

Representations with a reduced null cone

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In honor of Nolan Wallach

Abstract Let G be a complex reductive group and V a G -module. Let $\pi: V \rightarrow V//G$ be the quotient morphism defined by the invariants and set $\mathcal{N}(V) := \pi^{-1}(\pi(0))$. We consider the following question. Is the null cone $\mathcal{N}(V)$ reduced, i.e., is the ideal of $\mathcal{N}(V)$ generated by G -invariant polynomials? We have complete results when G is SL_2 , SL_3 or a simple group of adjoint type, and also when G is semisimple of adjoint type and the G -module V is irreducible.

Keywords: Null cone • Null fiber • Quotient morphism • Semisimple groups • Representations

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1 Introduction

Let G be a reductive complex algebraic group and let V be a finite-dimensional G -module. Let $\pi: V \rightarrow V//G$ be the categorical quotient morphism given by the G -invariant functions on V , and let

$$\mathcal{N} := \pi^{-1}(\pi(0)) = \{v \in V \mid \overline{Gv} \ni 0\} \subset V$$

be the *null cone*. We say that V is *coreduced* if \mathcal{N} is reduced. This means that the ideal $I(\mathcal{N}) \subset \mathcal{O}(V)$ of the set \mathcal{N} is generated by the invariant functions

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$\mathfrak{m}_0 := \mathcal{O}(V)^G \cap I(\mathcal{N})$, the homogeneous maximal ideal of $\mathcal{O}(V)^G$, and so $I(\mathcal{N}) = I_G(\mathcal{N}) := \mathfrak{m}_0 \mathcal{O}(V)$, where we use $\mathcal{O}(X)$ to denote the regular functions on a variety X . If it is important to specify the group or representation involved, we will use notation such as $(V, G), \mathcal{N}(V), \mathcal{N}_G(V)$, etc.

We say that V is *strongly coreduced* if every fiber of π is reduced. We can reformulate this in terms of slice representations. Let Gv be a closed orbit. Then G_v is reductive and we have a splitting of G_v -modules,

$$V = T_v(Gv) \oplus N_v.$$

Then (N_v, G_v) is the *slice representation of G_v at v* . We show that the fiber $\pi^{-1}(\pi(v))$ is reduced if and only if (N_v, G_v) is coreduced (Remark 4.1). Hence V is strongly coreduced if and only if every slice representation of V is coreduced.

A main difficulty in our work is that, a priori, V may be coreduced but π may have a nonreduced fiber $F \neq \mathcal{N}$. We conjecture that this is never the case:

Conjecture 1. A coreduced G -module is strongly coreduced.

Recall that V is *cofree* if $\mathcal{O}(V)$ is a free module over $\mathcal{O}(V)^G$. Equivalently, $\pi: V \rightarrow V//G$ is flat. In the cofree case the associated cone to any fiber F (see [BK79] or [Kra84, II.4.2]) is the null cone. From this one can immediately see that \mathcal{N} reduced implies that F is reduced, so the conjecture holds when V is cofree. There is another case in which the associated cone of F is \mathcal{N} : the case in which the isotropy group H of the closed orbit $Gv \subset F$ has the same rank as G , i.e., contains a maximal torus T of G . This was first noticed by RICHARDSON [Ric89, Proposition 5.5]. If the slice representation of H at v is not coreduced, then neither is (V, G) (Proposition 5.1). For an irreducible representation V of G , having $V^T \neq 0$ means that the weights of V are in the root lattice; equivalently, the center of G acts trivially on V . Hence we have a representation of the adjoint group $G/Z(G)$. This explains why many of our results require that the group be adjoint, or that at least one of the irreducible components of our representation contains a zero weight vector.

For SL_2 and SL_3 we have a complete classification of the coreduced modules. In the case of SL_3 , the coreduced modules are either cofree (for which there is a finite list of possibilities) or one has the direct sum of arbitrarily many copies of \mathbb{C}^3 and $(\mathbb{C}^3)^*$ (§11). We would guess that something similar holds for representations of any simple algebraic group G . Besides the infinite series of coreduced representations involving the representations of the smallest dimension for SL_n and Sp_n (see §9), any coreduced module should be cofree (and for this there is a finite list of possibilities). If one replaces coreduced by *strongly* coreduced, then our techniques should be sufficient to establish this result. For irreducible representations we show that strongly coreduced and cofree are equivalent (§4), which is the first step. If G is adjoint, our classification of the coreduced representations produces a finite list of cofree representations. In case G is semisimple adjoint and not simple and the

representation is faithful and irreducible, our classification shows that there are finitely many coreduced representations, all of which are again cofree. In these cases the group has two simple factors (Theorem 8.3). We would also guess that one can add at most finitely many representations to these and still remain coreduced, but we have not carried out the details. In general, to carry the classification any further, new techniques are needed.

Here is a summary of the contents of this paper. In §2 we present elementary results and determine the coreduced representations of tori (Proposition 2.13). In Section 3 we show how to use covariants to prove that a null cone is not reduced and as an application we determine the coreduced representations of SL_2 (Theorem 3.7). In §4 we show that every cofree irreducible representation of a simple algebraic group is coreduced (Theorem 4.10) and that, sort of conversely, every irreducible representation of a simple group which is strongly coreduced is cofree (Theorem 4.12). In §5 we consider modules V with $V^T \neq 0$, T a maximal torus of G . We develop methods (based on weight multiplicities) to show that a slice representation at a zero weight vector is not coreduced (we say that V has a *bad toral slice*). We then show that V has a bad toral slice if all the roots of G appear in V with multiplicity two or more (Proposition 5.17).

In §6 we apply our techniques to find the maximal coreduced representations of the adjoint exceptional groups (Theorem 6.9). The case of F_4 is rather complicated and needs some heavy computations (see Appendix A). In §7 we do the same thing for the classical adjoint groups (Theorem 7.1), and in §8 we determine the irreducible coreduced representations of semisimple adjoint groups (Theorem 8.3). This is not straightforward, e.g., the representation $(\mathbb{C}^7 \otimes \mathbb{C}^7, G_2 \times G_2)$ is not coreduced, but showing this is difficult (see Appendix B).

In §9 we show that, essentially, the classical representations of the classical groups are coreduced (with a restriction for SO_n). This is a bit surprising, since these representations are often far from cofree. In §11 we classify the coreduced representations of SL_3 (not just PSL_3). To do this, we need to develop some techniques for finding irreducible components of null cones (see §10). These techniques should be useful in other contexts.

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2 Preliminaries and elementary results

We begin with some positive results. Let G be a reductive group. We do not assume that G is connected, but when we say that G is semisimple this includes connectivity. Let V be a finite-dimensional G -module.

Proposition 2.1. *Suppose that G is semisimple and that V satisfies one of the following conditions.*

- (1) $\dim V//G \leq 1$;
- (2) $V = \text{Ad } G$.

Then V is coreduced.

Proof. If $\dim V//G = 0$, then $\mathcal{N} = V$ is reduced. If $\dim V//G = 1$, then $\mathcal{O}(V)^G$ is generated by a homogeneous irreducible polynomial f and its zero set \mathcal{N} is reduced. If V is the adjoint representation of G , then \mathcal{N} is irreducible of codimension equal to the rank ℓ of G . Since \mathcal{N} is defined by ℓ homogeneous invariants and the rank of $d\pi$ is ℓ on an open dense subset of \mathcal{N} , it follows that \mathcal{N} is reduced and even normal ([Kos63]). □

Example 2.2. Suppose that G is finite and acts nontrivially on V . Then $\mathcal{N} = \{0\}$ (as set) is not reduced since not all the coordinate functions can be G -invariant.

Let V be a G -module. Then $V//G$ parameterizes the closed G -orbits in V . Let Gv be a closed orbit and let N_v be the slice representation of G_v . We say that Gv is a *principal orbit* and that G_v is a *principal isotropy group* if $\mathcal{O}(N_v)^{G_v} = \mathcal{O}(N_v^{G_v})$. In other words, N_v is the sum of a trivial representation and a representation N'_v with $\mathcal{O}(N'_v)^{G_v} = \mathbb{C}$. The principal isotropy groups form a single conjugacy class of G and the image of the principal orbits in $V//G$ is open and dense. We say that V is *stable* if $N'_v = (0)$; equivalently, there is an open dense subset of V consisting of closed orbits. If G is semisimple and there is a nonempty open subset of points with reductive isotropy group, then V is stable. In particular, if the general point in V has a finite isotropy group, then the representation is stable with finite principal isotropy groups.

Our example above generalizes to the following.

Remark 2.3. Let V be a G -module where $G/G^0 \neq \{e\}$. If G/G^0 acts nontrivially on the quotient $V//G^0$, then V is not coreduced. Note that, for example, G/G^0 acts nontrivially if the principal isotropy group of (V, G) is trivial.

Proposition 2.4. *Assume that (V, G^0) is not coreduced. Then (V, G) is not coreduced.*

Proof. The null cones for G^0 and G are the same (as sets). There is an $f \in I(\mathcal{N})$ which is not in $I_{G^0}(\mathcal{N})$. Hence $f \notin I_G(\mathcal{N})$ and (V, G) is not coreduced. □

For the next three more technical results we have to generalize the definition of coreducedness to pointed G -varieties.

Definition 2.5. A *pointed G -variety* is a pair (Y, y_0) where Y is an affine G -variety and y_0 is a fixed point. A pointed G -variety (Y, y_0) is *coreduced* if the fiber $\pi^{-1}(\pi(y_0))$ is reduced where $\pi: Y \rightarrow Y//G$ is the quotient morphism.

Lemma 2.6. *Let (X, x_0) be a pointed G -variety and $Y \subset X$ a closed G -stable subvariety containing x_0 . Assume that the ideal $I(Y)$ of Y is generated by G -invariant functions. Then (X, x_0) is coreduced if and only if (Y, x_0) is coreduced.*

Proof. Let $\mathfrak{m} \subset \mathcal{O}(X)^G$ be the maximal ideal of $\pi(x_0)$ where $\pi: X \rightarrow X//G$ is the quotient morphism, and let $\mathfrak{n} \subset \mathcal{O}(Y)^G$ denote the image of \mathfrak{m} . By assumption, the ideal $\mathfrak{m} \subset \mathcal{O}(X)$ contains $I(Y)$ and so $\mathcal{O}(Y)/(\mathfrak{n} \subset \mathcal{O}(Y)) \simeq \mathcal{O}(X)/(I(Y) + \mathfrak{m} \subset \mathcal{O}(X)) = \mathcal{O}(X)/(\mathfrak{m} \subset \mathcal{O}(X))$. \square

Example 2.7. Let V be a G -module. Denote by θ_n the n -dimensional trivial representation and let $F \subset V \oplus \theta_n$ be a G -stable hypersurface containing 0 . If G is semisimple, then F is defined by a G -invariant polynomial. Hence $(F, 0)$ is coreduced if and only if V is coreduced.

Lemma 2.8. *Let (X, x_0) be a pointed G -variety. Let H be a reductive group acting on X such that G sends H -orbits to H -orbits. Assume that every G -invariant function on X is H -invariant. If (X, x_0) is coreduced with respect to G , then so is $(X//H, \pi(x_0))$.*

Proof. Put $Y := X//H$. Then G acts on Y , because G preserves the H -invariant functions $\mathcal{O}(X)^H \subset \mathcal{O}(X)$. Suppose that f is an element of $\mathcal{O}(Y)$ which vanishes on the null fiber $\mathcal{N}_G(Y, \pi_H(x_0))$. By assumption, $\mathcal{N}_G(X, x_0) = \pi_H^{-1}(\mathcal{N}_G(Y, \pi_H(x_0)))$ and so $f \circ \pi_H$ vanishes on $\mathcal{N}_G(X, x_0)$. Hence $f \circ \pi_H = \sum_i a_i b_i$ where the a_i are G -invariant and vanish at x_0 . Since $f \circ \pi_H$ is H -invariant, we may average the b_i over H and assume that they are in $\mathcal{O}(X)^H$. But then $a_i = \bar{a}_i \circ \pi_H$ and $b_i = \bar{b}_i \circ \pi_H$ for unique $\bar{a}_i \in \mathcal{O}(Y)^G$ and $\bar{b}_i \in \mathcal{O}(Y)$. Thus $f = \sum_i \bar{a}_i \bar{b}_i$ and so $(Y, \pi_H(x_0))$ is coreduced. \square

Example 2.9. If (V, G) is a coreduced representation and $H \subset G$ a closed normal subgroup, then $(V//H, \pi_H(0))$ is coreduced (with respect to G/H).

Example 2.10. Let (X, x_0) be a pointed G -variety, let W be a G -module of dimension n and let $H = \text{SO}_n$ acting as usual on \mathbb{C}^n . Consider the pointed $(G \times H)$ -variety $(Y, y_0) := (X \times (W \otimes \mathbb{C}^n), (x_0, 0))$. Assume that $G \rightarrow \text{GL}(W)$ has image in $\text{SL}(W)$. We claim that if (Y, y_0) is a coreduced $(G \times H)$ -variety, then $(X \times S^2(W^*), (x_0, 0))$ is a coreduced G -variety.

By Classical Invariant Theory (see [Pro07]), the generators of the invariants of $(n\mathbb{C}^n, \text{SO}_n)$ are the inner product invariants f_{ij} of the copies of \mathbb{C}^n together with the determinant d . The relations are generated by the equality $d^2 = \det(f_{ij})$. Identifying $n\mathbb{C}^n$ with $W \otimes \mathbb{C}^n$ we see that the quadratic invariants transform by the representation $S^2(W^*)$ of G , the determinant d transforms by $\bigwedge^n(W^*) = \theta_1$, and the relation is G -invariant. Now applying Lemmas 2.8 and 2.6 gives the claim.

Lemma 2.11. *Let (Y, y_0) be a pointed G -variety and $Z \subset Y$ a closed G -stable subvariety containing y_0 . Suppose that there is a G -equivariant retraction $p: (Y, y_0) \rightarrow (Z, y_0)$. If (Y, y_0) is coreduced, then so is (Z, y_0) .*

Proof. Clearly, if y is in the null cone of Y , then $p(y)$ is in the null cone of Z . Thus if $f \in \mathcal{O}(Z)$ vanishes on the null cone of Z , then $\tilde{f} := p^* f \in \mathcal{O}(Y)$ vanishes

on the null cone of Y . By hypothesis we have that $\widetilde{f} = \sum_i a_i b_i$ where the a_i are invariants vanishing at y_0 . Restricting to Z we get a similar sum for f . Hence Z is coreduced. \square

- Example 2.12.** (1) If $(V_1 \oplus V_2, G)$ is a coreduced representation, then so is (V_i, G) , $i = 1, 2$.
 (2) If (V, G) is a coreduced representation and $H \subset G$ a closed normal subgroup, then (V^H, G) is also coreduced.
 (3) Let V_i be a G_i module, $i = 1, 2$. Then $(V_1 \oplus V_2, G_1 \times G_2)$ is coreduced if and only if both (V_1, G_1) and (V_2, G_2) are coreduced. Here we use that $\mathcal{N}(V_1 \oplus V_2) = \mathcal{N}(V_1) \times \mathcal{N}(V_2)$.

We finish this section with the case of tori which is quite easy. We will then see in Section 5 that this case can be applied to representations containing zero weights.

Proposition 2.13. *Let V be a T -module where T is a torus. Let $\alpha_1, \dots, \alpha_n$ be the nonzero weights of V . Then V is coreduced if and only if the solutions of*

$$\sum_i m_i \alpha_i = 0, \quad m_i \in \mathbb{N},$$

are generated by solutions where the m_i are zero or one.

It is well-known that the monoid of relations $\sum_i m_i \alpha_i = 0, m_i \in \mathbb{N}$ is generated by the *indecomposable* relations, i.e., by relations which cannot be written as a sum of two nontrivial relations. So a necessary and sufficient condition for coreducedness is that the indecomposable relations $\sum_i n_i \alpha_i = 0, n_i \in \mathbb{N}$, satisfy $n_i = 0$ or 1 .

Proof. We may assume that $V^T = 0$. Let x_1, \dots, x_n be a weight basis of V^* corresponding to the α_i . Suppose that there is an indecomposable relation where, say, $m_1 \geq 2$. Then the monomial $x_1 x_2^{m_2} \dots x_n^{m_n}$ vanishes on the null cone, but it is not in the ideal of the invariants. Hence our condition is necessary.

On the other hand, suppose that the indecomposable relations are of the desired form. Now any polynomial vanishing on $\mathcal{N}(V)$ is a sum of monomials with this property, and a monomial p vanishing on $\mathcal{N}(V)$ has a power which is divisible by an invariant monomial q without multiple factors. But then p is divisible by q and so $\mathcal{N}(V)$ is reduced. \square

Corollary 2.14. *Let $T = \mathbb{C}^*$. Then $\mathcal{N}(V)$ is reduced if and only if $\mathcal{O}(V)^T = \mathbb{C}$ or the nonzero weights of V are $\pm k$ for a fixed $k \in \mathbb{N}$.*

3 The method of covariants

In this section we explain how covariants can be used to show that a representation is not coreduced. As a first application we classify the coreduced representations of SL_2 .

Let G be a reductive group and V a representation of G . A G -equivariant morphism $\varphi: V \rightarrow W$ where W is an irreducible representation of G is called a *covariant of V of type W* . Clearly, covariants of type W can be added and multiplied with invariants and thus form an $\mathcal{O}(V)^G$ -module $\text{Cov}(V, W)$ which is known to be finitely generated (see [Kra84, II.3.2 Zusatz]).

A nontrivial covariant $\varphi: V \rightarrow W$ defines a G -submodule $\varphi^*(W^*) \subset \mathcal{O}(V)$ isomorphic to the dual W^* , and every irreducible G -submodule of $\mathcal{O}(V)$ isomorphic to W^* is of the form $\varphi^*(W^*)$ for a suitable covariant φ . Moreover, φ vanishes on the null cone \mathcal{N} if and only if $\varphi^*(W^*) \subset I(\mathcal{N})$. We say that φ is a *generating covariant* if φ is not contained in $\mathfrak{m}_0 \text{Cov}(V, W)$, or equivalently, if $\varphi^*(W^*)$ is not contained in $I_G(\mathcal{N})$. Thus we obtain the following useful criterion for non-coreducedness.

Proposition 3.1. *If φ is a generating covariant which vanishes on \mathcal{N} , then V is not coreduced.*

Remark 3.2. Let $f \in \mathcal{O}(V)^G$ be a generating homogeneous invariant of positive degree, i.e., $f \in \mathfrak{m}_0 \setminus \mathfrak{m}_0^2$. Then the differential $df: V \rightarrow V^*$ is a generating covariant. In fact, using the contraction $(df, \text{Id}) = \text{deg } f \cdot f$, we see that if $df = \sum_i f_i \varphi_i$ where the f_i are homogeneous nonconstant invariants, then $f = \frac{1}{\text{deg } f} \sum_i f_i (\varphi_i, \text{Id}) \in \mathfrak{m}_0^2$.

Example 3.3. Let G be SL_2 and $V = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ where $\mathfrak{sl}_2 = \text{Lie } \text{SL}_2$ is the Lie algebra of SL_2 . Then the null cone $\mathcal{N}(V)$ consists of commuting pairs of nilpotent matrices, and so the covariant

$$\varphi: \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2, (A, B) \mapsto AB - \frac{1}{2} \text{tr}(AB) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

vanishes on $\mathcal{N}(V)$, i.e., $\varphi^*(\mathfrak{sl}_2) \subset I(\mathcal{N})$. But $\varphi^*(\mathfrak{sl}_2)$ is bihomogeneous of degree $(1, 1)$ and therefore is not contained in $I_G(\mathcal{N})$ because there are no invariants of degree 1.

Example 3.4. Let G be SO_4 and $V = \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4$. The weights of \mathbb{C}^4 relative to the maximal torus $T = \text{SO}_2 \times \text{SO}_2$ are $\pm \varepsilon_1, \pm \varepsilon_2$, and the degree 2 invariants (dot products) $q_{ij}: (v_1, v_2, v_3) \mapsto v_i \cdot v_j, 1 \leq i \leq j \leq 3$, generate the invariant ring. Let V_{++} denote the span of the positive weight vectors and let V_{+-} denote the span of the weight vectors corresponding to ε_1 and $-\varepsilon_2$. Then $\mathcal{N} = GV_{++} \cup GV_{+-}$, and an easy calculation shows that every homogeneous covariant $V \rightarrow \mathbb{C}^4$ of degree 3 vanishes on the null cone \mathcal{N} . Now, using LiE (see [vLCL92]), one finds that there are 19 linearly independent homogeneous covariants of type \mathbb{C}^4 of degree 3, whereas there are 6 linearly independent invariants in degree 2, and obviously 3 linear covariants of type \mathbb{C}^4 . Therefore, there is at least one generating covariant of type \mathbb{C}^4 in degree 3 and so V is not coreduced. (See Theorem 9.1(4) for a more general statement.)

We now use our method to classify the cofree SL_2 -modules. Starting with the natural representation on \mathbb{C}^2 we get a linear action on the coordinate ring $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x, y]$ where x has weight 1 with respect to the standard torus $T = \mathbb{C}^* \subset \text{SL}_2$.

The homogeneous components $R_m := \mathbb{C}[x, y]_m$ of degree m give all irreducible representations of SL_2 up to isomorphism. A binary form $f \in R_m$ will be written as

$$f = \sum_{i=0}^m a_i \binom{m}{i} x^i y^{m-i}$$

so that the corresponding coordinate functions x_i are weight vectors of weight $m - 2i$. The null cone of R_m consists of those forms f that have a linear factor of multiplicity strictly greater than $\frac{m}{2}$. More generally, for any representation V of SL_2 , we have

$$\mathcal{N} = SL_2 V_+,$$

where V_+ is the sum of all weight spaces of strictly positive weight. In particular, \mathcal{N} is always irreducible.

Example 3.5. The binary forms of degree 4 have the following invariants:

$$A := x_0x_4 - 4x_1x_3 + 3x_2^2 \quad \text{and} \quad H := \det \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$$

classically called ‘‘Apolare’’ and ‘‘Hankelsche Determinante’’ which generate the invariant ring (see [Sch68]). It is easy to see that the null cone $\mathcal{N}(R_4) = SL_2(\mathbb{C}x^3y \oplus \mathbb{C}x^4)$ is the closure of the 3-dimensional orbit of x^3y and thus has codimension 2. A simple calculation shows that the Jacobian $Jac(H, A)$ has rank 2 at x^3y . It follows that $\mathcal{N}(R_4)$ is a reduced complete intersection. (One can deduce from $\text{rank } Jac(H, A) = 2$ that A, H generate the invariants.)

