. Rota. "The invariant theory of binary forms." *Bull. Amer. Math. Soc. (N.S.)* 10, no. 1 (1984): 27–85.

[10] MacMahon, P. A. "On perpetuants." *Am. J. Math.* 7, no. 1 (1884): 26–46.

[11] MacMahon, P. A. "Memoir on seminvariants." *Am. J. Math.* 8, no. 1 (1885): 1–18.

[12] MacMahon, P. A. "A second paper on perpetuants." *Am. J. Math.* 7, no. 3 (1885): 259–63.

[13] MacMahon, P. A. "The perpetuant invariants of binary quantics." *Proc. London Math. Soc.* 26 (1894/95): 262–84.

[14] Nagata, M. "On the 14-th problem of Hilbert." *Am. J. Math.* 81 (1959): 766–72.

[15] Nagata, M. *Lectures on the Fourteenth Problem of Hilbert*. Bombay: Tata Institute of Fundamental Research, 1965.

[16] Popoviciu Draisma, M. "Invariants of binary forms." PhD thesis, University of Basel. https://edoc.unibas.ch/33424/, 2013.

[17] Procesi, C. *Lie Groups—An Approach through Invariants and Representations*. Universitext. New York: Springer, 2007.

[18] Rota, G.-C. "Two turning points in invariant theory." *Math. Intelligencer* 21, no. 1 (1999): 20–7.

[19] Rota, G.-C. and B. D. Taylor. "The classical umbral calculus." *SIAM J. Math. Anal.* 25, no. 2 (1994): 694–711.

[20] Stroh, E. "Über die symbolische Darstellung der Grundsyzyganten einer binären Form sechster Ordnung und eine Erweiterung der Symbolik von Clebsch." *Math. Ann.* 36, no. 2 (1890): 262–303.

[21] Sylvester, J. J. "On subvariants, i.e. semi-invariants to binary quantics of an unlimited order." *Am. J. Math.* 5, no. 1–4 (1882): 79–136.

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[22] Wood, P. W. “On the irreducibility of perpetuant types.” *Proc. London Math. Soc. (2)* 1 (1904): 480–4.

[23] Wood, P. W. “Alternative expressions for Perpetuant type forms.” *Proc. London Math. Soc. (2)* 3 (1905): 334–44.

[24] Wood, P. W. “On the reducibility of Covariants of binary quantics of infinite order.” *Proc. London Math. Soc. (2)* 5 (1907): 177–96.

[25] Young, A. and P. W. Wood. “Perpetuant syzygies.” *Proc. London Math. Soc. (2)* 2 (1905): 221–65.

[26] Young, A. “Ternary perpetuants.” *Proc. London Math. Soc. (2)* 22 (1924): 171–200.

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