Example 3.6. The representation $kR_1 = R_1 \oplus R_1 \oplus \dots \oplus R_1$ (k copies) can be identified with the space $M_{2 \times k}(\mathbb{C})$ of $2 \times k$ -matrices where SL_2 acts by left multiplication. Then the null cone \mathcal{N} is the closed subset of matrices of rank ≤ 1 , which is the determinantal variety defined by the vanishing of the 2×2 -minors $M_{ij} = x_{1i}x_{2j} - x_{2i}x_{1j}$, $1 \leq i < j \leq k$. It is known that the ideal of \mathcal{N} is in fact generated by the minors M_{ij} . This is an instance of the so-called Second Fundamental Theorem, see [Pro07, Chap. 11, Section 5.1]. Thus \mathcal{N} is reduced, and the minors M_{ij} generate the invariants.

Theorem 3.7. *Let V be a nontrivial coreduced representation of SL_2 where $V^{SL_2} = (0)$. Then V is isomorphic to R_2, R_3, R_4 or $kR_1, k \geq 1$.*

It is shown in [Dix81] that the representations R_n are not coreduced for $n \geq 5$. Our proof is based on the following results.

Lemma 3.8. *Let V be a representation of SL_2 and $\varphi: V \rightarrow R_m$ a homogeneous covariant of degree d .*

- (1) If $d > m$, then $\varphi(\mathcal{N}) = (0)$.
- (2) If $\pm \text{Id}$ acts trivially on V and $2d > m$, then $\varphi(\mathcal{N}) = (0)$.

Proof. Let $V_+ \subset V$ be the sum of the positive weight spaces. Since $\mathcal{N} = \text{SL}_2 V_+$ it suffices to show that φ vanishes on V_+ . Choose coordinates x_1, \dots, x_n on V consisting of weight vectors and let $z = x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ be a monomial occurring in a component of φ . Then $\sum_i k_i = d$ and the weight of z occurs in R_m . Since $m < d$ the monomial z must contain a variable x_i with a weight ≤ 0 , and so, z vanishes on V_+ . This proves (1).

For (2) we remark that V contains only even weights and so a variable x_i with nonpositive weight has to appear in z as soon as $2d > m$. □

Lemma 3.9. *Let V be a nontrivial representation of SL_2 not isomorphic to R_1, R_2, R_3 or R_4 . Then the principal isotropy group is either trivial or equal to $\{\pm \text{Id}\}$.*

Proof. This is well known for the irreducible representations $R_j, j \geq 5$. Let T and U denote the usual maximal torus and maximal unipotent subgroup of SL_2 . Denote by H_i the generic stabilizer of $R_i, i = 1, 2, 3$ and 4. Then we have $H_1 = U, H_2 = T, H_3 = \left\{ \begin{bmatrix} \xi & 0 \\ 0 & \xi^2 \end{bmatrix} \mid \xi^3 = 1 \right\} \simeq \mathbb{Z}/3$ and

$$H_4 = \left\{ \begin{bmatrix} \xi & 0 \\ 0 & \xi^3 \end{bmatrix} \mid \xi^4 = 1 \right\} \cup \left\{ \begin{bmatrix} 0 & \xi \\ -\xi^3 & 0 \end{bmatrix} \mid \xi^4 = 1 \right\} \simeq \tilde{Q}_8,$$

the group of quaternions of order 8. It is easy to see that the generic stabilizer of H_1 and H_3 on any nontrivial representations of SL_2 is trivial, and that the generic stabilizer of H_2 and H_4 on the $R_{2j}, j > 0$, is $\{\pm \text{Id}\}$. □

Proof (of Theorem 3.7). For $V = R_2$ or R_3 the quotient $V // \text{SL}_2$ is one-dimensional and so both are coreduced. In Examples 3.5 and 3.6 we have seen that R_4 and kR_1 are coreduced. So it remains to show that any other representation V of SL_2 is not coreduced.

By Lemma 3.9 we can assume that the principal isotropy group is trivial or $\{\pm \text{Id}\}$. In the first case, V contains a closed orbit isomorphic to SL_2 . By Frobenius reciprocity, we know that the multiplicity of R_m in $\mathcal{O}(\text{SL}_2)$ is equal to $\dim R_m = m + 1$. This implies that the rank of the $\mathcal{O}(V)^G$ -module $\text{Cov}(V, R_m)$ is at least $m + 1$. Since we may assume that V is not kR_1 , removing all but one copy of R_1 gives a module V' which still has trivial principal isotropy groups. By Example 2.12(1) it is sufficient to show that V' is not coreduced. Now there has to be a generating homogeneous covariant $\varphi: V' \rightarrow R_1$ of degree > 1 . By Lemma 3.8(1) and Proposition 3.1 it follows that V' is not coreduced, hence neither is V .

Now assume that the principal isotropy group is $\{\pm \text{Id}\}$. As above this implies that the rank of $\text{Cov}(V, R_2)$ is at least 3. Since $R_2 \oplus R_2$ is not coreduced (Example 3.3) we can assume that V contains at most one summand isomorphic to R_2 . It follows that there is a generating homogeneous covariant $\varphi: V \rightarrow R_2$ of degree > 1 , and the claim follows by Lemma 3.8(2) and Proposition 3.1. □

4 Cofree representations

Let G be a (connected) reductive group, V a G -module and $\pi: V \rightarrow V//G$ the quotient morphism.

Remark 4.1. Let Gv be a closed orbit with slice representation (N_v, G_v) . Then, by LUNA’s slice theorem, the fiber $F := \pi^{-1}(\pi(v))$ is isomorphic to $G \times^{G_v} \mathcal{N}(N_v)$ which is a bundle over $G/G_v \simeq Gv$ with fiber $\mathcal{N}(N_v)$. Hence F is reduced if and only if N_v is coreduced.

If the fiber $F = \pi^{-1}(z)$ is reduced, then F is smooth in a dense open set $U \subset F$ which means that the rank of the differential $d\pi_u$ equals $\dim V - \dim_u F$ for $u \in U$. Thus we get the following criterion for non-coreducedness.

Lemma 4.2. *If X is an irreducible component of $\mathcal{N}(V)$ and the rank of $d\pi$ on X is less than the codimension of X in V , then V is not coreduced.*

Recall that a G -module V is said to be *cofree* if $\mathcal{O}(V)$ is a free $\mathcal{O}(V)^G$ -module. Equivalently, $\mathcal{O}(V)^G$ is a polynomial ring (V is *coregular*) and the codimension of $\mathcal{N}(V)$ is $\dim V//G$. See [Sch79] for more details and a classification of cofree representations of simple groups.

Proposition 4.3. *Let V be a cofree G -module. If the null cone is reduced, then so is every fiber of $\pi: V \rightarrow V//G$, and every slice representation of V is coreduced.*

Proof. Since V is cofree, the map π is flat. By [Gro67, 12.1.7], note that the set $\{v \in V \mid \pi^{-1}(\pi(v)) \text{ is reduced at } v\}$ is open in V . But this set is a cone. Thus if the null cone is reduced, then so is any fiber of π , and every slice representation is coreduced. □

For a cofree representation V the (schematic) null cone $\mathcal{N}(V)$ is a complete intersection. Using SERRE’s criterion [Mat89, Ch. 8] one can characterize quite precisely when $\mathcal{N}(V)$ is reduced.

Proposition 4.4. *Let V be a cofree G -module. Then V is coreduced if and only if $\text{rank } d\pi = \text{codim } \mathcal{N}(V)$ on an open dense subset of $\mathcal{N}(V)$.*

Example 4.5. Let $G = \text{SL}_n$ and $V := S^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n$. The quotient map $\pi: V \rightarrow \mathbb{C}^2$ is given by the two invariants $f := \det(q)$ and $h := q(v)$ of bidegrees $(n, 0)$ and $(1, 2)$ where $(q, v) \in V$. An easy calculation shows that for $n = 2$ the differential $d\pi$ has rank ≤ 1 on the null cone. Hence

$$(S^2(\mathbb{C}^2)^* \oplus \mathbb{C}^2, \text{SL}_2)$$

is not coreduced, which we already know from Theorem 3.7.

We claim that for $n \geq 3$ the null cone is irreducible and reduced. Set $q_k := x_k^2 + x_{k+1}^2 + \dots + x_n^2$,

$$X_k := \{q_k\} \times \{v \in \mathbb{C}^n \mid q_k(v) = 0\} \subset V \text{ and } X_{n+1} := \{0\} \times \mathbb{C}^n.$$

Then $\mathcal{N}(V) = \bigcup_{k=2}^{n+1} \text{SL}_n \cdot X_k$. Since $\dim \text{SL}_n \cdot X_k = \dim \text{SL}_n q_k + n - 1$ for $2 \leq k \leq n$ we get $\text{codim } \text{SL}_n \cdot X_2 = 2 < \text{codim } \text{SL}_n \cdot X_k$ for all $k > 2$, and so $\mathcal{N}(V) = \overline{\text{SL}_n \cdot X_2}$ is irreducible. Moreover,

$$df_{(q_2,v)}(q, w) = a_{11} \quad \text{and} \quad dh_{(q_2,v)}(q, w) = \sum_{i=2}^n 2v_i y_i + q(v),$$

where $v = (v_1, \dots, v_n)$, $q = \sum a_{ij} x_i x_j$ and $w = (y_1, \dots, y_n)$. It follows that the two linear maps $df_{(q_2,v)}$ and $dh_{(q_2,v)}$ are linearly independent on a dense open set of X_2 , hence the claim.

In order to see that the null cone is reduced in a dense set we can use the following result due to KNOP [Kno86]. Recall that the *regular sheet* \mathcal{S}_V of a representation (V, G) of an algebraic group is the union of the G -orbits of maximal dimension.

Proposition 4.6. *Let (V, G) be a representation of a semisimple group and let $\pi: V \rightarrow V//G$ be the quotient map. Assume that $x \in V$ belongs to the regular sheet and that $\pi(x)$ is a smooth point of the quotient. Then π is smooth at x .*

Corollary 4.7. *Let (V, G) be a cofree representation of a semisimple group. Assume that the regular sheet \mathcal{S}_V of V meets the null cone $\mathcal{N}(V)$ in a dense set. Equivalently, assume that every irreducible component of the null cone contains a dense orbit. Then (V, G) is coreduced.*

Let θ be an automorphism of finite order of a semisimple group H and let G denote the identity component of the fixed points H^θ . Given any eigenspace V of θ on the Lie algebra \mathfrak{g} of G , we have a natural representation of G on V . These representations (V, G) are called θ -representations. They have been introduced and studied by VINBERG, see [Vin76]. Among other things he proved that θ -representations are cofree and that every fiber of the quotient map contains only finitely many orbits. As a consequence of Corollary 4.7 above we get the following result.

Corollary 4.8. *Every θ -representation (V, G) where G is semisimple is coreduced.*

Remark 4.9. The corollary above was first established by PANYUSHEV [Pan85].

Finally we can prove the main result of this section.

Theorem 4.10. *An irreducible cofree representation V of a simple group G is coreduced.*

Proof. It follows from the classification (see [Pop76, KPV76, Sch79]) that the only irreducible cofree representations of the simple groups that are not θ -representations (or have one-dimensional quotient) are the spin representation of Spin_{13} and the half-spin representations of Spin_{14} . For these representations, GATTI-VINIBERGH I [GV78] show that every irreducible component of the null cone has a dense orbit. \square

Example 4.11. Here is an example of a *cofree but not coreduced* representation. Let $G = \mathrm{SL}_2$ and let $V = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \simeq 2R_2$. Example 3.3 shows that V is not coreduced. Here is a different proof. Each copy of R_2 has a weight basis $\{x^2, xy, y^2\}$ relative to the action of the maximal torus $T = \mathbb{C}^*$. The null cone is $U^-(\mathbb{C}x^2 \oplus \mathbb{C}x^2)$ where U^- is the maximal unipotent subgroup of G opposite to the usual Borel subgroup. One can easily see that $d\pi_{(\alpha x^2, \beta x^2)}$ is nontrivial only on the vectors $(\gamma y^2, \delta y^2)$, giving a rank of two. But V is cofree with $\mathrm{codim} \mathcal{N}(V) = 3$. Thus V is not coreduced.

We can prove a sort of converse to the theorem above. Recall that V is *strongly coreduced* if every fiber of π is reduced; equivalently, every slice representation of V is coreduced.

Theorem 4.12. *A strongly coreduced irreducible representation of a simple group G is cofree.*

If G is simple, then we use the ordering of Bourbaki [Bou68] for the simple roots α_j of G and we let φ_j denote the corresponding fundamental representations. We use the notation ν_j to denote the 1-dimensional representation of \mathbb{C}^* with weight j .

Proof. We use the techniques of [KPV76] (but we follow the appendix of [Sch78]). Let V be non-coregular (which is the same as V not being cofree by the classifications [Pop76, KPV76]). If V is $\varphi_1^3(\mathbf{A}_3)$ or $\varphi_2^3(\mathbf{A}_3)$, then there is a closed orbit with finite stabilizer whose slice representation is not coregular. Thus the slice representation is certainly not trivial, hence V is not strongly coreduced. Otherwise, there is a copy $T = \mathbb{C}^* \subset \mathrm{SL}_2 \subset G$ and a closed orbit Gv , $v \in V^T$, such that the identity component G_v^0 of the stabilizer G_v of v has rank 1. Moreover, one of the following occurs:

- (1) $G_v^0 = T$ and the slice representation of G_v , restricted to T , has at least 3 pairs of nonzero weights $\pm a, \pm b, \pm c$ (where we could have $a = b = c$).
- (2) The module is $\varphi_1\varphi_2(\mathbf{A}_3)$ or $\varphi_2\varphi_3(\mathbf{A}_3)$, G_v centralizes $G_v^0 = T$ and the slice representation is $\theta_2 + \nu_1 + \nu_{-1} + \nu_2 + \nu_{-2}$ where θ_n denotes the n -dimensional trivial representation.
- (3) $G_v^0 = \mathrm{SL}_2$ and the slice representation of G_v , restricted to T , contains at least 4 pairs of weights $\pm a, \pm b, \dots$

If, in case (1) above, the weights are not of the form $\pm a$ for a fixed a , then the G -module V is not strongly coreduced by Corollary 2.14. The same remark holds in case (3). Of course, in case (2), the module is not strongly coreduced. We went through the computations again and saw in which cases the weights of the slice representations were of the form $\pm a$ for a fixed a . One gets a list of representations as follows. (The list is complete up to automorphisms of the group.)

- (4) $\varphi_i(\mathbf{A}_n)$, $5 \leq i, 2i \leq n + 1$.
- (5) $\varphi_n(\mathbf{B}_n)$, $n \geq 7$.
- (6) $\varphi_n(\mathbf{D}_n)$, $n \geq 9$.
- (7) $\varphi_i(\mathbf{C}_n)$, $3 \leq i \leq n, n \geq 5$.

For the groups of type **A** and **C**, consider $SL_2 \subset G$ such that the fundamental representation restricted to SL_2 is $2R_1 + \theta_1$. For the groups of type **B** and **D** consider $SL_2 \subset G$ such that the fundamental representation restricted to SL_2 is $4R_1 + \theta$. Then using the techniques of [KPV76] one sees that there is a closed orbit in V^{SL_2} whose stabilizer has identity component SL_2 such that the slice representation restricted to SL_2 contains at least two copies of R_2 . Hence the slice representation is not coreduced. \square

We know from the [KPV76] and [Pop76] that an irreducible representation of a simple group is cofree if and only if it is coregular. Thus we get the following corollary.

Corollary 4.13. *Let (V, G) be an irreducible representation of a simple group. Then the following are equivalent:*

$$(V, G) \text{ cofree} \iff (V, G) \text{ coregular} \iff (V, G) \text{ strongly coreduced.}$$

5 The method of slices and multiplicity of weights

Let G be a reductive group and $T \subset G$ a maximal torus. It is well known that the orbit Gv is closed for any zero weight vector $v \in V^T$. We say that V is a G -module with a zero weight if $V^T \neq (0)$. The basic result for such modules is the following.

Proposition 5.1. *Let V be a G -module with a zero weight. If the slice representation at a zero weight vector is not coreduced, then neither is V .*

For the proof we use the following result originally due to RICHARDSON [Ric89, Proposition 5.5]. If $X \subset V$ is a closed subset of a vector space V , then the associated cone CX of X is defined to be the zero set of $\{\text{gr } f \mid f \in I(X)\}$ where $\text{gr } f$ denotes the (nonzero) homogeneous term of f of highest degree. If V is a G -module and X a closed subset of a fiber $F \neq \mathcal{N}(V)$ of the quotient map, then $CX = \overline{\mathbb{C}^*X} \setminus \mathbb{C}^*X$ (cf. [BK79, §3]).

Lemma 5.2. *Suppose that G_v has the same rank as G . Then the associated cone of $F := \pi^{-1}(\pi(v))$ is equal to $\mathcal{N}(V)$.*

Proof. We know that the associated cone of every fiber of π is contained in $\mathcal{N}(V)$. For the reverse inclusion we can assume that $T \subset G_v$. Let $v_0 \in \mathcal{N}(V)$. Then $\overline{Tgv_0} \ni 0$ for a suitable $g \in G$. This implies that $\overline{T(gv_0 + v)} \ni v$ and so $\mathbb{C}gv_0 + v \subset F$. The lemma follows since $gv_0 \in \overline{\mathbb{C}^*(\mathbb{C}gv_0 + v)}$. \square

Proof (of Proposition 5.1). Suppose that $\mathcal{N}(V)$ is reduced, and let $0 \neq f \in I(F)$ where F is as in the lemma above. Then the leading term $\text{gr } f$ lies in the ideal of $\mathcal{N}(V)$, so that there are homogeneous $f_i \in \mathfrak{m}_0$ and homogeneous $h_i \in \mathcal{O}(V)$ such that $\text{gr } f = \sum_i f_i h_i$ where $\deg f_i + \deg h_i = \deg \text{gr } f$ for all i .

Then $\widetilde{f} := \sum_i (f_i - f_i(v))h_i$ lies in $I_G(F)$ and $\text{gr } f = \text{gr } \widetilde{f}$. Replacing f by $f - \widetilde{f}$ we are able to reduce the degree of f . Hence by induction we can show that $f \in I_G(F)$. Thus F is reduced. \square

Remark 5.3. The proof above shows that the conclusion of Proposition 5.1 holds in a more general situation. *If the slice representation at $v \in V$ is not coreduced and the associated cone of $\pi^{-1}(\pi(v))$ is equal to $\mathcal{N}(V)$, then V is not coreduced.*

Example 5.4. We use the proposition above to give another proof that the irreducible representations R_{2m} of SL_2 are not coreduced for $m \geq 3$ (see Theorem 3.7). We have $R_{2m}^T = \mathbb{C}x^m y^m$, and a zero weight vector v has stabilizer $T \simeq \mathbb{C}^*$ if m is odd and $N(T)$ if m is even. The slice representation of T at v has the weights $\pm 4, \dots, \pm 2m$ (each with multiplicity one), and so, for $m \geq 3$, we have at least the weights ± 4 and ± 6 . But then the slice representation restricted to T is not coreduced by Corollary 2.14, hence neither are the representations R_{2m} of SL_2 for $m \geq 3$.

Let G be semisimple with Lie algebra \mathfrak{g} . If μ is a dominant weight of G , let $V(\mu)$ denote the corresponding simple G -module. Recall that the following are equivalent:

- (i) $V(\mu)$ has a zero weight;
- (ii) All weights of $V(\mu)$ are in the root lattice;
- (iii) μ is in the root lattice;
- (iv) The center of G acts trivially on $V(\mu)$.

Remark 5.5. Let V be a nontrivial simple G -module with a zero weight. Then the short roots are weights of $V(\mu)$ and the highest short root is the smallest nontrivial dominant weight. This follows from the following result due to STEINBERG, see [Ste98]. (We thank John Stembridge for informing us of this result.)

Lemma 5.6. *Let $\nu < \mu$ be dominant weights. Then there are positive roots β_i , $i = 1, \dots, n$, such that*

- (1) $\mu - \nu = \beta_1 + \beta_2 + \dots + \beta_n$, and
- (2) $\mu - \beta_1 - \dots - \beta_j$ is dominant for all $j = 1, \dots, n$.

Example 5.7. Let G be a semisimple group and \mathfrak{g} its Lie algebra. If V is a G -module with a zero weight, then the representation of G on $\mathfrak{g} \oplus V$ is not coreduced. This is a special case of a result of PANYUSHEV, see [Pan99, Theorem 4.5].

Proof. Let $T \subset G$ be a maximal torus and α a (short) root. Put $T_\alpha := (\ker \alpha)^0$. Then, for a generic $x \in \text{Lie } T_\alpha \subset \mathfrak{g}$ we have $G_x = \text{Cent}_G T_\alpha = G_\alpha \cdot T_\alpha$ where $G_\alpha \simeq \text{SL}_2$ or $\simeq \text{PSL}_2$, and so the slice representation at $x \in \mathfrak{g}$ is $\text{Lie}(G_x) \simeq \mathfrak{sl}_2 + \theta_{\ell-1}$ where $\ell = \text{rank } G$. Now consider the slice representation (S, G_x) at $(x, 0) \in \mathfrak{g} \oplus V$. Since V has a zero weight all short roots are weights of V (Remark 5.5) and thus the fixed points S^{T_α} are of the form $(\mathfrak{sl}_2 \oplus W, G_\alpha)$ where W is a nontrivial SL_2 -module. Using Example 2.12(2) the claim follows from Theorem 3.7. \square

If there is a $v \in V^T$ such that $G_v^0 = T$, then we can use Proposition 2.13 to show that the slice representation at v is not coreduced by giving an indecomposable relation of the weights of the slice representation which involves coefficients ≥ 2 . We will see that this is a very efficient method to prove non-coreducedness in many cases.

The lemma below follows from [Ric89, Proposition 3.3]. For completeness we give a proof.

Lemma 5.8. *Let G be a semisimple group and let V be a G -module. Then all the roots of \mathfrak{g} are weights of V if and only if there is a zero weight vector $v \in V^T$ whose isotropy group is a finite extension of the maximal torus T of G .*

Proof. Clearly if $(G_v)^0 = T$, then the roots of \mathfrak{g} are weights of V . Conversely, assume all the roots appear and let α be a root of \mathfrak{g} . The weight spaces, with weight a multiple of α , form a submodule of V for the action of the corresponding copy of SL_2 . Since α occurs as a weight of V , this module is not the trivial module. Hence there is a $v \in V^T$ such that $x_\alpha(v) \neq 0$ where $x_\alpha \in \mathfrak{g}$ is a root vector of α . Thus the kernel of x_α is a proper linear subspace of V^T and there is a $v \in V^T$ which is not annihilated by any x_α . Then the isotropy subalgebra of v is \mathfrak{t} . \square

Definition 5.9. We say that a representation V of G has a *toral slice* if there is a $v \in V^T$ such that $G_v^0 = T$. We say that V has a *bad slice* if there is a $v \in V^T$ such that the slice representation at v restricted to G_v^0 is not coreduced, and that V has a *bad toral slice* if, in addition, $G_v^0 = T$.

Now Proposition 5.1 can be paraphrased by saying that *a representation with a bad slice is not coreduced*.

Example 5.10. Consider the representation $(S^{3k}(\mathbb{C}^3), SL_3)$, $k \geq 2$. Then the isotropy group of the zero weight vector is a finite extension of the maximal torus T of SL_3 , and the slice representation W of the torus contains the highest weight $2k\alpha + k\beta$ as well as the weights $-k\alpha$ and $-k\beta$. Thus there is the “bad” relation

$$(2k\alpha + k\beta) + 2(-k\alpha) + (-k\beta) = 0,$$

and so V has a bad toral slice.

Example 5.11. The following representations are not coreduced.

- (1) $G = SL_2 \times SL_2$ on $(\mathfrak{sl}_2 \otimes \mathfrak{sl}_2) \oplus (R_i \otimes R_j)$, $i + j \geq 1$;
- (2) $G = SL_2 \times SL_2 \times SL_2$ on $(\mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathfrak{sl}_2 \otimes \mathfrak{sl}_2)$.

Proof. (1) Let $t \in \mathfrak{sl}_2$ be a nonzero diagonal matrix. The stabilizer of

$$v = t \otimes t \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$$

is $\mathbb{C}^* \times \mathbb{C}^*$. If i is odd, then the slice representation contains the weights $(\pm 2, \pm 2)$ and $(\pm 1, 0)$ or $(\pm 1, \pm 1)$, and so we find the bad relations

$$(2, 2) + (2, -2) + 4(-1, 0) = 0 \quad \text{or} \quad (2, 2) + 2(-1, -1) = 0.$$

The same argument applies if j is odd. If i and j are both even and $i > 0$, then the slice representation contains the weights $(\pm 2, \pm 2)$ and $(\pm 2, 0)$, and so we find the bad relation

$$(2, 2) + (2, -2) + 2(-2, 0) = 0.$$

- (2) The stabilizer of $v = t \otimes t \otimes x + x \otimes t \otimes t$ where $0 \neq x \in \mathbb{C}$ is $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$. The slice representation contains the weights $(\pm 2, \pm 2, 0)$ and $(0, \pm 2, 0)$, and we can proceed as in (1).

□

Example 5.12. Consider the second fundamental representation of Sp_6 : $\varphi_2(\mathbb{C}_3) = \bigwedge_0^2 \mathbb{C}^6 := \bigwedge^2(\mathbb{C}^6)/\mathbb{C}\beta$ where $\beta \in \bigwedge^2(\mathbb{C}^6)$ is the invariant form. It has the isotropy group $\mathrm{Sp}_2 \times \mathrm{Sp}_4$ with slice representation $\bigwedge_0^2 \mathbb{C}^4 + \theta_1$. We claim that $(2 \bigwedge_0^2 \mathbb{C}^6, \mathrm{Sp}_6)$ is not coreduced, although it is cofree ([Sch79]). In fact, the slice representation is $(2 \bigwedge_0^2 \mathbb{C}^4 + \mathbb{C}^2 \otimes \mathbb{C}^4 + \theta_2, \mathrm{Sp}_2 \times \mathrm{Sp}_4)$. Quotienting by Sp_2 we get a hypersurface $F \subset 3 \bigwedge_0^2 \mathbb{C}^4 + \theta_3$ defined by an Sp_4 -invariant function. Now the claim follows from Example 2.7, because $(3 \bigwedge_0^2 \mathbb{C}^4, \mathrm{Sp}_4) = (3\mathbb{C}^5, \mathrm{SO}_5)$ is not coreduced as we will see in Theorem 9.1(4).

Next we want to show that a representation V is not coreduced if the weights contain all roots with multiplicity at least 2. This needs some preparation.

Lemma 5.13. *Let (V, G) and (W, H) be two representations. Let $v \in V$ and $w \in W$ be nonzero zero weight vectors with slice representations $(N_V \oplus \theta_n, G_v)$ and $(N_W \oplus \theta_m, H_w)$ where $N_V^{G_v} = 0$ and $N_W^{H_w} = 0$. Then the slice representation $N_{V \otimes W}$ of $G_v \times H_w$ at $v \otimes w$ contains*

$$(V^{\oplus(m-1)} \oplus N_V, G_v) \oplus (W^{\oplus(n-1)} \oplus N_W, H_w) \oplus ((\mathfrak{g}/\mathfrak{g}_v \oplus N_V) \otimes (\mathfrak{h}/\mathfrak{h}_w \oplus N_W), G_v \times H_w).$$

Proof. The lemma follows from the decomposition $(V, G_v) = (\mathfrak{g}/\mathfrak{g}_v \oplus N \oplus \theta_n)$ and similarly for (W, H_w) , and the fact that

$$T_{v \otimes w}((G \times H)v \otimes w) = \mathfrak{g}(v) \otimes w + v \otimes \mathfrak{h}(w) \subset \mathfrak{g}/\mathfrak{g}_v \otimes \theta_m + \theta_n \otimes \mathfrak{h}/\mathfrak{h}_w. \quad \square$$

Corollary 5.14. (1) *The two slice representations (N_V, G_v) and (N_W, H_w) occur as subrepresentations of the slice representation at $v \otimes w$. In particular, if (V, G) has a bad slice, then so does $(V \otimes W, G \times H)$.*

(2) *The slice representation at $v \otimes w$ contains $N_V \otimes N_W, \mathfrak{g}/\mathfrak{g}_v \otimes \mathfrak{h}/\mathfrak{h}_w, \mathfrak{g}/\mathfrak{g}_v \otimes N_W$ and $N_V \otimes \mathfrak{h}/\mathfrak{h}_w$.*

(3) *If $n > 1$ (resp. $m > 1$), then the slice representation contains a copy of W (resp. V).*

Remark 5.15. Since G_v and H_w have maximal rank, the isotropy group of $v \otimes w$ can at most be a finite extension of $G_v \times H_w$. Note also that the corollary generalizes in an obvious way to a representation of the form

$$(V_1 \otimes V_2 \otimes \cdots \otimes V_k, G_1 \times G_2 \times \cdots \times G_k),$$

where each (V_i, G_i) is a representation with a zero weight.

Proposition 5.16. *Let $G = G_1 \times \cdots \times G_k$ be a product of simple groups and $V = V_1 \otimes \cdots \otimes V_k$ a simple G -module where $k > 1$. Assume that all roots of G occur in V . Then V is coreduced if and only if G is of type $A_1 \times A_1$ and $V = \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$.*

Proof. By Lemma 5.8 the product $T = T_1 \times \cdots \times T_k$ of the maximal tori appears as the connected component of the isotropy group of an element $v_1 \otimes \cdots \otimes v_k \in V^T$ where v_i is a generic element in $V_i^{T_i}$. Denote by $W_i := N_{V_i}$ the slice representation at v_i . Then the tensor products $W_{i_1} \otimes \cdots \otimes W_{i_m}$ where $i_1 < \cdots < i_m$ appear as subrepresentations of the slice representation at v (see Remark 5.15 above).

First, assume that $k > 2$. Choose simple roots α, β, γ of G_1, G_2, G_3 , respectively. Then

$$(\alpha + \beta) + (\beta + \gamma) + (\alpha + \gamma) + 2(-\alpha - \beta - \gamma) = 0$$

is an indecomposable relation with a coefficient > 1 .

Now assume that $k = 2$ and that $\text{rank } G_1 > 1$ and choose two adjacent simple roots α, β of G_1 so that $\alpha + \beta$ is again a root. Let γ be a simple root of G_2 . Then the relation

$$(\alpha + \gamma) + (\alpha - \gamma) + 2(\beta - \gamma) + 2(-(\alpha + \beta) + \gamma) = 0$$

is indecomposable, but contains coefficients > 1 .

As a consequence, G is of type

$$A_1 \times A_1,$$

and $V = R_{2r} \otimes R_{2s}$. Calculating the representation of the maximal torus of $A_1 \times A_1$ at the zero weight vector shows that V is coreduced only for $r = s = 1$. \square

Proposition 5.17. *Let G be a semisimple group and let V be a G -module. Assume that all roots of G are weights of V with multiplicity at least 2. Then V admits a bad toral slice.*

Proof. Choose a generic element v of the zero weight space V^T of V . Then $(G_v)^0 = T$ by Lemma 5.8, and all roots occur in the slice representation W of T at v as well as the highest weights of V . We will show that there is a bad relation.

If not all simple factors of G are of type A , then there is always a root α which expressed in terms of simple roots has some coefficient ≥ 2 : $\alpha = \sum_i n_i \alpha_i$ where

$\{\alpha_1, \dots, \alpha_r\}$ is a set of simple roots of \mathfrak{g} , $n_i \in \mathbb{N}$, and $n_j > 1$ for some j . But then $\alpha + \sum_i n_i(-\alpha_i) = 0$ is a bad relation and thus N is not coreduced.

We may thus assume that G is of type

$$A_{n_1} \times A_{n_2} \times \dots \times A_{n_k},$$

for $n_1 \geq n_2 \geq \dots \geq n_k \geq 1$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots, $n := n_1 + n_2 + \dots + n_k$. We can assume that the highest weights of the irreducible components of V are all of the form $\lambda = \sum_i n_i \alpha_i$ where $n_i \in \{0, 1\}$. It is easy to see that such a weight is dominant if and only if λ is a sum of highest roots. Thus each irreducible component V_k of V is a tensor product of certain \mathfrak{sl}_{n_j} 's. Now it follows from the previous proposition that either V_k is isomorphic to \mathfrak{sl}_j or isomorphic to $\mathfrak{sl}_2 \otimes \mathfrak{sl}_2$. If $n_1 > 1$, then $\mathfrak{sl}_{n_1} \oplus \mathfrak{sl}_{n_1}$ must occur and so V is not coreduced (Example 5.7). The remaining cases where G is of type $A_1 \times A_1 \times \dots \times A_1$ follow immediately from Example 5.11. □

We finish this section with a criterion for the non-coreducedness of an irreducible representation of a simple group. We begin with a lemma about weights and multiplicities. Let U denote a maximal unipotent subgroup of G .

Lemma 5.18. *Let μ, ν be nonzero dominant weights of \mathfrak{g} .*

- (1) *If there is a weight of $V(\mu)$ of multiplicity m , then there are nonzero weights in $V(\mu + \nu)$ with multiplicity $\geq m$.*
- (2) *Suppose that zero is a weight of $V(\mu)$. Then the multiplicities of the nonzero weights of $V(\mu)$ are bounded below by the multiplicities of the (short) roots.*

Proof. Let $v_\mu \in V(\mu), v_\nu \in V(\nu)$ be highest weight vectors. Recall that the coordinate ring $\mathcal{O}(G/U)$ is a domain and contains every irreducible representation of G exactly once. Therefore, the multiplication with v_ν is injective and sends $V(\mu)$ into $V(\mu + \nu)$, i.e., $V(\mu) \otimes \mathbb{C}v_\nu \hookrightarrow V(\mu + \nu)$ as a T -submodule, and we have (1).

For (2), recall that the highest short root is the smallest dominant weight (Remark 5.5). Looking at root strings (see Remark 5.5 and Lemma 5.6) we see that the multiplicity of the highest short root has to be at least that of an arbitrary nonzero weight. □

The following criterion will be constantly used for the classifications in the following sections. Let G be a simple group. We use the notation φ, ψ, \dots for irreducible representations of G and denote by $\varphi\psi$ the Cartan product of φ and ψ .

Criterion 5.19. *Let φ, ψ be irreducible representations of G with a zero weight. Then $\varphi\psi$ has a bad toral slice in the following cases.*

- (i) *φ has a bad toral slice.*
- (ii) *$\varphi\psi$ contains a nonzero weight of multiplicity > 1 .*
- (iii) *The zero weight of φ has multiplicity > 1 .*

Proof. As in the proof above, every nonzero weight vector $w \in \psi$ defines an embedding $\varphi \hookrightarrow \varphi\psi$ which shows that $\varphi\psi$ contains all sums of two (short) roots

and therefore all roots. Thus $\varphi\psi$ has a toral slice. In case (i) we choose for $w \in \psi$ a weight vector of weight 0 and obtain a T -equivariant embedding $\varphi \hookrightarrow \varphi\psi$ which shows that $\varphi\psi$ has a bad toral slice.

In case (ii) the (short) roots occur in $\varphi\psi$ with multiplicity at least 2. Now let α be a short root. Then 2α and α occur in a toral slice representation, and we have the bad relation $(2\alpha) + 2(-\alpha) = 0$.

Finally, (iii) implies (ii) by Lemma 5.18(1). □

Remark 5.20. Let G be of type A, D or E so that all roots have the same length. If $\varphi = \varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_k}$ is a coreduced representation with a zero weight, then either $k = 1$ or all φ_{i_j} are multiplicity-free. In all other cases, φ has a bad toral slice. (If $k > 1$ and if one of the φ_{i_j} has a weight space of multiplicity ≥ 2 , then the roots occur in φ with multiplicity ≥ 2 , by Lemma 5.18, and thus φ is not coreduced, by Proposition 5.17.)

6 Coreduced representations of the exceptional groups

Let G be an exceptional simple group. In this section we classify the coreduced representations V of G which contain a zero weight. We know that each irreducible summand of V is coreduced (Example 2.12(1)). We show that all coreduced representations with a zero weight are contained in *maximal* coreduced representations all of which we determine. The types E_n and G_2 are easy consequences from what we have done so far, but the type F_4 turns out to be quite involved.

Proposition 6.1. *Let G be a simple group of type E and let V be a G -module with a zero weight. If V is coreduced, then V is the adjoint representation of G . Any other V with a zero weight has a bad toral slice.*

Proof. Since the groups of type E are simply laced, every irreducible representation φ with a zero weight contains all roots and thus has a toral slice. Now it follows from Lemma 6.2 below that every representation of the form $\varphi \oplus V$ where V is nontrivial has a bad toral slice. Hence a coreduced representation with a zero weight is irreducible.

- (a) Let $G = E_8$. One can check with LiE that the fundamental representations of G , except for the adjoint representation $\varphi_1(E_8)$, contain the roots with multiplicity ≥ 2 . Since the zero weight of $\varphi_1(E_8)$ has multiplicity ≥ 2 , it follows from Criterion 5.19 that every irreducible representation except for the adjoint representation has a bad toral slice.
- (b) Let $G = E_7$. Of the fundamental representations only $\varphi_1 = \mathfrak{g} = \text{Ad } G$, φ_3 , φ_4 and φ_6 are representations of the adjoint group. Using LiE one shows that every fundamental representation, except for φ_1 and the 56-dimensional representation φ_7 , has a nonzero weight of multiplicity at least 6. Hence, by Remark 5.20, the only other candidates for a coreduced representation besides

φ_1 are φ_7^{2k} , $k \geq 1$. But φ_7^2 , contains the roots with multiplicity 5. Thus every irreducible representation except for the adjoint representation has a bad toral slice.

- (c) Let $G = E_6$. From the fundamental representations only $\varphi_2 = \mathfrak{g}$ and φ_4 are representations of the adjoint group. By LiE, φ_3, φ_4 and φ_5 have nonzero weights of multiplicity at least 5, and $\varphi_1^2, \varphi_1\varphi_6$ and φ_6^2 have nonzero weights of multiplicities at least 4. Thus all irreducible representations with a zero weight, except for the adjoint representation φ_2 , have a bad toral slice. □

Lemma 6.2. *Let G be a simple group of type E , V a G -module and $T \subset G$ a maximal torus. Then V , considered as a representation of T , is not coreduced.*

Proof. We have to show that the weights $\Lambda = \{\lambda_i\}$ of V admit a “bad relation,” i.e., an indecomposable relation $\sum_i n_i \lambda_i = 0$ where $n_i \geq 0$ and at least one $n_j \geq 2$ (Proposition 2.13). This is clear if Λ contains the roots, in particular for all representations of E_8 .

For E_7 we first remark that $\omega_1, \omega_3, \omega_4, \omega_6$ are in the root lattice and $\omega_7 \prec \omega_2, \omega_5$ in the usual partial order. This implies that for every dominant weight λ we have either $\omega_1 \prec \lambda$ or $\omega_7 \prec \lambda$. Thus the weights of V either contain the roots or the Weyl orbit of ω_7 . Using LiE one calculates the Weyl orbit of ω_7 and shows that there is a “bad relation” among these weights.

Similarly, for E_6 one shows that for a dominant weight λ not in the root lattice, one has either $\omega_1 \prec \lambda$ or $\omega_6 \prec \lambda$. Then, using LiE, one calculates the Weyl orbit of ω_1 and shows that there is a “bad relation” among these weights. Since ω_6 is dual to ω_1 its weights also have a “bad relation.” □

We prepare to consider F_4 . The following result will be used several times in connection with slice representations at zero weight vectors.

Lemma 6.3. *Let G be semisimple and let V be a G -module where $V^G = 0$. Let $H \subset G$ be a maximal connected reductive subgroup which fixes a nonzero point $v \in V$. Then $Gv \subset V$ is closed with stabilizer a finite extension of H .*

Proof. Since H is maximal, $N_G(H)/H$ is finite so that Gv is closed [Lun75]. Similarly, G_v can only be a finite extension of H . □

For the maximal subgroups of the simple Lie groups see the works of Dynkin [Dyn52b, Dyn52a].

Example 6.4. Let $V = \varphi_2(\mathbb{C}_n)$, $n \geq 3$. Then $H := \mathbb{C}_1 \times \mathbb{C}_{n-1}$ is a maximal subgroup of \mathbb{C}_n where $(\varphi_1(\mathbb{C}_n), \mathbb{C}_1 \times \mathbb{C}_{n-1}) = \varphi_1(\mathbb{C}_1) \oplus \varphi_1(\mathbb{C}_{n-1})$. Now H fixes a line in V . Thus a finite extension of H (actually H itself) is the stabilizer of a closed orbit, and one easily sees that the slice representation is $\theta_1 + \varphi_2(\mathbb{C}_{n-1})$. By induction one sees that the principal isotropy group of $\varphi_2(\mathbb{C}_n)$ is a product of n copies of SL_2 .

Example 6.5. Let $G = F_4$ which is an adjoint group. Now $\varphi_1 = \mathfrak{g}$ and φ_4 is the irreducible 26-dimensional representation whose nonzero weights are the short roots. The representations φ_2 and φ_3 contain the roots with multiplicities at least two.

Moreover, $\varphi_1^2, \varphi_1\varphi_4$ and φ_4^2 contain the roots with multiplicities at least 3. Hence every irreducible representation of G except for φ_1 and φ_4 has a bad toral slice.

Proposition 6.6. *The representations $\varphi_1(\mathbf{F}_4)$ and $2\varphi_4(\mathbf{F}_4)$ are the maximal coreduced representations of \mathbf{F}_4 . Moreover, the representation $2\varphi_4(\mathbf{F}_4)$ contains a dense orbit in the null cone.*

Proof. The sum $\varphi_1 + \varphi_4$ is not coreduced because the slice representation of the maximal torus is not coreduced (the nonzero weights are the short roots and these contain a bad relation). This leaves us to consider copies of φ_4 . We know that $2\varphi_4$ is cofree ([Sch79]). So it suffices to show that $3\varphi_4$ is not coreduced and that $2\varphi_4$ contains a dense orbit in the null cone. For both statements we use some heavy calculations which are given in [Appendix A](#), see Proposition A.1. \square

Example 6.7. Let $G = \mathbf{G}_2$ which is an adjoint group. The fundamental representation φ_1 of dimension 7 and φ_2 (adjoint representation) are the only coreduced irreducible representations. This follows from Criterion 5.19, because φ_1^2 contains a nonzero weight of multiplicity ≥ 2 .

Proposition 6.8. *Let $G = \mathbf{G}_2$. Then $2\varphi_1$ and the adjoint representation φ_2 are the maximal coreduced representations of G .*

Proof. See [Sch88] for the invariant theory of \mathbf{G}_2 . The invariants of $2\varphi_1$ are just the SO_7 -invariants, so this representation is coreduced (see Theorem 9.1(4)). Now $\bigwedge^3(\varphi_1)$ contains a copy of φ_1 , and it is easy to see that the corresponding covariant vanishes on the null cone of $3\varphi_1$. In fact, this holds for any covariant of type φ_1 of degree ≥ 3 . Since the covariant is alternating of degree 3, it cannot be in the ideal of the quadratic invariants. More precisely, we have $S^2(\varphi_1 \otimes \mathbb{C}^3)^G = \theta_1 \otimes S^2\mathbb{C}^3$, and so in $S^3(\varphi_1 \otimes \mathbb{C}^3)$ we have

$$S^2(\varphi_1 \otimes \mathbb{C}^3)^G \cdot (\varphi_1 \otimes \mathbb{C}^3) = \varphi_1 \otimes (S^2\mathbb{C}^3 \cdot \mathbb{C}^3),$$

and this space does not contain $\varphi_1 \otimes \bigwedge^3 \mathbb{C}^3$. Thus $3\varphi_1$ is not coreduced.

To see that $\varphi_1 + \varphi_2$ is not coreduced we choose a nontrivial zero weight vector in $\varphi_2 = \mathfrak{g}$ which is annihilated by a short root α . Then the isotropy group has rank 2 and semisimple rank 1, and the slice representation contains two copies of (R_2, \mathbf{A}_1) , hence is not coreduced (Theorem 3.7). \square

Let us summarize our results.

Theorem 6.9. *The following are the maximal coreduced representations of the exceptional groups containing a zero weight.*

- (1) For \mathbf{E}_n : the adjoint representations $\varphi_2(\mathbf{E}_6), \varphi_1(\mathbf{E}_7), \varphi_1(\mathbf{E}_8)$.
- (2) For \mathbf{F}_4 : $\mathrm{Ad} \mathbf{F}_4 = \varphi_1(\mathbf{F}_4)$ and $2\varphi_4(\mathbf{F}_4)$.
- (3) For \mathbf{G}_2 : $\mathrm{Ad} \mathbf{G}_2 = \varphi_2(\mathbf{G}_2)$ and $2\varphi_1(\mathbf{G}_2)$.

Remark 6.10. The proofs above and in [Appendix A](#) show that if an irreducible representation (V, G) of an adjoint exceptional group G is not coreduced, then V has a bad slice.

7 Coreduced representations of the classical groups

In this section we classify the coreduced representations V of the *simple adjoint groups* of classical type. If G is adjoint and simply laced, i.e., of type A or D , then a reducible representation V is not coreduced by [Proposition 5.17](#), and so the maximal coreduced representations are all irreducible. We will see that this is also true for G of type $C_n, n \geq 3$, but not for type B_n .

The case of SL_2 has been settled in [Theorem 3.7](#) even without assuming that the center acts trivially. So we may assume that the rank of G is at least 2.

Theorem 7.1. *Let G be a simple classical group of rank at least 2. Then, up to automorphisms, the following representations are the maximal coreduced representations of $G/Z(G)$.*

- (1) $G = A_n, n \geq 2$: $\text{Ad } A_n = \varphi_1\varphi_n, \varphi_2^2(A_3), \varphi_1^3(A_2)$;
- (2) $G = B_n, n \geq 2$: $\text{Ad } B_n = \varphi_2(B_n)$ (φ_2^2 if $n=2$), $\varphi_1^2(B_n), n\varphi_1(B_n)$;
- (3) $G = C_n, n \geq 3$: $\text{Ad } C_n = \varphi_1^2(C_n), \varphi_2(C_n), \varphi_4(C_4)$;
- (4) $G = D_n, n \geq 4$: $\text{Ad } D_n = \varphi_2(D_n), \varphi_1^2(D_n)$.

In [Section 4](#) we showed that every irreducible cofree representation of a simple group is coreduced. Looking at the list above and the one in [Theorem 6.9](#) we see that we have the following partial converse.

Corollary 7.2. *Let G be a simple adjoint group and V an irreducible representation of G . Then V is coreduced if and only if V is cofree.*

We start with type $A_n, n \geq 2$. Recall that $\varphi_p := \bigwedge^p \mathbb{C}^{n+1}, p = 1, \dots, n$.

Lemma 7.3. *Consider the representations φ_p and φ_q of SL_{n+1} where $1 \leq p \leq q \leq n$ and $n \geq 2$. Then there is a nonzero weight of $\varphi_p\varphi_q$ of multiplicity ≥ 2 except in the cases*

- (1) φ_1^2 or φ_n^2 ,
- (2) $\varphi_1\varphi_n$,
- (3) $\varphi_2^2(SL_4)$,

where the zero weight has multiplicity greater than one in (2) and (3).

Proof. It is easy to calculate that the weight $2\varepsilon_1 + \dots + 2\varepsilon_{p-1} + \varepsilon_p + \dots + \varepsilon_{q+1}$ occurs in $\varphi_p \otimes \varphi_q$ with multiplicity $q - p + 2$ and that it occurs in $\varphi_{p-1} \otimes \varphi_{q+1}$ once. Since $\varphi_p \otimes \varphi_q = \varphi_p\varphi_q + \varphi_{p-1} \otimes \varphi_{q+1}$ we see that our weight occurs with multiplicity $q - p + 1$ in $\varphi_p\varphi_q$. This gives us a nonzero weight of multiplicity at least two, except in the following two cases:

- (1) $\varphi_1\varphi_n$ where the above weight is the zero weight, and
- (2) φ_p^2 where $1 \leq p \leq n$.

However, in the second case, we can suppose, by duality, that $2p \leq n + 1$. If $2p \leq n$, then one sees as above that $\varepsilon_1 + \dots + \varepsilon_{2p}$ occurs with multiplicity $\frac{1}{p} \binom{2p}{p-1}$ which is at least 2 as long as $p > 1$. If $2p = n + 1$, then one computes that $\varepsilon_1 - \varepsilon_2 = 2\varepsilon_1 + \varepsilon_3 + \dots + \varepsilon_{2p}$ occurs with multiplicity $\frac{1}{p-1} \binom{2p-2}{p-2}$ which is ≥ 2 as long as $p > 2$. Thus the only possibilities are φ_1^2 and $\varphi_2^2(\text{SL}_4)$. \square

The next lemma was proved by STEMBRIDGE. We give a slightly different version of his proof.

Lemma 7.4. *Let φ be an irreducible representation of PSL_{n+1} , $n \geq 2$. Then the roots of G occur with multiplicity at least two in φ , except in the following cases.*

- (1) The adjoint representation $\varphi_1\varphi_n$;
- (2) $\varphi_1^{k(n+1)}(\text{SL}_{n+1})$ or its dual, $k = 1, 2, \dots$;
- (3) $\varphi_2^2(\text{SL}_4) = \varphi_1^2(\text{D}_3)$.

Proof. The representation φ has highest weight $\lambda = \sum_i \lambda_i \omega_i$ where the ω_i are the fundamental dominant weights and $\sum_i i \lambda_i$ is a multiple of n . Now, Lemma 5.18 together with Lemma 7.3 above implies that the only irreducible representations of PSL_{n+1} containing the roots with multiplicity one are those listed. \square

Proposition 7.5. *Let $n \geq 2$. The nontrivial irreducible coreduced representations of PSL_{n+1} are the adjoint representation $\varphi_1\varphi_n$, $\varphi_2^2(\text{SL}_4)$, $\varphi_1^3(\text{SL}_3)$ and $\varphi_2^3(\text{SL}_3)$. All other irreducible representations admit a bad toral slice.*

Proof. By Proposition 5.17 we know that the only candidates for coreduced irreducible representations of PSL_n are those listed in Lemma 7.4 above. So it remains to show that $S^{km}(\mathbb{C}^m)$ is not coreduced for $m > 3$ and for $m = 3, k > 2$. For $m \geq 4$ the slice representation at a generic fixed point of the maximal torus T contains the weights $\beta_i := km\varepsilon_i$ and the weight $\alpha := -k(2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \dots + \varepsilon_{m-2})$ of the monomial $(x_3 \cdots x_{m-2} x_{m-1}^2 x_m^2)^k$ which satisfy the indecomposable relation $m\alpha + 2\beta_1 + 2\beta_2 + \beta_3 + \dots + \beta_{m-2} = 0$, and so the slice representation is not coreduced.

For $m = 3$ and $k > 1$ we have the weights $\beta_i := 3k\varepsilon_i$ and the weight $\alpha := -3(k-1)\varepsilon_1 - 3(k-2)\varepsilon_2$ of the monomial $x_2^3 x_3^{3(k-1)}$ which satisfy the indecomposable relation $k\alpha + (k-1)\beta_1 + (k-2)\beta_2 = 0$. Again it follows that the slice representation is not coreduced. \square

Now we look at type B_n .

Proposition 7.6. *Let $G = \text{SO}_{2n+1}$ be the adjoint group of type B_n , $n \geq 2$. Then the only nontrivial irreducible coreduced representations are the adjoint representation φ_2 , the standard representation φ_1 and φ_1^2 . All other irreducible representations admit a bad toral slice.*

The representations φ_2 and φ_1^2 are maximal coreduced, whereas $k\varphi_1$ is coreduced if and only if $k \leq n$.

Proof. The highest weights of irreducible representations of G are just sums of the highest weights $\omega_1, \dots, \omega_{n-1}, 2\omega_n$ of the representations $W_\ell := \bigwedge^\ell(\mathbb{C}^{2n+1})$ for $1 \leq \ell \leq n$. For $\ell = 2m + 1$ or $2m$, $m \geq 2$, one can compute that the weights of W_ℓ contain the roots of G with multiplicity $\binom{n-2}{m-1}$ which is at least 2. Thus W_ℓ admits a non-coreduced slice representation of a maximal torus and is therefore not coreduced for $\ell \geq 4$. For $\ell = 3$, hence $n \geq 3$, we have the weights $\pm \varepsilon_i \pm \varepsilon_j \pm \varepsilon_k$ where $1 \leq i < j < k \leq n$ as well as the weights $\pm \varepsilon_i$, $1 \leq i \leq n$ where the latter have multiplicity ≥ 2 . Now the relation

$$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + (-\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2(-\varepsilon_2) + 2(-\varepsilon_3) = 0 \tag{1}$$

is indecomposable and so the slice representation of the maximal torus is not coreduced.

Now let V be an irreducible representation of G with highest weight $\lambda = \sum_i m_i \omega_i$. If $m_i > 0$ for some $i \geq 3$ then, by Criterion 5.19, V has a non-coreduced slice representation of a maximal torus, and thus is not coreduced.

Hence we are left with $\lambda = r\omega_1 + s\omega_2$ where s is even in case $n = 2$. Let us first assume that $n > 2$. Since φ_2^2 contains the roots with multiplicity $\geq \dim W_2^T = n \geq 3$ and since the weights of $\varphi_1\varphi_2$ contain the indecomposable weight relation $(2\varepsilon_1 + \varepsilon_2) + 2(-\varepsilon_1) + (-\varepsilon_2) = 0$ and the short roots occur with multiplicity > 1 , we are reduced to the highest weights $r\omega_1$. If $r \geq 3$, we have the roots and the weights $3\varepsilon_1$ and $-2\varepsilon_1$ which lead one to see that the slice representation is not coreduced.

The arguments in the case $n = 2$ are the same (one has to replace φ_2 by φ_2^2 everywhere).

Finally, we have to look at direct sums of φ_1 , φ_2 and φ_1^2 . We will see in Theorem 9.1(2) that $k\varphi_1$ is coreduced if and only if $k \leq n$. Since φ_2 and φ_1^2 contain all roots it remains to show that $\varphi_1^2 + \varphi_1$ and $\varphi_2 + \varphi_1$ are not coreduced. First, consider φ_1^2 , $n \geq 4$. The subgroup $\text{SO}_3 \times \text{SO}_{2n-2}$ is maximal in SO_{2n+1} , it has rank n and has slice representation $\varphi_1^4(\mathbf{A}_1) \oplus \varphi_1^2(\mathbf{D}_{n-1}) + \theta_1$. If we add a copy of $\varphi_1(\mathbf{B}_n)$, then we have a subrepresentation $(\varphi_1^4 + \varphi_1^2, \mathbf{A}_1)$ which is not coreduced. The details work out similarly for $n = 2$ and $n = 3$. We are left with $\text{Ad } G + \varphi_1$. The slice representation of the group $\text{SO}_3 \times (\text{SO}_2)^{n-1}$ contains two copies of the standard representation of SO_3 on \mathbb{C}^3 which is not coreduced (Theorem 3.7). Hence $\text{Ad } G + \varphi_1$ is not coreduced. \square

For type \mathbf{D}_n we get the following result. Recall that only irreducible representations of PSO_{2n} can be coreduced (Proposition 5.17).

Proposition 7.7. *Let $G = \text{PSO}_{2n}$ be the adjoint group of type \mathbf{D}_n , $n \geq 4$. Then the only nontrivial coreduced representations are the adjoint representation φ_2 , φ_1^2 , $\varphi_3^2(\mathbf{D}_4)$ and $\varphi_4^2(\mathbf{D}_4)$, and these are maximal coreduced. All other irreducible representations admit a bad toral slice.*

Proof. The highest weights of representations of SO_{2n} are just sums of the highest weights $\omega_1, \dots, \omega_{n-2}, \omega_{n-1} + \omega_n$ of the representations $W_\ell := \bigwedge^\ell(\mathbb{C}^{2n})$ for $1 \leq \ell \leq n-1$ and twice the highest weights ω_{n-1} and ω_n of the two half-spin representations. Moreover, $\varphi_{n-1}^2 \oplus \varphi_n^2 \simeq W_n := \bigwedge^n(\mathbb{C}^{2n})$.

The representations W_{2m} for $m > 1$ contain the roots of G with multiplicity $\binom{n-2}{m-1} \geq 2$. The representations W_k for k odd, $k > 1$, have no zero weights, but they contain the weights of φ_1 with multiplicity greater than one. Hence the Cartan products $W_k W_\ell$ for $k, \ell \leq n - 1$ odd, $k + \ell \geq 4$, contain the adjoint representation more than once, so that the representations are not coreduced. We already know that φ_1^2 is coreduced and by Criterion 5.19 no power φ_1^{2k} is coreduced for $k \geq 2$.

It remains to consider those representations φ of PSO_{2n} that are Cartan products with φ_{n-1}^2 or φ_n^2 . If $n \geq 6$ is even, then both contain the roots at least three times, hence φ is not coreduced. If $n = 4$, then φ_3^2 and φ_4^2 are outer isomorphic to φ_1^2 which is coreduced. If φ is not exactly one of these representations, then it is not coreduced by Criterion 5.19. If n is odd, then φ_{n-1}^2 and φ_n^2 both contain the weights of W_1 at least three times, and so φ contains the roots with multiplicity at least 3 and is not coreduced. □

For type C_n we will use the following lemma.

Lemma 7.8. *Let H_1, \dots, H_4 be copies of SL_2 and let $V_i \simeq \mathbb{C}^2$ have the standard action of H_i . Let $H = \prod_i H_i$ and $V = \bigoplus_{i < j} V_{ij}$ where $V_{ij} = V_i \otimes V_j$. Then (V, H) is not coreduced.*

Proof. Consider the subrepresentation $V' := V_{12} \oplus V_{14} \oplus V_{23} \oplus V_{34} \oplus V_{24}$. We have the quotient mapping (by H_1) from $V_{12} \oplus V_{14}$ to $V'_{24} \oplus \theta_2$ where V'_{24} is another copy of V_{24} . The image is a hypersurface F defined by an equation saying that the invariant of $(V'_{24}, H_2 \times H_4)$ is the product of the coordinate functions on θ_2 . By Lemmas 2.6 and 2.8 (see Examples 2.7 and 2.9) the representation $V'_{24} \oplus V_{23} \oplus V_{34} \oplus V_{24} \oplus \theta_2$ of $H_2 \times H_3 \times H_4$ is coreduced if V' is coreduced. Quotienting by the action of H_3 , we similarly obtain a representation $(V'_{24} \oplus V''_{24} \oplus V_{24} \oplus \theta_4, H_2 \times H_4) \simeq (3\mathbb{C}^4 \oplus \theta_4, \text{SO}_4)$ which is not coreduced (Example 3.4). Hence (V', H) and (V, H) are not coreduced. □

The fundamental representations φ_i of C_n are given by $\varphi_1 = \mathbb{C}^{2n}$, $\varphi_2 = \bigwedge^2 \mathbb{C}^{2n} / \mathbb{C}\beta$, and $\varphi_i = \bigwedge^i \mathbb{C}^{2n} / \beta \wedge \bigwedge^{i-2} \mathbb{C}^{2n}$ for $i = 3, \dots, n$ where $\beta \in \bigwedge^2 \mathbb{C}^{2n}$ is the invariant form. They can be realized as the irreducible subspaces $\bigwedge_0^i(\mathbb{C}^{2n}) \subset \bigwedge^i(\mathbb{C}^{2n})$ of highest weight $\omega_i := \varepsilon_1 + \dots + \varepsilon_i$. The generators of the representations of the adjoint group $G = \text{PSp}_{2n}$ are the φ_i for i even and the $\varphi_i \varphi_j$ for i and j odd.

Proposition 7.9. *Let $G = \text{PSp}_{2n}$ be the adjoint group of type C_n , $n \geq 3$. Then the nontrivial irreducible coreduced representations of G are the adjoint representation φ_1^2 , φ_2 and $\varphi_4(\mathbb{C}_4)$, and these are all maximal. Moreover, all other irreducible representations admit a bad slice.*

Proof. (a) First consider the case of $\varphi_i \varphi_j$ where i and j are odd. We may suppose that $j \geq 3$. Then φ_j contains the weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ (it is a dominant weight which is the highest weight of φ_j minus a sum of positive roots). By the action of the Weyl group we have all the weights $\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$. In φ_i (and φ_j) we similarly have all the weights $\pm \varepsilon_k$. Thus $\varphi_i \varphi_j$ contains the roots $2\varepsilon_1$ and $\varepsilon_1 - \varepsilon_2$,

hence all the roots. Moreover, we have the following indecomposable relation of weights in $\varphi_i \varphi_j$ (none of which are roots):

$$(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + (2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) + 2(-\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3) + 2(-\varepsilon_1 - 2\varepsilon_2 - \varepsilon_3) = 0. \tag{2}$$

Hence $\varphi_i \varphi_j$ has a bad toral slice and is therefore not coreduced. The same holds for every Cartan product of $\varphi_i \varphi_j$ with any other representation of G .

Now φ_1^4 is a representation of G , but since φ_1^2 contains the trivial representation n times, φ_1^4 contains the adjoint representation at least n times, hence has a bad toral slice and is not coreduced. Therefore, the adjoint representation φ_1^2 is the only coreduced irreducible representation $\varphi = \varphi_{i_1} \varphi_{i_2} \cdots \varphi_{i_m}$ of G where at least one i_k is odd.

- (b) Now we consider representations φ_{2i} , $2i \leq n$. These representations, one can show as above, contain the short roots of G . But the long roots do not occur. Hence the connected component of the isotropy group at a generic zero weight vector is covered by a product $H := \prod_{j=1}^n H_j$, where each H_j is the copy of SL_2 in G corresponding to the positive long root $2\varepsilon_j$. If $n \geq 5$ and $2i \geq 4$, then the slice representation contains the subrepresentation

$$\bigoplus_{1 \leq j < k \leq n} V_{jk} \quad \text{where} \quad V_{jk} := (\mathbb{C}^2 \otimes \mathbb{C}^2, H_j \times H_k),$$

which is not coreduced (Lemma 7.8). Finally, one easily sees that any product $\varphi_{2i} \varphi_{2j}$ contains all the roots as well as the zero sum of weights given above in equation (2). This includes the case where a factor is φ_2 or φ_4 . Hence the irreducible coreduced representations of G are as claimed.

- (c) It remains to show that the coreduced representations of G are all irreducible. As seen above, the connected component of the isotropy group at a generic zero weight vector of φ_2 is covered by a product $H := \prod_{j=1}^n H_j$, where each H_j is the copy of SL_2 in G corresponding to the positive long root $2\varepsilon_j$. If we add another copy of φ_2 or the adjoint representation φ_1^2 , then the slice representation contains $\bigoplus_{1 \leq j < k \leq n} V_{jk}$ where $V_{jk} := (\mathbb{C}^2 \otimes \mathbb{C}^2, H_j \times H_k)$, which is not coreduced for $n \geq 4$. The same holds if $n = 4$ and we add a copy of $\varphi_4(\mathbb{C}_4)$. This proves the claim for $n \geq 4$, because φ_1^2 and $\varphi_4(\mathbb{C}_4)$ contain all roots. For $\varphi_2(\mathbb{C}_3) + \varphi_1^2(\mathbb{C}_3)$ we have the slice representation of $H = H_1 \times H_2 \times H_3$ on $V_{12} \oplus V_{13} \oplus V_{23} \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \theta_2$ where the H_i are copies of SL_2 and the V_{ij} are as above. Consider the subrepresentation $(V', H_1 \times H_2) := (V_{12} \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2, H_1 \times H_2)$. The principal isotropy group of \mathfrak{h}_1 is $\mathbb{C}^* \times H_2$ where \mathbb{C}^* acts on V_{12} with weights ± 1 . Let \mathfrak{h}'_2 denote a second copy of \mathfrak{h}_2 . Then the quotient of V_{12} by \mathbb{C}^* is a quadratic hypersurface in $\mathfrak{h}'_2 + \theta_1$ which equates the quadratic invariant of \mathfrak{h}'_2 and the square of the coordinate function on θ_1 . Thus, as in Lemma 7.8, the fact that the representation $(\mathfrak{h}_2 + \mathfrak{h}'_2 + \theta_1, H_2)$ is not coreduced (Example 5.12) implies that V' is not coreduced, hence neither is $\varphi_2(\mathbb{C}_3) + \varphi_1^2(\mathbb{C}_3)$. Finally, $2\varphi_2(\mathbb{C}_3)$ is not coreduced as we have seen in Example 5.12. □

Remark 7.10. The proofs above show that if an irreducible representation (V, G) of an adjoint classical group G is not coreduced, then V has a bad slice. We have already seen in the previous section that the same holds for the exceptional groups (Remark 6.10).

8 Irreducible coreduced representations of semisimple groups

In this section we classify the irreducible coreduced representations of adjoint semisimple groups.

Example 8.1. The representation $(\mathbb{C}^{2n+1} \otimes \mathbb{C}^{2m+1}, \text{SO}_{2n+1} \times \text{SO}_{2m+1})$ is the isotropy representation of a symmetric space. (Consider the automorphism θ of $\text{SO}_{2(n+m+1)}$ given by conjugation with $\begin{bmatrix} \text{Id}_{2n+1} & \\ & -\text{Id}_{2m+1} \end{bmatrix}$.) It now follows from [KR71, Theorem 14, p. 758] that this representation is coreduced for all $n, m \geq 1$. Of course, this is also an example of a θ -representation, hence coreduced by Corollary 4.8.

Example 8.2. The representation $(V, G \times H) = (\mathbb{C}^3 \otimes \varphi_1(\mathbf{G}_2), \text{SO}_3 \times \mathbf{G}_2)$ is coreduced. In fact, (V, H) is cofree and the quotient $V//H$ is the SO_3 -module $\varphi_1^4 \oplus \theta_2$ which is cofree and coreduced. Hence $(V, G \times H)$ is cofree too. Now the proper nontrivial slice representations of $(3\varphi_1, \mathbf{G}_2)$ are $(2\mathbb{C}^3 + 2(\mathbb{C}^3)^* + \theta_3, \text{SL}_3)$ (coreduced by Theorem 9.1) and $(2\mathbb{C}^2 + \theta_6, \text{SL}_2)$ (coreduced by Theorem 3.7). Thus every fiber of $\pi: V \rightarrow V//H$ is reduced, except for the zero fiber, which has codimension 7. Thus the null cone of $(V, G \times H)$, which has codimension 4, is reduced off of a subset of V of codimension 7, hence $(V, G \times H)$ is coreduced.

Surprisingly, these two examples are the only irreducible coreduced representations besides those where G is simple.

Theorem 8.3. *The coreduced irreducible representations of a semisimple non-simple adjoint group are*

$$(\varphi_1(\mathbf{B}_n) \otimes \varphi_1(\mathbf{B}_m), \mathbf{B}_n \times \mathbf{B}_m) \quad \text{and} \quad (\varphi_1^2(\mathbf{A}_1) \otimes \varphi_1(\mathbf{G}_2), \mathbf{A}_1 \times \mathbf{G}_2).$$

The proof needs some preparation. We first construct a list of non-coreduced representations which will help to rule out most candidates.

Example 8.4. Let $(V, G) = (\mathbb{C}^n \otimes \mathbb{C}^m + \mathbb{C}^n, \text{SO}_n \times \text{SO}_m)$ where $m, n \geq 2$. We show that V is not coreduced. There are three cases. Recall that

$$(S^2(\mathbb{C}^n) \oplus \mathbb{C}^n, \text{SO}_n)$$

is not coreduced even for $n = 2$.

- (1) $n < m$. Quotienting by the action of SO_m we obtain $S^2(\mathbb{C}^n) \oplus \mathbb{C}^n$ which is not coreduced, hence neither is V (Example 2.9).
- (2) $n = m$. By Example 2.10 the representation V is not coreduced since quotienting by O_m we obtain the non-coreduced representation $S^2(\mathbb{C}^n) \oplus \mathbb{C}^n$.
- (3) $n > m$. We have at most n copies of \mathbb{C}^n , so by Example 2.10 we may quotient by the action of O_n to arrive at the representation

$$S^2(\mathbb{C}^m) \oplus \mathbb{C}^m$$

which is not coreduced. Hence V is not coreduced.

We have seen in Corollary 5.14 that for two representations (V, G) and (W, H) with a zero weight, if (V, G) has a bad slice, then so does $(V \otimes W, G \times H)$. Together with Remarks 6.10 and 7.10 this implies that we need only consider tensor products of the irreducible coreduced representations (V, G) of the simple adjoint groups. They fall into five types.

- (1) $(V, G) = \varphi_1^2(\mathbf{A}_1) = (\mathbb{C}^3, SO_3)$.
- (2) $(V, G) = \varphi_1^4(\mathbf{A}_1)$ or there is a slice representation (W, H) where $H^0 = T$ is a maximal torus of G (of rank at least 2) and W contains weight spaces of roots α, β and $-(\alpha + \beta)$ or W contains θ_2 and weight spaces $\pm\alpha$.
- (3) $(V, G) = \varphi_1(\mathbf{B}_n), n \geq 2$.
- (4) $(V, G) = \varphi_1(\mathbf{G}_2)$.
- (5) $(V, G) = \varphi_4(\mathbf{F}_4), \varphi_4(\mathbf{C}_4),$ or $\varphi_2(\mathbf{C}_n), n \geq 3$.

Note that the representations $\varphi_1^2(\mathbf{D}_n), n \geq 3$ and $\varphi_1^2(\mathbf{B}_n), n \geq 2$, are of type (2) as is the representation $\varphi_1^3(\mathbf{A}_2)$. We consider tensor products of the various types of representations.

Lemma 8.5. *Let (V_1, G_1) be of type (2) and let (V_2, G_2) be of arbitrary type. Then $(V_1 \otimes V_2, G_1 \times G_2)$ has a bad slice.*

Proof. We leave the case that (V_1, G_1) or (V_2, G_2) is $\varphi_1^4(\mathbf{A}_1)$ to the reader. It will be clear from our techniques what to do in this case. Let T_1 be a maximal torus of G_1 fixing v_1 . First, assume that the weights of the slice representation at v_1 contain roots α, β and $-\alpha - \beta$. Let T_2 be a maximal torus of G_2 . Suppose first that $(V_2, G_2) = (\mathbb{C}^3, SO_3)$. Let $v_2 \in V_2$ be a zero weight vector and let γ be a nonzero weight of (V_2, T_2) . Then by Corollary 5.14 the slice representation of $T_1 \times T_2$ at $v_1 \otimes v_2$ contains the weights

$$-\gamma + \alpha, \gamma + \beta, \gamma - \alpha - \beta, \text{ and } -\gamma - \alpha - \beta.$$

We have the minimal zero sum

$$2(-\gamma + \alpha) + 2(\gamma + \beta) + (\gamma - \alpha - \beta) + (-\gamma - \alpha - \beta) = 0,$$

hence the slice representation of $T_1 \times T_2$ is not coreduced. The same argument works in case (V_2, G_2) is of type (2). Now suppose that $(V_2, G_2) = \varphi_1(\mathbf{B}_n)$, $n \geq 2$. Then we have a slice representation of $\mathrm{SO}_{2n} \times T_1$ containing the irreducible components $\mathbb{C}^{2n} \otimes \mathbb{C}_\alpha$, $\mathbb{C}^{2n} \otimes \mathbb{C}_\beta$ and $\mathbb{C}^{2n} \otimes \mathbb{C}_{-\alpha-\beta}$. Quotienting by SO_{2n} we obtain a representation of T_1 with weights

$$2\alpha, 2\beta, \alpha + \beta, -\alpha, -\beta \text{ and } -2\alpha - 2\beta.$$

Hence the slice representation is not coreduced. The same argument works in case (V_2, G_2) is of type (5). For type (4) we get a slice representation of $\mathrm{SL}_3 \times T_1$ containing

$$\mathbb{C}^3 \otimes \mathbb{C}_\alpha, \mathbb{C}^3 \otimes \mathbb{C}_\beta, (\mathbb{C}^3)^* \otimes \mathbb{C}_{-\alpha-\beta}, \text{ and } (\mathbb{C}^3)^* \otimes \mathbb{C}_\alpha,$$

and quotienting by SL_3 we obtain a T_1 -representation with weights $-\beta, -\alpha, 2\alpha$ and $\alpha + \beta$. Hence we have a non-coreduced slice representation.

Finally, assume that the slice representation at v_1 contains θ_2 and weights $\pm\alpha$ and that (V_2, G_2) is of arbitrary type. Let $\pm\gamma$ be nonzero weights of V_2 . Because of the θ_2 , the slice representation at $v_1 \otimes v_2$ contains the weights of V_2 (Corollary 5.14). Hence we have weights $\pm\alpha \pm \gamma$ and $\pm\gamma$. and the minimal bad relation

$$(\alpha + \gamma) + (-\alpha + \gamma) - 2(\gamma) = 0.$$

Thus $(V_1 \otimes V_2, G_1 \times G_2)$ is not coreduced. □

We are left with type (1) and types (3–5).

Lemma 8.6. *Suppose that (V_1, G_1) is of type (1) or (3) or (5) and that (V_2, G_2) is of type (5). Then $(V_1 \otimes V_2, G_1 \times G_2)$ has a bad slice.*

Proof. First assume that (V_1, G_1) is $\varphi_1(\mathbf{B}_n)$, $n \geq 1$ (type (1) or type (3)). If (V_2, G_2) is $\varphi_4(\mathbf{F}_4)$, then there is a (principal) slice representation of \mathbf{D}_4 on θ_2 where $(\varphi_4(\mathbf{F}_4), \mathbf{D}_4) = (\varphi_1 + \varphi_3 + \varphi_4 + \theta_2)$ while (V_1, G_1) has a slice representation of SO_{2n} on θ_1 where $(V_1, \mathrm{SO}_{2n}) = (\mathbb{C}^{2n} + \theta_1, \mathrm{SO}_{2n})$. By Corollary 5.14 there is a subrepresentation of a slice representation of $(V_1 \otimes V_2, G_1 \times G_2)$ which is of the form $(\mathbb{C}^{2n} \otimes \mathbb{C}^8 \oplus \mathbb{C}^{2n}, \mathrm{SO}_{2n} \times \mathrm{SO}_8)$. It follows from Example 8.4 that the slice representation is not coreduced.

If (V_2, G_2) is $\varphi_2(\mathbf{C}_m)$, $m \geq 4$, then there is a slice representation $(W, H) = (\theta_2 + \varphi_2, \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathbf{C}_{m-2})$ where $(\varphi_2(\mathbf{C}_m), H)$ contains

$$(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathrm{SL}_2 \times \mathrm{SL}_2) \simeq (\mathbb{C}^4, \mathrm{SO}_4).$$

There is a non-coreduced subrepresentation of the slice representation of $(V_1 \otimes V_2, G_1 \times G_2)$ of the form $(\mathbb{C}^{2n} \otimes \mathbb{C}^4 \oplus \mathbb{C}^{2n}, \mathrm{SO}_{2n} \times \mathrm{SO}_4)$. The case of $\varphi_2(\mathbf{C}_3)$ is only notationally different and the case of $\varphi_4(\mathbf{C}_4)$ is similar. Finally, if (V_1, G_1) is of type (5), then the same techniques produce a non-coreduced slice representation at a zero weight vector. □

We leave the proof of the following to the reader.

Lemma 8.7. *A tensor product $(V_1 \otimes V_2, G_1 \times G_2)$ has a bad slice if (V_1, G_1) is $\varphi_1(\mathbf{G}_2)$ (type (4)) and (V_2, G_2) is of type (5).*

We are now left with the problem of tensor products of representations of types (1), (3) and (4). First, we handle types (1) and (3).

Proposition 8.8. *Let $3 \leq 2k + 1 \leq 2m + 1 \leq 2n + 1$ and*

$$(V, G) = (\mathbb{C}^{2n+1} \otimes \mathbb{C}^{2m+1} \otimes \mathbb{C}^{2k+1}, \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m+1} \times \mathrm{SO}_{2k+1}).$$

Then the slice representation at the zero weight vector is not coreduced.

Proof. The slice representation at the zero weight vector is

$$(W, H) = (\mathbb{C}^{2n} \otimes \mathbb{C}^{2m} \otimes \mathbb{C}^{2k} + \mathbb{C}^{2n} \otimes \mathbb{C}^{2m} + \mathbb{C}^{2m} \otimes \mathbb{C}^{2k} + \mathbb{C}^{2n} \otimes \mathbb{C}^{2k}, \mathrm{SO}_{2n} \times \mathrm{SO}_{2m} \times \mathrm{SO}_{2k}).$$

If $k > 1$, consider the subrepresentation $\mathbb{C}^{2m} \otimes \mathbb{C}^{2k} \oplus \mathbb{C}^{2n} \otimes \mathbb{C}^{2k}$. Quotienting by $\mathrm{SO}_{2m} \times \mathrm{SO}_{2n}$ we get $(2S^2(\mathbb{C}^{2k}), \mathrm{SO}_{2k})$ which is not coreduced. Using Example 2.9 we see that (V, G) is not coreduced.

Now assume that $k = 1$ but $m > 1$. We have a subrepresentation

$$\mathbb{C}^{2n} \otimes \mathbb{C}^{2m} \oplus \mathbb{C}^{2m} \otimes \mathbb{C}_\nu \oplus \mathbb{C}^{2m} \otimes \mathbb{C}_{-\nu},$$

where the $\mathbb{C}_{\pm\nu}$ are irreducible representations of $\mathrm{SO}_{2k} \simeq \mathbb{C}^*$ of weight ± 1 . Quotienting by O_{2n} we obtain the representation

$$(S^2(\mathbb{C}^{2m}) \oplus \mathbb{C}^{2m} \otimes \mathbb{C}_\nu \oplus \mathbb{C}^{2m} \otimes \mathbb{C}_{-\nu}, \mathrm{SO}_{2m} \times \mathrm{SO}_2).$$

Let $\pm\varepsilon_1, \dots, \pm\varepsilon_m$ be the weights of \mathbb{C}^{2m} for the action of the maximal torus T of SO_{2m} . Then the slice representation of $S^2(\mathbb{C}^{2m})$ at a generic zero weight vector is, up to trivial factors, the sum of the $\mathbb{C}_{\pm 2\varepsilon_i}$. Hence we have a slice representation of $T \times \mathrm{SO}_2$ containing

$$\mathbb{C}_{-2\varepsilon_1} \oplus \mathbb{C}_{-2\varepsilon_2} \oplus (\mathbb{C}_{\varepsilon_1} \otimes \mathbb{C}_\nu) \oplus (\mathbb{C}_{\varepsilon_2} \otimes \mathbb{C}_{-\nu}).$$

This last representation is not coreduced.

Now assume that $n \geq m = k = 1$. We rename the weight ε_1 of $\mathrm{SO}_{2m} = \mathrm{SO}_2$ to be just ε . Then we have the subrepresentation

$$(\mathbb{C}^{2n} \otimes \mathbb{C}_\varepsilon) \oplus (\mathbb{C}^{2n} \otimes \mathbb{C}_\nu) \oplus (\mathbb{C}_{-\varepsilon} \otimes \mathbb{C}_{-\nu}).$$

Quotienting by O_{2n} we get a representation

$$\mathbb{C}_{2\varepsilon} \oplus \mathbb{C}_{2\nu} \oplus (\mathbb{C}_\varepsilon \otimes \mathbb{C}_\nu) \oplus (\mathbb{C}_{-\varepsilon} \otimes \mathbb{C}_{-\nu})$$

of $\mathbb{C}^* \times \mathbb{C}^*$ which is not coreduced. □

Proposition 8.9. *Let $(V, G) = (\mathbb{C}^{2n+1} \otimes \mathbb{C}^7, \mathrm{SO}_{2n+1} \times \mathbf{G}_2)$, $n \geq 2$, or*

$$(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^7, \mathrm{SO}_3 \times \mathrm{SO}_3 \times \mathbf{G}_2).$$

Then (V, G) has a bad slice.

Proof. We leave the latter case to the reader. In the former case we have the slice representation (minus the trivial factor)

$$(W, H) = (\mathbb{C}^{2n} \otimes (\mathbb{C}^3 \oplus (\mathbb{C}^3)^*), \mathrm{SO}_{2n} \times \mathrm{SL}_3).$$

If $n \geq 3$, then quotienting by O_{2n} we obtain the representation

$$(S^2(\mathbb{C}^3) \oplus S^2(\mathbb{C}^{3*}) \oplus \mathbb{C}^3 \otimes \mathbb{C}^{3*}, \mathrm{SL}_3)$$

which is not coreduced.

We are left with the case $(W, H) = (\mathbb{C}^4 \otimes (\mathbb{C}^3 \oplus \mathbb{C}^{3*}), \mathrm{SO}_4 \times \mathrm{SL}_3)$. Consider a 1-parameter subgroup ρ of $\mathrm{SO}_4 \times \mathrm{SL}_3$ whose action on \mathbb{C}^4 has weights ± 1 and on \mathbb{C}^3 has weights $2, 0, -2$. Then Z_ρ , the span of the positive weight vectors, has dimension 12 (which is not surprising since (W, H) is self-dual of dimension 24). Note that Z_ρ is in the null cone and is stable under a Borel subgroup B of H . Now one can show that the dimension of $U^- Z_\rho$ is $17 = 12 + \dim U^-$, the maximal possible, where U^- is the maximal unipotent subgroup of H opposite B . Hence $H Z_\rho$ is a component of the null cone (see section 10 for more details).

The positive weights of ρ on W are 1 and 3 and the negative weights are -1 and -3 . This implies that the differential of an invariant of degree > 4 vanishes on Z_ρ , hence on $H Z_\rho$. But we have only 4 generating invariants in degree at most 4, and so the null cone is not reduced along $H Z_\rho$, because $\mathrm{codim} H Z_\rho = 7$. \square

We are left with the case $\mathbf{G}_2 \times \mathbf{G}_2$ acting on $\mathbb{C}^7 \otimes \mathbb{C}^7$.

Proposition 8.10. *The representation $(\mathbb{C}^7 \otimes \mathbb{C}^7, \mathbf{G}_2 \times \mathbf{G}_2)$ is not coreduced.*

We have two proofs of this, and both need some computations. They are given in [Appendix B](#).

9 Classical invariants

Classical Invariant Theory describes the invariants of copies of the standard representations of the classical groups, e.g., the $\mathrm{GL}(V)$ - or $\mathrm{SL}(V)$ -invariants of $mV \oplus nV^*$ or the $\mathrm{Sp}(V)$ -invariants of mV where $mV := V^{\oplus m}$ denotes the direct sum of m copies of V . In this context we will prove the following theorem.

Theorem 9.1. (1) *The representation $(pV \oplus qV^*, \mathrm{GL}(V))$ is coreduced for all $p, q \geq 0$. The null cone is irreducible if and only if $p + q \leq n$.*

- (2) The representation $(pV \oplus qV^*, \text{SL}(V))$ is coreduced for all $p, q \geq 0$. The null cone is irreducible in the following cases: $p + q \leq n$ or $(p, q) = (n, 1)$ or $(p, q) = (1, n)$.
- (3) The representations $(mV, \text{Sp}(V))$ are coreduced for all $m \geq 0$, and the null cone is irreducible and normal.
- (4) The representations $(mV, \text{O}(V))$, $(mV, \text{SO}(V))$ are coreduced if and only if $2m \leq \dim V$. The null cone is irreducible and normal for $2m < \dim V$.

The basic reference for this section is [Sch87]. Denote by T_m , B_m and U_m the subgroups of GL_m consisting of diagonal, upper triangular, and upper triangular unipotent matrices. If λ is a dominant weight, i.e., $\lambda = \sum_{i=1}^m \lambda_i \varepsilon_i \in X(T_n) = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$, we denote by ψ_λ or $\psi_\lambda(m)$ the corresponding irreducible representation of GL_m . In the following, we will only deal with polynomial representations of GL_m , so that $\lambda_i \geq 0$ for all i . Set $|\lambda| := \sum \lambda_i$ and define the height of a dominant weight by $\text{ht}(\lambda) := \max\{i \mid \lambda_i > 0\}$.

The famous CAUCHY formula describes the decomposition of the symmetric powers of a tensor product where we consider $\psi_\lambda(m) \otimes \psi_\mu(k)$ as a representation of $\text{GL}_m \times \text{GL}_k$ (see [Sch87, (1.9) Theorem]).

Proposition 9.2.

$$S^d(\mathbb{C}^m \otimes \mathbb{C}^k) = \bigoplus_{|\lambda|=d, \text{ht}(\lambda) \leq \min\{m, k\}} \psi_\lambda(m) \otimes \psi_\lambda(k).$$

If λ is a dominant weight of height r , then $\psi_\lambda(m)$ makes sense for any $m \geq r$. In fact, ψ_λ is a functor and $\psi_\lambda(V)$ is a well-defined $\text{GL}(V)$ -module for every vector space V of dimension $\geq r$. In particular, if $\rho: G \rightarrow \text{GL}(V)$ is a representation of a reductive group G , then all $\psi_\lambda(V)$ for $\text{ht}(\lambda) \leq \dim V$ are representations of G as well. From the CAUCHY formula we thus get

$$\mathcal{O}(mV)_d = S^d(\mathbb{C}^m \otimes V^*) = \bigoplus_{|\lambda|=d, \text{ht}(\lambda) \leq \min(m, \dim V)} \psi_\lambda(m) \otimes \psi_\lambda(V^*)$$

as a representation of $\text{GL}_m \times G$. Taking U_m -invariants we find

$$\mathcal{O}(mV)_d^{U_m} = S^d(\mathbb{C}^m \otimes V^*)^{U_m} = \bigoplus_{|\lambda|=d, \text{ht}(\lambda) \leq \min(m, \dim V)} \psi_\lambda(V^*), \tag{*}$$

where the torus $T_m \subset \text{GL}_m$ acts on $\psi_\lambda(V^*)$ with weight λ . Thus the algebra $\mathcal{O}(mV)^{U_m}$ is \mathbb{Z}^m -graded, and the homogeneous component of weight λ is the G -module $\psi_\lambda(V^*)$. In particular, $\mathcal{O}(mV)^{U_m}$ is multiplicity-free as a $\text{GL}(V)$ -module. It follows that the product $\psi_\lambda(V^*) \cdot \psi_\mu(V^*)$ in $\mathcal{O}(mV)$ is equal to $\psi_{\lambda+\mu}(V^*)$. This leads to the following definition.

Definition 9.3. Let G be a connected reductive group and let A be a G -algebra, i.e., a commutative \mathbb{C} -algebra with a locally finite and rational action of G by algebra

automorphisms. Two simple submodules $U, V \subset A$ are called *orthogonal* if the product $U \cdot V \subset A$ is either zero or simple and isomorphic to the Cartan (highest weight) component of $U \otimes V$.

The result above can therefore be expressed by saying that all irreducible $GL(V)$ -submodules of $\mathcal{O}(mV)^{U_m}$ are orthogonal to each other. The following crucial result is due to BRION [Bri85, Lemme 4.1].

Proposition 9.4. *Let A be a G -algebra and let $V_1, V_2, W \subset A$ be simple submodules. Assume that V_1, V_2 are both orthogonal to W . Then any simple factor of $V_1 \cdot V_2$ is orthogonal to W .*

We will also need the following result about U -invariants (see [Kra84, III.3.3]).

Proposition 9.5. *Let G be a connected reductive group, $U \subset G$ a maximal unipotent subgroup, and let A be a finitely generated G -algebra. Then A is reduced, resp. a domain, resp. normal if and only if A^U is reduced, resp. a domain, resp. normal.*

Another consequence of formula (*) is that $\mathcal{O}(mV)^{U_m} = \mathcal{O}(nV)^{U_n}$ for all $m \geq n = \dim V$.

We start with the groups $GL(V)$ and $SL(V)$ acting on $W := pV \oplus qV^*$. It is known that the $GL(V)$ -invariants are generated by the bilinear forms

$$f_{ij}: (v_1, \dots, v_p, \xi_1, \dots, \xi_q) \mapsto \xi_j(v_i).$$

If V_i^* is the i th copy of V^* in $W^* \subset \mathcal{O}(W)$ and V_j the j th copy of V , then $V_i^* \cdot V_j = \mathfrak{sl}(V) \oplus \mathbb{C}f_{ij}$ in $\mathcal{O}(W)$, and so V_i^* and V_j are orthogonal in $\mathcal{O}(W)/I$ where I is the ideal generated by the invariants f_{ij} . It follows from Proposition 9.4 above that all simple submodules of $\mathcal{O}(pV)$ are orthogonal to all simple submodules in $\mathcal{O}(qV^*)$ modulo I . Thus the $GL(V)$ -homomorphism

$$\mathcal{O}(pV)^{U_p} \otimes \mathcal{O}(qV^*)^{U_q} \rightarrow (\mathcal{O}(pV \oplus qV^*)/I)^{U_p \times U_q}$$

is surjective, and the same holds if we take invariants under $U := U_p \times U_V \times U_q \subset GL_p \times GL(V) \times GL_q$ where $U_V \subset GL(V)$ is a maximal unipotent subgroup. This also shows that the $(U_p \times U_q)$ -invariants do not change once $p \geq n$ or $q \geq n$, so that we can assume that $p, q \leq n$.

Now we have $\mathcal{O}(pV)^U = \mathbb{C}[x_1, \dots, x_p]$ where $x_i \in \wedge^i V^*$ is a highest weight vector. Similarly, $\mathcal{O}(qV^*)^U = \mathbb{C}[y_1, \dots, y_q]$, and thus we get a surjective homomorphism

$$\varphi: \mathbb{C}[x_1, \dots, x_p, y_1, \dots, y_q] \rightarrow (\mathcal{O}(pV \oplus qV^*)/I)^U. \tag{**}$$

Proof (of both Theorem 9.1(1) and (2)). We claim that the kernel of φ is generated by the products $x_r y_s$ where $r + s > n$. This implies that we have an isomorphism

$$(\mathcal{O}(pV \oplus qV^*)/I)^U \simeq \mathbb{C}[x_1, \dots, x_p, y_1, \dots, y_q]/(x_i y_j \mid i + j > n),$$

and so $\mathcal{N}_{p,q} := \mathcal{N}(pV \oplus qV^*)$ is reduced, by Proposition 9.5. We also see that the ideal $(x_i y_j \mid i + j > n)$ is prime if and only if it is (0) , i.e., when $p + q \leq n$. This proves the theorem for $\text{GL}(V)$.

To prove the claim we first remark that the kernel of φ is spanned by monomials, because φ is equivariant under the action of the maximal torus $T_p \times T_q$. Moreover, it is not difficult to see that $\varphi(x_r y_s) = 0$ if $r + s > n$; see [Sch87, Remark 1.23(2)].

Now let $f := x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q}$ be a monomial which is not in the ideal $(x_i y_j \mid i + j > n)$. Then $r + s \leq n$ where $r := \max(p_i)$ and $s := \max(q_j)$. If (e_1, \dots, e_n) is a basis of V and (e_1^*, \dots, e_n^*) the dual basis of V^* , then we can assume that $x_i = e_{n-i+1}^* \wedge e_{n-i+2}^* \wedge \cdots \wedge e_n^*$ and $y_j = e_1 \wedge e_2 \wedge \cdots \wedge e_j$. Now it is clear that the monomial f does not vanish at the point $w := (0, \dots, 0, e_{n-r+1}, \dots, e_n, e_1^*, \dots, e_s^*, 0, \dots, 0)$ which is in the null cone $\mathcal{N}_{p,q}$.

For the group $\text{SL}(V)$ there are more invariants, namely the determinants

$$d_{i_1 \dots i_n} := \det \begin{bmatrix} v_{i_1} \\ \vdots \\ v_{i_n} \end{bmatrix} \text{ where } i_1 < i_2 < \cdots < i_n, \text{ and } d_{j_1 \dots j_n}^* := \det \begin{bmatrix} \xi_{j_1} \\ \vdots \\ \xi_{j_n} \end{bmatrix} \text{ where } j_1 < j_2 < \cdots < j_n.$$

These invariants only appear if $p \geq n$, resp. $q \geq n$. In particular, we have the same invariants and the same null cone in case $p, q < n$. From the surjectivity of the map φ in $(**)$ above we see that there remain only the cases where either $p = n$ and $q \leq n$, or $q = n$ and $p \leq n$. Let J denote the ideal generated by the $\text{SL}(V)$ -invariants. Then $J^U = I^U + (x_n)$ if $p = n > q$, $J^U = I^U + (y_n)$ if $p < n = q$, and $J^U = I^U + (x_n, y_n)$ if $p = n = q$. Hence

$$(\mathcal{O}(pV \oplus qV^*)/J)^U \simeq \mathbb{C}[x_1, \dots, x_{p'}, y_1, \dots, y_{q'}]/(x_i y_j \mid i + j > n)$$

where $p' := \min(p, n - 1)$ and $q' := \min(q, n - 1)$. The rest of the proof is as above. □

Next we study the case where V is a symplectic space, i.e., V is equipped with a nondegenerate skew form β , $\dim V = 2n$. We have the group $G := \text{Sp}(V) \subset \text{GL}(V)$ which preserves β . Then the invariants of mV are generated by the bilinear maps

$$\beta_{ij}: (v_1, \dots, v_n) \mapsto \beta(v_i, v_j), \quad 1 \leq i < j \leq m.$$

We denote by $\psi_k := \bigwedge_0^k V^* \subset \bigwedge^k V^*$ ($k = 1, \dots, n$) the fundamental representations of $\text{Sp}(V)$ where $\bigwedge^k V^* = \bigwedge_0^k V^* \oplus \beta \wedge \bigwedge^{k-2} V^*$. We know from equation $(*)$ that $\mathcal{O}(mV)_k^{U_m}$ contains a unique copy of $\bigwedge^k V^*$ for $k \leq \min(m, n)$.

Lemma 9.6. *Let $I \subset \mathcal{O}(mV)$ be the ideal generated by the invariants β_{ij} . Then in $\mathcal{O}(mV)^{U_m}$ we have*

- (1) $\bigwedge^k V^* = \psi_k \pmod{I}$ for $k = 1, \dots, \min(m, n)$;
- (2) $\bigwedge^k V^* \cdot \bigwedge^\ell V^* = \psi_k \psi_\ell \pmod{I}$ for $1 \leq k \leq \ell \leq \min(m, n)$.

Proof. Part (1) is clear since $\psi_{k+2} = \bigwedge^{k+2} V^* / \beta \bigwedge^k V^*$. For part (2) let $x_1, \dots, x_n \in V^*$ correspond to the positive weights $\varepsilon_1, \dots, \varepsilon_n$ and let y_1, \dots, y_n correspond to the $-\varepsilon_j$. A simple submodule occurring in $\psi_k \cdot \psi_\ell$ has a highest weight vector containing a unique term $\gamma := x_1 \wedge \dots \wedge x_k \cdot \alpha$ where α is an ℓ -fold wedge product of a certain number of x_i and y_j . But the only possibility for obtaining a highest weight of $\text{Sp}(V)$ is $\alpha = x_1 \wedge \dots \wedge x_q \wedge y_{k-r+1} \wedge \dots \wedge y_k$ where $q \leq \ell$ and $r = \ell - q$. This gives the highest weight of a (unique) copy of $\psi_p \psi_q$ where $p = k - r$.

Suppose that $r > 0$. We have an element β_r in $(\bigwedge^r(V^*) \otimes \bigwedge^r(V^*))^{\text{Sp}(V)}$ where $\beta_r(v_1 \wedge \dots \wedge v_r, w_1 \wedge \dots \wedge w_r) = \det(\beta(v_i, w_j))$. Here the v_i and w_j are elements of V . It is easy to see that β_r projects to a nontrivial invariant element β'_r of $\psi_r \psi_r$, and that $\beta'_r \in I^r$. Then the product of β'_r with $\psi_p \psi_q \subset \psi_p \cdot \psi_q$ is a copy of $\psi_p \psi_q$ in $\psi_k \psi_\ell$, and we have (2). \square

Proof (of Theorem 9.1(3)). It follows from the lemma above and Proposition 9.4 that all simple submodules in $\mathcal{O}(mV)^{U_m}$ are orthogonal and the covariants are generated by $\psi_1, \dots, \psi_{m'}$ where $m' := \min(m, n)$. Let $U_V \subset \text{Sp}(V)$ be a maximal unipotent subgroup and let $x_k \in \bigwedge_0^k V^* \subset \mathcal{O}(mV)_k^{U_m}$ be a highest weight vector. Then we have a surjective homomorphism

$$\varphi: \mathbb{C}[x_1, \dots, x_{m'}] \rightarrow (\mathcal{O}(mV)/I)^{U_m \times U_V}.$$

If $W \subset V$ is a maximal isotropic subspace, then $W^{\oplus m}$ is contained in the null cone of mV , and for a suitable choice of W , the function x_k does not vanish on $W^{\oplus m}$ for $k \leq m'$. This implies that φ is an isomorphism, because the action of T_m on $\mathbb{C}[x_1, \dots, x_{m'}]$ has one-dimensional weight spaces, and so the kernel of φ is linearly spanned by monomials. Now the theorem for Sp_n follows from Proposition 9.5. \square

Finally, let V be a quadratic space, i.e., an n -dimensional vector space with a nondegenerate quadratic form q . The $\text{O}(V)$ -invariants of mV are generated by the bilinear maps

$$q_{ij}: (v_1, \dots, v_m) \mapsto q(v_i, v_j), \quad 1 \leq i \leq j \leq m.$$

The $\text{SO}(V)$ -modules $\psi_k := \bigwedge^k V^*$ are simple if $2k < n$. For $n = 2m$ $\psi_m := \bigwedge^m V^*$ is simple as an $\text{O}(V)$ -module, but decomposes as $\psi_m = \psi_m^+ \oplus \psi_m^-$ as an $\text{SO}(V)$ -module.

Lemma 9.7. *Let $2m \leq n$ and let $I \subset \mathcal{O}(mV)$ be the ideal generated by the invariants q_{ij} . Then in $\mathcal{O}(mV)^{U_m}$ we have*

- (1) $\psi_k \cdot \psi_\ell = \psi_k \psi_\ell \pmod{I}$ for $1 \leq k \leq \ell \leq \min(m, \frac{n-1}{2})$;
- (2) If $n = 2m$, then $\psi_m^+ \cdot \psi_m^- = 0 \pmod{I}$.

Proof. Let $n = 2s$ or $2s + 1$ so that $m \leq s$. We consider a weight basis x_1, \dots, x_s and y_1, \dots, y_s (and a zero weight element z if n is odd). First suppose that n is even. For (1) we can then proceed as in the symplectic case. The only difference is that we

use the invariant bilinear form q to generate an element q'_r lying in $(\psi_r \psi_r)^{\text{SO}(V)}$ and in I^r . As for (2), the highest weight vectors are $x_1 \wedge \cdots \wedge x_m$ and $x_1 \wedge \cdots \wedge x_{m-1} \wedge y_m$. Their product is the image of $q'_1 \otimes \psi_{m-1} \psi_{m-1}$ in $\psi_m^+ \psi_m^-$. The argument of (1) shows that any other irreducible occurring in $\psi_m^+ \cdot \psi_m^-$ also lies in I .

Now suppose that n is odd. Then the argument for (1) above goes through except when the zero weight vector appears in the expression for α . So suppose that $\alpha = x_1 \wedge \cdots \wedge x_{\ell-1} \wedge z$. Then

$$x_1 \wedge \cdots \wedge x_k \cdot \alpha + (x_1 \wedge \cdots \wedge x_{\ell-1} \wedge x_{\ell+1} \cdots \wedge x_k \wedge z) \cdot (x_1 \wedge \cdots \wedge x_\ell)$$

is a vector in $\psi_k \cdot \psi_\ell$. It is obtained from $(x_1 \wedge \cdots \wedge x_k) \cdot (x_1 \wedge \cdots \wedge x_\ell)$ by applying elements of U^- . Hence we don't have a new irreducible component of $\psi_k \cdot \psi_\ell$. \square

Proof (of Theorem 9.1(4)). Choose highest weight vectors $x_k \in \bigwedge^k V^*$ for $2k < n$ and $x_m^+ \in \psi_m^+, x_m^- \in \psi_m^-$ for $n = 2m$. The preceding lemma and Proposition 9.4 show that the induced maps

$$\mathbb{C}[x_1, \dots, x_m] \rightarrow (\mathcal{O}(mV)/I)^{U_m \times U_V} \text{ for } 2m < n, \text{ and}$$

$$\mathbb{C}[x_1, \dots, x_m^+, x_m^-]/(x_m^+ x_m^-) \rightarrow (\mathcal{O}(mV)/I)^{U_m \times U_V} \text{ for } 2m = n$$

are surjective. The weights $\nu(x_k)$ of the highest weight vectors (with respect to $T_m \times T_V$, T_V a maximal torus of $\text{SO}(V)$) are linearly independent, except that in case $n = 2m$ we have $\nu(x_m^+) + \nu(x_m^-) = 2\nu(x_{m-1})$. It follows that the algebras on the left-hand side are multiplicity free, and so the kernels of the two maps are spanned by monomials. But it is easy to see that none of the x_k, x_m^\pm vanish on the null cone, and so the two maps are isomorphisms. Again using Proposition 9.5 we obtain the theorem for the groups $\text{O}(V)$ and $\text{SO}(V)$ in the case where $2m \leq n$.

It remains to show that the null cone is not reduced for $2m > n$. Let $n = 2k$. (The case $n = 2k - 1$ is similar and will be left to the reader.) We may take $m = k + 1$. Then in degree $k + 1$ we find the submodule $M := \bigwedge^{k+1} \mathbb{C}^{k+1} \otimes \bigwedge^{k+1} V^*$, by CAUCHY'S formula (Proposition 9.2). The $\text{SO}(V)$ -module $\bigwedge^{k+1} V^*$ is simple and isomorphic to $\psi_{k-1} = \bigwedge^{k-1} V$. We claim that M vanishes on the null cone \mathcal{N} , but is not contained in the ideal I generated by the invariants.

The first part is clear, because $\mathcal{N} = \text{O}(V) \cdot (k + 1)W$ where $W \subset V$ is a maximal isotropic subspace, and every function $f_1 \wedge \cdots \wedge f_{k+1}$ vanishes on $(k + 1)W$ because $\dim W = k$.

For the second part, we remark that the module ψ_{k-1} appears the first time in degree $k - 1$, in the form $\bigwedge^{k-1} \mathbb{C}^{k+1} \otimes \bigwedge^{k-1} V^*$. If $M \subset I$, then M must belong to the product

$$\mathcal{O}((k + 1)V)_2^{\text{SO}(V)} \cdot (\bigwedge^{k-1} \mathbb{C}^{k+1} \otimes \bigwedge^{k-1} V^*)$$

which is a quotient of $(S^2(\mathbb{C}^{k+1}) \otimes \bigwedge^{k-1} \mathbb{C}^{k+1}) \otimes \bigwedge^{k-1} V^*$. But the tensor product $S^2(\mathbb{C}^{k+1}) \otimes \bigwedge^{k-1} \mathbb{C}^{k+1}$ does not contain the "determinant" $\bigwedge^{k+1} \mathbb{C}^{k+1}$ as a GL_{k+1} -module. \square

10 Non-reduced components of the null cone

We need some information about null cones (see [Hes80, KW06, Ric89] for more details). Let G be a connected reductive complex group, $T \subset G$ a maximal torus and V a G -module. Let $X(T) = \text{Hom}(T, \mathbb{C}^*)$ denote the character group of T and let $Y(T) = \text{Hom}(\mathbb{C}^*, T)$ denote the group of 1-parameter subgroups of T . Then $Y(T)$ and $X(T)$ are dually paired: $\langle \rho, \mu \rangle = n$ if $\mu(\rho(t)) = t^n$. For any $\rho \in Y(T)$ we set

$$Z_\rho := \{v \in V \mid \lim_{t \rightarrow 0} \rho(t)v = 0\} = \bigoplus_{\mu \in X(T), \langle \rho, \mu \rangle > 0} V_\mu$$

where $V_\mu \subset V$ denotes the weight space of weight μ . These Z_ρ are called *positive weight spaces*. Then the Hilbert–Mumford theorem says that \mathcal{N} is the union of the sets GZ_ρ , $\rho \in Y(T)$. In fact, one needs only a finite number of elements of $Y(T)$. Pick a system of simple roots for G . Then using the action of the Weyl group, we can assume that any given ρ is positive when paired with the simple roots $\alpha_1, \dots, \alpha_\ell \in X(T)$, $\ell = \dim T$. In fact, we can always assume that the pairings are strictly positive and that ρ only takes the value 0 on the zero weight. We call such elements of $Y(T)$ *generic*. Now Z_ρ is stable under the action of the Borel B , thus GZ_ρ is closed in the Zariski topology, and GZ_ρ is irreducible. Thus there are finitely many generic ρ_i such that the sets GZ_{ρ_i} are the irreducible components of \mathcal{N} .

Remark 10.1. We will use this description of the null cone to show that a given homogeneous covariant $\tau: V \rightarrow W$ of degree d vanishes on the null cone, generalizing Lemma 3.8. It suffices to show that τ vanishes on Z_ρ for the relevant generic ρ 's. Denote by μ_1, \dots, μ_m the weights of Z_ρ . If $\tau \neq 0$ on Z_ρ , then the highest weight μ of W is of the form $\sum_i d_i \mu_i$ where $\sum_i d_i = d$. (This follows from the B -equivariance of τ .) Hence τ vanishes if μ cannot be expressed as such a sum.

Let $\Lambda(V)$ denote the set of weights of V . For $\rho \in Y(T)$, let Λ_ρ denote the subset of $\Lambda(V)$ of elements which pair strictly positively with ρ . A subset $\Lambda \subset \Lambda(V)$ is called *admissible* if $\Lambda = \Lambda_\rho$ for a generic ρ . In this case set $Z_\Lambda := Z_\rho$. We will often switch between looking at generic elements of $Y(T)$ (or $Y(T) \otimes \mathbb{Q}$) and corresponding subsets $\Lambda \subset \Lambda(V)$. We say that an admissible Λ is *dominant* if GZ_Λ is a component of the null cone.

Here is a way to show that the null cone \mathcal{N} is not reduced.

Proposition 10.2. *Let $\Lambda \subset \Lambda(V)$ be dominant and let $W \subset V$ be a T -stable complement of Z_Λ . Assume that for any $z \in Z_\Lambda$ the differential $d\pi_z$ restricted to W has rank $< \text{codim}_V GZ_\Lambda$, or, equivalently; there is a subspace $W' \subset W$ of dimension $> \text{codim}_{GZ_\Lambda} Z_\Lambda$ such that the differential of any invariant vanishes on W' . Then no point of $GZ_\Lambda \subset \mathcal{N}$ is reduced.*

Proof. Either condition implies that the rank of $d\pi_z$ is less than the codimension of GZ_Λ for any $z \in Z_\Lambda$. \square

Remark 10.3. Let (v_1, \dots, v_n) be a basis of V consisting of weight vectors of weight μ_1, \dots, μ_n , and let (x_1, \dots, x_n) be the dual basis. If $f = x_{i_1} \cdots x_{i_d}$ is a monomial of weight zero, then an x_i such that $\mu_i \notin \Lambda$ has to appear. If two such x_i appear in f , then clearly $(df)|_{Z_\Lambda} = 0$. This gives our first method to show that \mathcal{N} is not reduced.

- (1) Let Λ' be the complement of Λ in $\Lambda(V)$. Let $d \in \mathbb{N}$ be minimal such that every zero weight monomial f containing exactly one factor x_i corresponding to a weight from Λ' has degree $\leq d$.
- (2) Show that there are not enough invariants of degree $\leq d$, i.e., show that the number of invariants of degree $\leq d$ is strictly less than the codimension of GZ_Λ .

If W is irreducible of highest weight λ , we denote the highest weight of the dual representation, W^* by λ^* . The next result will give us another way to see if the null cone is not reduced. It uses the method of covariants introduced in section 3 (see Proposition 3.1).

Proposition 10.4. *Let $\varphi: V \rightarrow W$ be a covariant, where W is irreducible of highest weight λ . Let $\Lambda \subset \Lambda(V)$ be admissible and assume that φ does not vanish on GZ_Λ . Then $\lambda^* \in \mathbb{N}\Lambda$.*

Proof. Let W^* be the subspace of $\mathcal{O}(V)$ corresponding to φ . Let f be a highest weight vector of W^* . Then f has weight λ^* and f does not vanish on GZ_Λ by assumption. It follows that f contains a monomial $m = x_{i_1}x_{i_2} \cdots x_{i_d}$ where the corresponding v_{i_k} all belong to Z_Λ , i.e., $\lambda^* = \mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_d} \in \mathbb{N}\Lambda$. \square

Remark 10.5. This proposition will be used in the following way.

- (1) Find a suitable highest weight λ and an integer d such that λ^* cannot be written as a sum of more than d weights from Λ .
- (2) Show that there are generating covariants of type W_λ in degree $> d$.

By the proposition above this implies that the generating covariants from (2) vanish on GZ_Λ . In order to apply Proposition 3.1 one has to fix d and check (1) for any admissible Λ .

We finish this section by giving some criteria to find the dominant Λ among the admissible ones. Let Λ_1 and Λ_2 be admissible subsets of $\Lambda(V)$. Set $Z_i := Z_{\Lambda_i}$, $i = 1, 2$. We say that Λ_2 dominates Λ_1 , and we write $\Lambda_1 \leq \Lambda_2$, if $GZ_1 \subset GZ_2$. Given $\sigma \in \mathcal{W}$, the Weyl group of G , let $\Lambda_1^{(\sigma)} := \{\lambda \in \Lambda_1 \mid \sigma(\lambda) \in \Lambda_2\}$ and let $Z_1^{(\sigma)}$ denote the sum of the weight spaces with weights in $\Lambda_1^{(\sigma)}$.

Lemma 10.6. *Let Λ_1 and Λ_2 be admissible. Then Λ_2 dominates Λ_1 if and only if there is a $\sigma \in \mathcal{W}$ such that $BZ_1^{(\sigma)}$ is dense in Z_1 .*

Proof. Suppose that $\Lambda_1 \leq \Lambda_2$. Let $z \in Z_1$. Then there is a $g \in G$ such that $gz \in Z_2$. Write $g = u\sigma b$ where $b \in B$, $u \in U$ and $\sigma \in \mathcal{W}$ (Bruhat decomposition). Since b and u preserve the Z_i , we see that $bz \in Z_1^{(\sigma)}$. Thus Z_1 is the union of the constructible subsets $BZ_1^{(\sigma)}$, $\sigma \in \mathcal{W}$, and one of them must be dense.

Conversely, suppose that some $BZ_1^{(\sigma)}$ is dense in Z_1 . Since $\sigma(BZ_1^{(\sigma)})$ lies in GZ_2 and GZ_2 is closed, we see that $GZ_1 \subset GZ_2$. \square

The condition that $BZ_1^{(\sigma)}$ is dense in Z_1 has some consequences for the weights of $Z_1^{(\sigma)}$. Denote by Φ^+ the set of positive roots, i.e., the weights of $\mathfrak{b} := \text{Lie } B$.

Lemma 10.7. *Let Z be a B -module and $Z' \subset Z$ a T -stable subspace. If BZ' is dense in Z , then*

$$\Lambda(Z) \subset \Lambda(Z') + (\Phi^+ \cup \{0\}).$$

In particular, $\Lambda(Z')$ contains the set $\Omega := \{\lambda \in \Lambda(Z) \mid \lambda \notin \Lambda(Z) + \Phi^+\}$.

Proof. The tangent map of $B \times Z' \rightarrow Z$ at a point (e, z_0) has the form $(X, v) \mapsto Xz_0 + v$, and so $\mathfrak{b}Z' + Z' = Z$. If $z \in Z'$ is a weight vector of weight λ , then $\mathfrak{b}z \subset \bigoplus_{\omega \in \Phi^+ \cup \{0\}} Z_{\lambda+\omega}$, hence $\Lambda(Z)$ is as claimed. \square

Proposition 10.8. *Let $\Lambda_1, \Lambda_2 \subset \Lambda(V)$ be admissible subsets. Define $\Omega_1 := \{\lambda \in \Lambda_1 \mid \lambda \notin \Lambda_1 + \Phi^+\}$ and suppose that $\mathbb{Q}_{\geq 0}\Omega_1$ contains the simple roots. Then $\Lambda_1 \leq \Lambda_2$ implies that $\Lambda_1 \subset \Lambda_2$.*

Proof. Let σ be as in Lemma 10.6. Then $\Lambda_1^{(\sigma)}$ contains Ω_1 by Lemma 10.7. This in turn implies that Λ_2 is positive on $\sigma(\alpha_j)$, $j = 1, \dots, \ell$. Thus each $\sigma(\alpha_j)$ is a positive root and so σ is the identity. Hence $\Omega_1 \subset Z_2$ and thus $\Lambda_1 \subset \Lambda_2$. \square

Corollary 10.9. *Suppose that $G = \text{SL}_3$ with simple roots α and β . Let $\Lambda = \Lambda_\rho \subset \Lambda(V)$ be admissible and maximal with respect to set inclusion. Suppose that Λ contains nonzero weights of the form $\lambda_1 := -a\alpha + b\beta$ and $\lambda_2 := c\alpha - d\beta$ where the coefficients a, b, c and d are nonnegative rational numbers. Then Λ is dominant.*

Proof. Let $\Omega \subset \Lambda$ be the minimal elements. We may assume that λ_1 and λ_2 are in Ω . Clearly $b, c \neq 0$. If $a = 0$ or $d = 0$, then the hypotheses of Proposition 10.8 are satisfied. If $a, d \neq 0$, then $\langle \rho, \lambda_1 \rangle > 0$ and $\langle \rho, \lambda_2 \rangle > 0$ forces $bc - ad > 0$. Thus the inverse of the matrix $\begin{pmatrix} c & -d \\ -d & b \end{pmatrix}$ has positive entries, so that the hypotheses of Proposition 10.8 are satisfied and Λ is dominant. \square

See Example 11.2 below for a calculation of components of a null cone.

11 Coreduced representations of SL_3

In this section we classify the coreduced representations of $G = \text{SL}_3$ (Theorems 11.10 and 11.12).

We denote the representation $V := \varphi_1^r \varphi_2^s$ by $V[r, s]$, $r, s \in \mathbb{N}$, and its highest weight by $[r, s]$. Let α and β be the simple roots of G . We denote a weight $p\alpha + q\beta$ of a representation of G by (p, q) where $p, q \in (1/3)\mathbb{Z}$ and $p + q \in \mathbb{Z}$. Hence $\alpha = (1, 0) = [2, -1]$, $\beta = (0, 1) = [-1, 2]$, and so

$$[r, s] = \left(\frac{2r + s}{3}, \frac{r + 2s}{3} \right) \text{ and } (p, q) = [2p - q, 2q - p].$$

Moreover, $[r, s]$ is in the root lattice if and only if $r \equiv s \pmod 3$.

We leave the following lemma to the reader (see Lemma 5.6).

Lemma 11.1. *Let $V := V[r, s]$ be an irreducible representation of G and set $(p, q) = [r, s]$.*

- (1) *The dominant weights of $V[r, s]$ are the weights $[r', s']$ obtained starting with $[r, s]$ and using the following inductive process: $[r', s']$ gives rise to $[r' - 2, s' + 1]$ if $r' \geq 2$ and to $[r' + 1, s' - 2]$ if $s' \geq 2$. Finally, $[1, 1]$ gives rise to $[0, 0]$. Equivalently, the dominant weights of $V[r, s]$ are those of the form $(k, l) := (p - a, q - b)$ where $a, b \in \mathbb{N}$, $0 \leq k \leq 2l$ and $0 \leq l \leq 2k$.*
- (2) *The Weyl group orbit of the dominant weight (k, l) is*
 - a. $(k, l), (l - k, l), (k, k - l), (l - k, -k), (-l, k - l), (-l, -k)$ if $k \neq 2l$ and $l \neq 2k$,
 - b. $(2l, l), (-l, l), (-l, -2l)$ if $k = 2l$ and
 - c. $(k, 2k), (k, -k)$ and $(-2k, -k)$ if $l = 2k$.
- (3) *Let (p, q) be dominant, $p \neq q$, and let $\mathcal{W} \cdot (p, q)$ be the Weyl group orbit of (p, q) . Then*

$$\max \left\{ \frac{-k}{\ell} \mid (k, \ell) \in \mathcal{W} \cdot (p, q) \right\} = \frac{\min(p, q)}{|p - q|},$$

$$\min \left\{ \frac{-k}{\ell} \mid (k, \ell) \in \mathcal{W} \cdot (p, q), \frac{-k}{\ell} > 0 \right\} = \frac{|p - q|}{\min(p, q)}.$$

Suppose that $\Lambda(V)$ is not contained in the root lattice. Then let Λ_α denote the weights (p, q) of V where $p > 0$. We define Λ_β similarly. Note that Λ_α is stable under the simple reflection σ_β and that Λ_β is stable under the simple reflection σ_α .

Example 11.2. Consider the module $V = V[3, 1]$. Then the dominant weights are $[3, 1], [1, 2], [2, 0]$ and $[0, 1]$. Thus the weights of V are

- (1) $(7/3, 5/3), (-2/3, 5/3), (7/3, 2/3), (-2/3, -7/3), (-5/3, 2/3), (-5/3, -7/3)$ (the \mathcal{W} -orbit of $[3, 1]$);
- (2) $(4/3, 5/3), (1/3, 5/3), (4/3, -1/3), (1/3, -4/3), (-5/3, -1/3), (-5/3, -4/3)$ (the \mathcal{W} -orbit of $[1, 2]$);
- (3) $(4/3, 2/3), (-2/3, 2/3), (-2/3, -4/3)$ (the \mathcal{W} -orbit of $[2, 0]$);
- (4) $(1/3, 2/3), (1/3, -1/3), (-2/3, -1/3)$ (the \mathcal{W} -orbit of $[0, 1]$).

Let $\rho \in Y(T) \otimes \mathbb{Q}$ be generic. We may assume that $\rho(\alpha) = 1$ and, of course, we have $\rho(\beta) > 0$. Then $\rho(\beta)$ has to avoid the values $2/5, 5/2, 4, 1/4$ and 1 , so there are six cases to consider.

Case 1: Let Λ_1 correspond to $2/5 < \rho(\beta) < 1$. It is easy to see that Λ_1 is maximal.

Then Λ_1 is dominant by Corollary 10.9 since $(-2/3, 5/3)$ and $(1/3, -1/3)$ are ρ -positive.

Case 2: Let Λ correspond to $0 < \rho(\beta) < 1/4$. Then $\Lambda = \Lambda_\alpha$ is $\sigma := \sigma_\beta$ -stable so that $\sigma(\Lambda^{(\sigma)}) = \sigma(\Lambda \cap \Lambda_1)$. Now $\Lambda \cap \Lambda_1$ is $\Lambda \setminus \{(1/3, -4/3)\}$, hence $\Lambda^{(\sigma)}$ is $\Lambda \setminus \{(1/3, 5/3)\}$ where $(1/3, 5/3)$ has multiplicity one. Thus $UZ_\Lambda^{(\sigma)}$ is dense in Z_Λ so that $\Lambda < \Lambda_1$. (One can also see directly that U^-Z_1 has Z_Λ in its closure.)

Now it is easy to calculate that $\dim GZ_\Lambda < \dim GZ_1$, so that $\Lambda = \Lambda_\alpha$ is not dominant.

Case 3: Let Λ correspond to $1/4 < \rho(\beta) < 2/5$. Then $\Lambda \subset \Lambda_1$.

Case 4: Let Λ_2 correspond to $5/2 < \rho(\beta) < 4$. Then Λ_2 is maximal and $(-5/3, 2/3)$ and $(4/3, -1/3)$ are ρ -positive, so that Λ_2 is dominant by Corollary 10.9.

Case 5: Let Λ correspond to $1 < \rho(\beta) < 5/2$. Then $\Lambda \subset \Lambda_1$.

Case 6: Let Λ correspond to $\rho(\beta) > 4$. Then $\Lambda = \Lambda_\beta$ and as in Case 2 we see that $\Lambda < \Lambda_1$ and that Λ is not dominant.

Thus there are only two components of the null cone, GZ_{Λ_1} and GZ_{Λ_2} corresponding to cases 1 and 4. Note that neither Λ_α nor Λ_β is dominant.

Lemma 11.1 does not tell us anything about multiplicities of weights, but the following result gives us some lower bounds, which suffice for our uses. If $[r, s]$ is a weight of V , then we denote by $V_{[r,s]} \subset V$ the corresponding weight space.

Lemma 11.3. *Let $r = r_0 + r'$ and $s = s_0 + s'$ where $r' \equiv s' \pmod{3}$. Then every weight of $V[r_0, s_0]$ occurs in $V[r, s]$ with multiplicity at least the dimension of the zero weight space $V[r', s']_{[0,0]}$.*

Proof. As in Lemma 5.18 this follows from the fact that $\mathcal{O}(G/U)$ is a domain and that the product of the copies of $V[r_0, s_0]$ and $V[r', s']$ in $\mathcal{O}(G/U)$ is just the copy of $V[r, s]$ □

Example 11.4. Consider $V[3, 2]$. Then the multiplicity of $[1, 0]$ is at least the multiplicity of the zero weight in $V[2, 2]$, which is 3. The multiplicity of $[2, 1]$ is similarly seen to be at least 2. Thus the multiplicities of the dominant weights of $V[3, 2]$ are at least as follows: $[3, 2], [4, 0], [1, 3]$ and $[0, 2]$ with multiplicity one, $[2, 1]$ with multiplicity two and $[1, 0]$ with multiplicity three. In fact, these multiplicities are correct, except that $[0, 2]$ actually has multiplicity two.

In Example 11.2 we have seen that neither Λ_α nor Λ_β is dominant. But this is an exception as shown by the following result.

Lemma 11.5. *Let $V = V[r, s]$ where $r \geq s$.*

- (1) *If $r - s \equiv 1 \pmod{3}$, then Λ_β is dominant.*
- (2) *If $r - s \equiv 2 \pmod{3}$ and $[r, s] \neq [3, 1]$ or $[5, 0]$, then Λ_α is dominant.*

Proof. For $t > 0$ define $\rho_t \in Y(T)$ by $\rho_t(\alpha) = 1$ and $\rho_t(\beta) = t$, and set $\Lambda_t := \Lambda_{\rho_t}$. Define

$$\mathcal{T} := \{t > 0 \mid \rho_t(\lambda) = 0 \text{ for some } \lambda \in \Lambda(V[r, s]), \lambda \neq 0\}.$$

We have $\mathcal{T} = \{t_1, t_2, \dots, t_m\}$ where $0 < t_1 < t_2 < \dots < t_m$, so there are $m + 1$ admissible subsets $\Lambda^{(i)}$, $i = 0, \dots, m$, defined by $\Lambda^{(i)} := \Lambda_t$ for $t_i < t < t_{i+1}$, where $t_0 = 0, t_{m+1} = \infty$. Clearly, $\Lambda^{(0)} = \Lambda_\alpha, \Lambda^{(m)} = \Lambda_\beta$, and Λ_α (resp. Λ_β) is not maximal if and only if $\Lambda_\alpha \subset \Lambda^{(1)}$ (resp. $\Lambda_\beta \subset \Lambda^{(m-1)}$). Note that if $\rho_t((k, l)) = 0$, then $t = -k/l$.

(1) First suppose that $r - s \equiv 1 \pmod 3$ and let $(p, q) = [r, s]$. Then $[1, 0] = (2/3, 1/3)$ is a weight of V , and the α -string through $[1, 0]$ has the form

$$\Sigma = ((-q, 1/3), (-q + 1, 1/3), \dots, (2/3, 1/3), \dots, (q + 1/3, 1/3))$$

where $(-q, 1/3)$ is in the \mathcal{W} -orbit of $(q + 1/3, q)$. Note that $\#\Sigma = 2q + 4/3$. Since the case $V = V[1, 0]$ is obvious we can assume that $q \geq 4/3$, hence $\#\Sigma \geq 4$.

Claim 1: We have $t_m = 3q$ and $t_{m-1} = 3q - 3$, and these values are attained at the first two weights $(-q, 1/3)$ and $(-q + 1, 1/3)$ of the α -string Σ . In particular, $\Lambda_\beta \supset \Lambda^{(m-1)}$ and $\#(\Sigma \cap \Lambda^{(m')}) \leq \#\Sigma - 2$ for $m' \leq m - 2$.

This implies that Λ_β is dominant. In fact, suppose that $\Lambda_\beta < \Lambda$ for some admissible Λ . Set $Z_\beta := Z_{\Lambda_\beta}$. There is a $\sigma \in \mathcal{W}$ such that $BZ_\beta^{(\sigma)}$ is dense in Z_β and $\sigma(\Lambda_\beta^{(\sigma)}) \subset \Lambda$ (Lemma 10.6). Clearly $\Lambda_\beta^{(\sigma)}$ has to contain a subset Σ' of the α -string Σ which omits at most one element and contains $(-q, 1/3)$ (see Lemma 10.7). Since Σ' contains at least 3 elements it is easy to see that $\sigma = e$ and $\sigma = \sigma_\alpha$ are the only elements from \mathcal{W} which send Σ' to elements which have at least one positive α or β coefficient. Thus $\sigma(\Sigma') \subset \Lambda \cap \Sigma$. By the claim above, this implies that $\Lambda = \Lambda_\beta$ or $\Lambda = \Lambda^{(m-1)}$ and so $\Lambda \subset \Lambda_\beta$.

(2) Now suppose that $r - s \equiv 2 \pmod 3$. Then $[0, 1] = (1/3, 2/3)$ is a weight of V , and the β -string through $[0, 1]$ has the form

$$\Sigma = ((1/3, -q + 1/3), (1/3, -q + 4/3), \dots, (1/3, 2/3), \dots, (1/3, q))$$

where $(1/3, -q + 1/3)$ is in the \mathcal{W} -orbit of $(q - 1/3, q)$. Note that $\#\Sigma = 2q + 4/3$.

Claim 2: If $\#\Sigma \geq 6$ (i.e., $q \geq 8/3$), then $t_1 = 1/(3q - 1)$ and $t_2 = 1/(3q - 4)$, and these values are attained at the first two weights $(1/3, -q + 1/3)$ and $(1/3, -q + 4/3)$ of the β -string Σ . Moreover, $\Lambda_\alpha \supset \Lambda^{(1)}$ and $\#(\Sigma \cap \Lambda^{(m')}) \leq \#\Sigma - 2$ for $m' \geq 2$.

Now the same argument as above implies that Λ_α is dominant. Note that the condition $q \geq 8/3$ is satisfied for $[r, s] \neq [2, 0], [3, 1]$ or $[5, 0]$. For $V[2, 0]$ there are only two admissible sets, Λ_α and Λ_β , both are dominant and $\mathcal{N} = GZ_\alpha = GZ_\beta$.

(3) It remains to prove the two claims. Let $r - s \equiv 1 \pmod 3$. We use the first formula given in Lemma 11.1(3) for a dominant (p', q') :

$$\mu_{(p',q')} := \max \left\{ \frac{-k}{\ell} \mid (k, \ell) \in \mathcal{W} \cdot (p', q') \right\} = \frac{\min(p', q')}{|p' - q'|}$$

By assumption we have $q \geq 4/3$. If $(p', q') \leq (p, q)$ is dominant, then $|p' - q'| \geq 1/3$. Thus

$$t_m = \max(\mu_{(p',q')} \mid (p', q') \text{ dominant}, (p', q') \leq (p, q)) = \mu_{(q+1/3,q)} = 3q,$$

and this value is attained at a single weight of V , namely at the weight $(-q, 1/3) \in \mathcal{W} \cdot (q+1/3, q)$. It follows that t_{m-1} is either equal to $\mu_{(q-2/3,q-1)} = 3(q-1)$ or equal to $\mu_{(p',q)}$ for a suitable $p' \leq p, p' \neq q+1/3$. But then $p' = q-2/3$ or $p' = q+4/3$ and in both cases we get $\mu_{(p',q)} \leq 3(q-1)$, because $q \geq 4/3$. Hence $t_{m-1} = 3(q-1)$ and this value is attained at the weight $(-q+1, 1/3) \in \mathcal{W} \cdot (q-2/3, q-1)$. As a consequence, $\Lambda_\beta \supset \Lambda^{m-1} = \Lambda_\beta \setminus \{(-q, 1/3)\}$, and $(-q, 1/3), (-q+1, 1/3) \notin \Lambda^{m'}$ for $m' \leq m-2$. This proves Claim 1.

For $r - s \equiv 2 \pmod 3$ we use the second formula in Lemma 11.1(3) for a dominant (p', q') :

$$v_{(p',q')} := \min \left\{ \frac{-k}{\ell} \mid (k, \ell) \in \mathcal{W} \cdot (p', q'), \frac{-k}{\ell} > 0 \right\} = \frac{|p' - q'|}{\min(p', q')}.$$

The minimal values of $|p' - q'|$ are $1/3$ and $2/3$ and they are attained at $(q'-1/3, q')$ and $(q'+2/3, q')$. Thus, for a fixed q' the minimal values of $v_{(p',q')}$ are $1/(3q'-1)$ and $2/(3q')$. Since $q \geq 8/3 > 4/3$ we get

$$t_1 = \min(v_{(p',q')} \mid (p', q') \leq (p, q) \text{ dominant}) = v_{(q-1/3,q)} = 1/(3q-1),$$

and this value is attained at a single weight, namely at $(1/3, -q+1/3) \in \mathcal{W} \cdot (q-1/3, q)$. It follows that t_2 is either equal to $v_{(q+2/3,q)} = 2/(3q)$ or equal to $v_{(q-4/3,q-1)} = 1/(3q-4)$. Since $q \geq 8/3$ we get $3q-4 = (3/2)q + ((3/2)q-4) \geq (3/2)q$. Hence $t_2 = 1/(3q-4)$ and this value is attained at $(1/3, -q+4/3) \in \mathcal{W} \cdot (q-4/3, q-1)$. Now Claim 2 follows as above. \square

Remark 11.6. Let $\Lambda = \Lambda_\alpha$ or Λ_β . Then Z_Λ is stabilized by a parabolic subgroup of codimension 2, hence $\text{codim}_{GZ_\Lambda} Z_\Lambda \leq 2$.

We need the following estimate on the dimension of $S^3(V)^G$:

Proposition 11.7. *Let $r \geq s \geq 0$. Then*

- (1) *The multiplicity of $[r - s, 0]$ in $V[r, 0] \otimes V[0, s]$ is $\binom{s+2}{2}$.*
- (2) *The multiplicity of $[r - s, 0]$ in $V[r, s]$ is $s + 1$.*
- (3) *The multiplicity of $V[s, r]$ in $V[r, s] \otimes V[r, s]$ is at most $s + 1$.*
- (4) *The dimension of $S^3(V[r, s])^G$ is at most $s + 1$, hence there are at most $s + 1$ linearly independent cubic invariants of $V[r, s]$.*

Proof. Let e_1, e_2 and e_3 be the usual basis of \mathbb{C}^3 and let f_1, f_2, f_3 be the dual basis. Then the weight vectors of weight $[r-s, 0]$ in $V[r, 0] \otimes V[0, s]$ have basis the vectors $e_1^{r-t} m \otimes f_1^{s-t} m^*$, where $0 \leq t \leq s$ and m is a monomial of degree t in e_2 and e_3 and m^* is the same monomial in f_2 and f_3 . Thus the dimension of this weight space is $1 + \dots + (s + 1)$, giving (1).

Part (2) follows from the fact that

$$V[r, 0] \otimes V[0, s] = V[r, s] \oplus V[r - 1, 0] \otimes V[0, s - 1].$$

This is an immediate consequence of Pieri’s formula (see [Pro07, formula (10.2.2) in 9.10.2]).

The multiplicity of $V[s, r]$ in $V[r, s] \otimes V[r, s]$ is bounded by the multiplicity of the weight $[r, s] - (r, s)$ in $V[r, s]$ since $[r, s] + ([r, s] - (r, s)) = [s, r]$. Now $[r, s] - (r, s) = 1/3(-r + s, r - s)$ which is in the \mathcal{W} -orbit of $1/3(2r - 2s, r - s) = [r - s, 0]$. Thus (2) implies (3). Clearly (3) implies (4). \square

Example 11.8. Assume that $r \geq s \geq 1$ and that $r - s \equiv 2 \pmod 3$. Then the multiplicities of the weights of $V[0, 1]$ and $V[3, 1]$ in $V[r, s]$ are $\geq s$, and the multiplicities of the weights of $V[2, 0]$ are $\geq s + 1$ in case $r \geq 5$.

(In fact, for $V[3, 1]$ the multiplicities are $\geq \dim V[r - 3, s - 1]_{[0,0]}$ by Lemma 11.3 and $\dim V[r - 3, s - 1]_{[0,0]} \geq \dim V[r - 3, s - 1]_{[r-s-2,0]} = s$ by Proposition 11.7(2). The other cases follow by similar arguments.)

Proposition 11.9. *Let $V = V[r, s]$ where $r + s \geq 4$ or $(r, s) = (2, 1)$ or $(r, s) = (1, 2)$. Then there is an irreducible component \mathcal{N}_1 of \mathcal{N} such that the rank of $d\pi$ is less than the codimension of \mathcal{N}_1 in V on \mathcal{N}_1 . In particular, \mathcal{N} is not reduced.*

An immediate consequence is

Theorem 11.10. *Let V be an irreducible representation of $G = \mathrm{SL}_3$. Then V is coreduced if and only if V is on the following list:*

- (1) $V[1, 0], V[2, 0], V[3, 0]$;
- (2) $V[0, 1], V[0, 2], V[0, 3]$;
- (3) $V[1, 1]$.

Equivalently, V is coreduced if and only if it is cofree.

Proof (of Proposition 11.9). We may assume that $V = V[r, s]$ where $r \geq s$ and $V[r, s]$ does not appear in (1), (2) or (3) of the theorem. Let $(p, q) = [r, s]$. We are constantly applying Remarks 10.3 and 10.5.

Case 1: Assume that $r - s \equiv 1 \pmod 3$ and consider $\Lambda = \Lambda_\beta$ which is dominant by Lemma 11.5. Recall that $\mathrm{codim}_{GZ_\Lambda} Z_\Lambda \leq 2$. First suppose that $s \geq 1$ and $r > 2$. Then $[1, 3]$ and $[0, 2]$ are weights of V . Let $\rho \in Y(T)$ correspond to Λ_β . Then ρ is negative on the weights $(2/3, -2/3)$ and $(-4/3, -2/3)$ in the \mathcal{W} -orbit of $[0, 2]$, on the weights $(-7/3, -2/3)$ and $(5/3, -2/3)$ in the \mathcal{W} -orbit of $[1, 3]$ and on the weight $(-1/3, -2/3)$ in the \mathcal{W} -orbit of $[1, 0]$ which occurs with multiplicity

at least $s + 1$ since $[r - s, 0]$ has multiplicity $s + 1$ by Proposition 11.7(2). These negative weights can be paired with at most quadratic expressions in the positive weights (just look at the coefficients of β). Now there are at most $s + 1$ cubic invariants (and no quadratic invariants), hence \mathcal{N} is not reduced if $s \geq 1, r > 2$.

If $(r, s) = (2, 1)$, then we have the following negative weights: $(2/3, -2/3)$, $(-4/3, -2/3)$ and $(-1/3, -2/3)$ (with multiplicity 2). There is only a one-dimensional space of degree 3 invariants, and so \mathcal{N} is not reduced.

If $s = 0$, then the cases to consider are $V[4, 0], V[7, 0]$, etc. If $r \geq 7$, then we have a dominant weight $[1, 3]$ whose \mathcal{W} -orbit contains $(-7/3, -2/3)$ and $(5/3, -2/3)$. We still have $(2/3, -2/3), (-4/3, -2/3)$ and $(-1/3, -2/3)$. Since there is at most one degree 3 invariant, \mathcal{N} is not reduced.

We are left with the case of $V[4, 0]$. Here we have negative weights $(2/3, -2/3), (-4/3, -2/3)$ and $(-1/3, -2/3)$ as well as $(-1/3, -5/3)$ and $(-4/3, -5/3)$ in the \mathcal{W} -orbit of $[2, 1]$. Thus \mathcal{N} is not reduced since there are only two irreducible invariants of degree ≤ 6 (the Poincaré series of $\mathcal{O}(V)^G$ is $1 + t^3 + 2t^6 + \dots$).

Case 2: Assume that $r - s \equiv 2 \pmod 3$. For the cases $[r, s] = [3, 1]$ or $[5, 0]$ see Example 11.11 below. So we may assume that $\Lambda = \Lambda_\alpha$ is dominant. If $s \geq 1$ (and so $r \geq 5$), then among the dominant weights we have $[3, 1]$ with multiplicity at least s , $[2, 0]$ with multiplicity at least $s + 1$ and $[0, 1]$ with multiplicity at least s (see Example 11.8). The \mathcal{W} -orbit of $[3, 1]$ contains the weights $(-2/3, 5/3)$ and $(-2/3, -7/3)$ with negative α -coefficient, the \mathcal{W} -orbit of $[2, 0]$ contains $(-2/3, 2/3)$ and $(-2/3, -7/3)$ and the \mathcal{W} -orbit of $[0, 1]$ contains $(-2/3, -1/3)$. Since there is at most an $(s + 1)$ -dimensional space of degree 3 invariants, \mathcal{N} is not reduced. If $s = 0$ (and so $r \geq 5$), then we have the weights $[3, 1], [2, 0]$ and $[0, 1]$ with multiplicity one, and \mathcal{N} is not reduced because $\dim S^3(V)^G \leq 1$.

Case 3: If $r - s \equiv 0 \pmod 3$, then we are in the adjoint case and the claim follows from Proposition 7.5. □

Example 11.11. Let $V = V[3, 1]$. Then from Example 11.2 we see that there are two dominant Λ , one corresponding to $\rho(\alpha) = 1$ and $2/5 < \rho(\beta) < 1$ (choose $\rho(\beta) = 1/2$) and the other to $\rho(\alpha) = 1$ and $5/2 < \rho(\beta) < 4$ (choose $\rho(\beta) = 3$). Neither Λ_α nor Λ_β is dominant. Consider the case where $\rho(\beta) = 1/2$. Then the minimal positive weights (in terms of their ρ -value) are $(1/3, -1/3)$ and $(-2/3, 5/3)$, both having ρ -value $1/6$. Now consider the covariants of type $V[1, 0]$. The highest weight is $(2/3, 1/3)$ where $\rho(2/3, 1/3) = 5/6$. Thus the highest degree in which the covariant could occur in $S^*(V)$ and not vanish on GZ_ρ is 5. One gets the same bound in case $\rho(\beta) = 3$. The Poincaré series of the invariants is $1 + t^3 + \dots$ and for the $V[1, 0]$ covariants it is $4t^5 + 44t^8 + \dots$. Thus there are generating covariants in degree 8, which vanish on \mathcal{N} , so that \mathcal{N} is not reduced.

If $V = V[5, 0]$, then the calculations of Example 11.2 show that the dominant Λ again correspond to $\rho(\beta) = 1/2$ or 3. (The only new weights are $(10/3, 5/3), (-5/3, 5/3)$ and $(-5/3, -10/3)$ and they give rise to no new ratios.) Hence the highest degree in which the covariant $V[1, 0]$ could occur in $S^*(V)$ and not vanish on GZ_ρ is again 5. The covariant $V[1, 0]$ first occurs in degree 5, with multiplicity

one. But since the principal isotropy group of V is trivial, the $V[1, 0]$ covariants have to have generators in higher degree, and these necessarily vanish on \mathcal{N} . Thus \mathcal{N} is not reduced.

We now have the following result, which uses Theorem 11.10.

Theorem 11.12. *Let $G = \mathrm{SL}_3$ and V be a nontrivial reducible coreduced G -module with $V^G = 0$. Then, up to isomorphism and taking duals, we have the following list:*

- (1) $kV[1, 0] + \ell V[0, 1], k + \ell \geq 2$.
- (2) $V[2, 0] + V[0, 1]$.

Proof. We already know that the representations in (1) and (2) are coreduced by Theorem 9.1 and Example 4.5. We have to show that combinations not on the list are not coreduced.

Consider $V[1, 1]$ together with another irreducible. For $V[1, 1]$ there is a closed orbit with isotropy group double covered by $\mathrm{SL}_2 \times \mathbb{C}^*$. The slice representation (as representation of the double cover) is $\theta_1 + R_2$. If we add $V[1, 0]$ we get an additional copy of $R_1 \otimes \nu_1 + \nu_{-2}$ in the slice representation. Quotienting by \mathbb{C}^* we get the hypersurface in $\theta_1 + R_2 + R_2$ where the quadratic invariant of the second copy of R_2 vanishes. The hypersurface is not coreduced (see Example 3.3), hence $V[1, 1] + V[1, 0]$ is not coreduced. For $V[1, 1] + V[2, 0]$ one can easily see that the slice representation of the maximal torus is not coreduced, and for $V[3, 0]$ one uses the slice representation at the zero weight vector to rule out a coreduced sum involving $V[3, 0]$.

Next consider $2V[2, 0]$ which is cofree with generating invariants in bidegrees $(3, 0), (2, 1), (1, 2), (0, 3)$, and choose the 1-parameter subgroup ρ with weights $(1, 1, -2)$. Then one can easily see that the codimensions of GZ_ρ is $4 = \mathrm{codim} \mathcal{N}$ and that the rank of the differentials of the invariants is 2 on GZ_ρ . Hence the representation is not coreduced.

The representation $V[2, 0] + V[0, 2]$ is again cofree with generating invariants in degrees $(3, 0), (1, 1), (2, 2), (0, 3)$. For the same ρ one computes that the rank is 3, while the codimension of GZ_ρ is $4 = \mathrm{codim} \mathcal{N}$, so this possibility is ruled out. We cannot add $V[1, 0]$ to $V[2, 0]$ since the rank of the two generating invariants is only 1 on the null cone.

Finally, we have to show that $V := V[2, 0] + 2V[0, 1]$ (not cofree but coregular) is not coreduced. Consider the 1-parameter subgroup with weights $(2, -1, -1)$. This clearly gives a maximal dimensional component of the null cone and it has codimension $3 = \dim V // G - 1$. The 1-parameter subgroup with weights $(1, 1, -2)$ gives something of codimension 5, which is too small to be an irreducible component of \mathcal{N} since it is cut out by 4 functions. Hence \mathcal{N} is irreducible. But V has the slice representation $2R_2 + \theta_1$ of SL_2 whose null cone is not reduced but also has codimension three. Thus the associated cone to the fiber $F \simeq G \times^{\mathrm{SL}_2} \mathcal{N}(2R_2)$ is $\mathcal{N}(V)$. We know that F is not reduced (Example 3.3), hence by Remark 5.3, V is not coreduced. □

Appendix A Computations for F_4

Let G be a simple group of type F_4 and let $V = \varphi_4$ be the 26-dimensional representation of G . The main result of this appendix is the following proposition.

Proposition A.1. *The representation $(V^{\oplus n}, G)$ is coreduced if and only if $n \leq 2$. Moreover, V and $V \oplus V$ are both cofree and contain a dense orbit in the null cone.*

We will use the notation introduced in Section 10. The nonzero weights of V are the short roots of F_4 . Hence Z_ρ is the span of the positive short root spaces for any generic $\rho \in Y(T)$ which implies that the null cone $\mathcal{N}(V^{\oplus n})$ is irreducible for any n . We also know that $V \oplus V$ is cofree with $\dim(V \oplus V)//G = 8$ [Sch79], hence $\dim \mathcal{N}(V \oplus V) = 44$. Let us look at the following statements which imply the proposition.

- (a) $V \oplus V \oplus V$ is not coreduced.
- (b) $V \oplus V$ is coreduced.
- (c) There is a dense orbit in the null cone of $V \oplus V$.

Although we know that (c) implies (b) (Corollary 4.7) we will present direct proofs of all three claims. They are based on some explicit computations.

Proof (of statement (a)). There is a maximal subgroup of type B_4 of F_4 where $(\varphi_4(F_4), B_4) = \varphi_1 + \varphi_4 + \theta_1$. (For more see the discussion after Lemma A.2.) The slice representation of B_4 on $\varphi_4(F_4)$ is $\varphi_1(B_4) + \theta_1$. To prove that $3\varphi_4(F_4)$ is not coreduced, it suffices to prove that $V_1 := 3\varphi_1(B_4) + 2\varphi_4(B_4)$ is not coreduced. Now D_4 is a maximal subgroup of B_4 and $V' := 2\varphi_1(D_4) + 2\varphi_3(D_4) + 2\varphi_4(D_4)$ is a slice representation of V_1 at a zero weight vector. So we have to show that V' is not coreduced. Since our representations are self-dual, we will deal with the symmetric algebra $S(V')$ in place of $\mathcal{O}(V')$.

Recall that G is now D_4 and V' is as above. We have $\bigwedge^2 \varphi_1 = \bigwedge^2 \varphi_3 = \bigwedge^2 \varphi_4 = \varphi_2$, the adjoint representation. In the tensor product of three copies of φ_2 we have 7 copies of φ_2 , but only five of them are in the ideal generated by the invariants. (This can be checked using LiE). We will show now that every covariant of type φ_2 in $\bigwedge^2 \varphi_1 \otimes \bigwedge^2 \varphi_3 \otimes \bigwedge^2 \varphi_4 \subset S(V')_{(2,2,2)}$ vanishes on the null cone, i.e., vanishes on Z_ρ for every generic $\rho \in Y(T)$.

Recall that the weights of φ_1 are $\pm \varepsilon_i$, those of φ_3 are $1/2(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ where the number of minus signs is even. The weights of φ_4 look similar, but have an odd number of minus signs. We use the notation $(\pm \pm \pm \pm)$ for these weights.

There is an outer automorphism τ of D_4 of order 2 (coming from the Weyl group of B_4) which normalizes the maximal torus, fixes $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and sends ε_4 to $-\varepsilon_4$. If G is of type D_4 and if $\rho_i: G \rightarrow GL(V_i)$ denotes the i th fundamental representation φ_i , then $\rho_1 \circ \tau \simeq \rho_1, \rho_2 \circ \tau \simeq \rho_2$, and $\rho_3 \circ \tau \simeq \rho_4$. Thus there is a linear automorphism $\mu: V' \xrightarrow{\sim} V'$ which is τ -equivariant, i.e., $\mu(gv) = \tau(g)\mu(v)$. It follows that μ has the following properties:

- (1) μ sends G -orbits to G -orbits. In particular, $\mu(\mathcal{N}) = \mathcal{N}$.
- (2) $\mu(V'_\lambda) = V'_{\tau(\lambda)}$.
- (3) If $\psi: V' \rightarrow V''$ is a covariant of type φ_2 in $\wedge^2 \varphi_1 \otimes \wedge^2 \varphi_3 \otimes \wedge^2 \varphi_4$, then so is $\psi \circ \mu: V' \rightarrow V''$.

This implies that for every 1-PSG ρ we have $\mu(Z_\rho) = Z_{\tau(\rho)}$ and that if all $\psi: V' \rightarrow V''$ as in (3) vanish on Z_ρ , then they vanish on $Z_{\tau(\rho)}$ too.

As a consequence, we can assume that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \varepsilon_4 > 0$. This implies that the following weights are positive:

$$\begin{aligned} \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} &\subset \Lambda(\varphi_1), \\ \{(+ + + +), (+ - + -), (+ + - -)\} &\subset \Lambda(\varphi_3), \\ \{(+ + + -), (+ + - +), (+ - + +)\} &\subset \Lambda(\varphi_4). \end{aligned}$$

Since $(- + + -) < (- + + +)$ we see that there are only three cases of maximal positive weight spaces to be considered.

- (1) $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \{(+ + + +), (+ - + -), (+ + - -), (- + + -)\}$ and $\{(+ + + -), (+ - + +), (+ + - +), (- + + +)\}$;
- (2) $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \{(+ + + +), (+ - + -), (+ + - -), (+ - - +)\}$ and $\{(+ + + -), (+ - + +), (+ + - +), (+ - - -)\}$;
- (3) $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \{(+ + + +), (+ - + -), (+ + - -), (+ - - +)\}$ and $\{(+ + + -), (+ - + +), (+ + - +), (- + + +)\}$.

Now we have to calculate the positive weights in $\wedge^2 \varphi_1, \wedge^2 \varphi_3$ and $\wedge^2 \varphi_4$ which come from the positive weights in the two copies of φ_1, φ_3 , and φ_4 . For $\wedge^2 \varphi_1$ we get $\{\varepsilon_i + \varepsilon_j \mid i < j\}$ in all three cases. For the two others we find the following sets.

- (1) $\wedge^2 \varphi_3: \{\varepsilon_i + \varepsilon_j \mid i < j < 4\} \cup \{\varepsilon_i - \varepsilon_4 \mid i < 4\}; \wedge^2 \varphi_4: \{\varepsilon_i + \varepsilon_j \mid i < j\}$.
- (2) $\wedge^2 \varphi_3: \{\varepsilon_1 \pm \varepsilon_j \mid j > 1\}; \wedge^2 \varphi_4: \{\varepsilon_1 \pm \varepsilon_j \mid j > 1\}$.
- (3) $\wedge^2 \varphi_3: \{\varepsilon_1 \pm \varepsilon_j \mid j > 1\}; \wedge^2 \varphi_4: \{\varepsilon_i + \varepsilon_j \mid i < j\}$.

Now it is easy to see that in all three cases there is no way to write the highest weight $\varepsilon_1 + \varepsilon_2$ of φ_2 as a sum of three positive weights, one from each \wedge^2 . Hence (V', \mathbf{D}_4) is not coreduced and we have proved (a). □

Proof (of statement (b)). Since $V \oplus V$ is cofree the null cone is (schematically) a complete intersection. Therefore it suffices to find an element $v \in \mathcal{N}(V \oplus V)$ such that the differential $d\pi_v$ of the quotient morphism $\pi: V \oplus V \rightarrow Y$ at v has maximal rank $8 = \dim Y$.

The nonzero weights of $\varphi_4(\mathbf{F}_4)$ are $\pm\varepsilon_i, i = 1, \dots, 4$ (the nonzero weights of $\varphi_1(\mathbf{B}_4)$) and $(1/2)(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ (the weights of $\varphi_4(\mathbf{B}_4)$). We will abbreviate the latter weights as $(\pm\pm\pm\pm)$ from now on. The positive weights are the ε_i and the weights of $\varphi_4(\mathbf{B}_4)$ where the coefficient of ε_1 is positive. Let $v_{\pm i}$ denote a nonzero vector in the weight space of $\pm\varepsilon_i$, let v_0 denote a zero weight vector and let v_{++++} denote a nonzero vector in the weight space $(+ + + +)$ and similarly for $v_{+++-},$ etc. We claim that $d\pi$ has rank 8 at the point $v = (v_2 + v_3 + v_{+---} + v_{+----}) \in 2\varphi_1(\mathbf{B}_4) + 2\varphi_4(\mathbf{B}_4)$.

The invariants of $2\varphi_4(\mathbf{F}_4)$ are the polarizations of the degree 2 invariant and the degree 3 invariant of one copy of $\varphi_4(\mathbf{F}_4)$ together with an invariant of degree (2, 2) [Sch78]. The restriction of the degree 2 invariant to $\varphi_1(\mathbf{B}_4) + \varphi_4(\mathbf{B}_4)$ is the sum of the degree two invariants there (see [Sch78] for descriptions of the invariants of $(2\varphi_1 + 2\varphi_4, \mathbf{B}_4)$). Clearly the differentials of the degree 2 invariants of $2\varphi_4(\mathbf{F}_4)$ at v have rank 3 when applied to the subspace spanned by the vectors v_{-2}, v_{-3} in the two copies of $\varphi_1(\mathbf{B}_4)$. There is only one degree 3 generator in $(\varphi_1 + \varphi_4, \mathbf{B}_4)$ and it is the contraction of φ_1 with the copy of φ_1 in $S^2(\varphi_4)$. Another way to think of the invariant is as the contraction of φ_4 with the copy of φ_4 in $\varphi_1 \otimes \varphi_4$. Now the highest weight vector of the copy of φ_4 in $\varphi_1 \otimes \varphi_4$ is (up to some nonzero coefficients)

$$v_1 \otimes v_{-++++} + v_2 \otimes v_{+----} + v_3 \otimes v_{+--+} + v_4 \otimes v_{+--+} + v_0 \otimes v_{++++}.$$

From this one derives the form of the other weight vectors of $\varphi_4 \subset \varphi_1 \otimes \varphi_4$ and restricting to v one gets contributions to the weights $(+--+), (++--), (+-++)$ and $(+-+-)$. Thus the differential of the degree 3 invariant of $\varphi_1(\mathbf{F}_4)$ at v vanishes on φ_4 except on $v_{-+-}, v_{-++}, v_{+--}$ and v_{+--} . Now polarizing it is easy to see that the four generators of degree 3 have differential of rank 4 at v when applied to vectors in $2\varphi_4(\mathbf{B}_4)$.

There remains the generator of degree 4. Restricted to \mathbf{B}_4 one easily sees that the invariant is a sum of two generators (modulo products of the generators of degree 2), one of which is the invariant which contracts the copy of $\wedge^2(\varphi_1) \subset S^2(2\varphi_1)$ with the copy in $\wedge^2(\varphi_4) \subset S^2(2\varphi_4)$ and the other which is of degree 4 in $2\varphi_4(\mathbf{B}_4)$ (and doesn't involve $2\varphi_1$). Now the highest weight vector of $\wedge^2(\varphi_1) \subset \wedge^2(\varphi_4)$ is (up to nonzero scalars)

$$v_{++++} \wedge v_{+--+} + v_{+--+} \wedge v_{+--+}$$

from which it follows that the weight vector of weight $-\varepsilon_2 - \varepsilon_3$ does not vanish on $v_{+----} + v_{----+}$. Now in $\wedge^2(\varphi_1) \subset S^2(2\varphi_1)$ we have $v_2 \wedge v_3$ of weight $\varepsilon_2 + \varepsilon_3$. Hence the differential of the degree 4 invariant evaluated at v does not vanish on v_{----+} and the rank of the differentials of the 8 invariants is indeed 8. Thus $2\varphi_4(\mathbf{F}_4)$ is coreduced. \square

Proof (of statement (c)). Recall that the root system R of G has the following 3 parts A, B and C :

$$A = \{\pm\varepsilon_i\}, \quad B = \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j\}, \quad C = \left\{ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \right\}$$

with cardinality $\#A = 8, \#B = 24$ and $\#C = 16$. Thus

$$\mathfrak{g} = \text{Lie } G = \mathfrak{h} \oplus \bigoplus_{\beta \in A \cup B \cup C} \mathfrak{g}_\beta$$

where $\mathfrak{h} = \text{Lie } T$ is the Cartan subalgebra and $T \subset G$ a maximal torus. The weights $\Lambda = \Lambda_V$ of the representation V are the short roots $A \cup C$ together with the zero weight 0 . The nonzero weight spaces V_λ are 1-dimensional, and the zero weight space $V_0 = V^T$ has dimension 2. This implies the following.

Lemma A.2. *Let $\beta \in R$ be a root and $\lambda \in \Lambda$ a weight of V . If $\beta + \lambda$ is a weight of V , then $\mathfrak{g}_\beta V_\lambda$ is a nontrivial subspace of $V_{\beta+\lambda}$.*

Note that $\mathfrak{g}_\beta V_0 = V_\beta$ and $\mathfrak{g}_\beta V_{-\beta} \subset V_0$ is 1-dimensional for every short root $\beta \in A \cup C$.

The subspace $\mathfrak{g}' := \mathfrak{h} \oplus \bigoplus_{\beta \in A \cup B} \mathfrak{g}_\beta \subset \mathfrak{g}$ is the Lie algebra of a maximal subgroup $G' \subset G$ of type B_4 , and the representation V decomposes under G' into $V = \theta_1 \oplus \varphi_1(\mathbf{B}_4) \oplus \varphi_4(\mathbf{B}_4)$ where $\varphi_4(\mathbf{B}_4) = \bigoplus_{\gamma \in C} V_\gamma$, $\varphi_1(\mathbf{B}_4) = V_A \oplus \bigoplus_{\alpha \in A} V_\alpha$, and $V_A := \varphi_1(\mathbf{B}_4)^T \subset V_0$. It follows that $\mathfrak{g}_\alpha V_{-\alpha} = V_A$ for $\alpha \in A$, but $\mathfrak{g}_\gamma V_{-\gamma} \not\subset V_A$ for $\gamma \in C$, so that $\mathfrak{g}_\alpha V_{-\alpha} + \mathfrak{g}_\gamma V_{-\gamma} = V_0$.

The basic idea for the calculations is the following. To every vector $v \in V$ we define its *weight support* $\omega(v) \subset A \cup C \cup \{0_A, 0_C\}$ in the following way. Write v as a sum of weight vectors, $v = \sum_A v_\alpha + \sum_C v_\gamma + v_0$. Then

$$\omega(v) := \{\alpha \in A \mid v_\alpha \neq 0\} \cup \{\gamma \in C \mid v_\gamma \neq 0\} \cup \begin{cases} \emptyset & \text{if } v_0 = 0, \\ \{0_A\} & \text{if } v_0 \in V_A \setminus \{0\}, \\ \{0_C\} & \text{if } v_0 \in V_0 \setminus V_A. \end{cases}$$

This extends in an obvious way to the weight support of elements from $V \oplus V$. Now we look at a pair $v = (v', v'') = (v_{\alpha'} + v_{\gamma'}, v_{\alpha''} + v_{\gamma''}) \in V \oplus V$ where $\alpha', \alpha'' \in A$ and $\gamma', \gamma'' \in C$ are distinct weights. Define

$$\Omega(v) := \{\omega(x_\beta v) \mid \beta \in A \cup B \cup C\} \cup \{\alpha', \alpha'', \gamma', \gamma''\},$$

where $x_\beta \in \mathfrak{g}_\beta$ is a (nonzero) root vector. This is the set of weight supports of generators of $\mathfrak{g}v$ where we use that $\mathfrak{h}v = \mathbb{C}v_{\alpha'} \oplus \mathbb{C}v_{\gamma'} \oplus \mathbb{C}v_{\alpha''} \oplus \mathbb{C}v_{\gamma''}$.

Our problem can now be understood in the following way. We are given a matrix M of column vectors from which we want to calculate the rank. We replace M by the “support matrix” $\Omega(M)$ which is obtained from M by replacing each nonzero entry by 1. How can one find a lower bound for the rank of M from $\Omega(M)$?

There is an obvious procedure. We first look for a column of $\Omega(M)$ which contains a single 1, let us say in row i . Then we remove all other 1’s in row i and repeat this procedure as often as possible to obtain a matrix $\Omega(M)'$. It is clear that this “reduced” matrix $\Omega(M)'$ is again the support matrix of a matrix M' which is obtained by column reduction from M . This first step is called “column reduction.”

Now we apply row reduction to M' which amounts to looking at rows of $\Omega(M)'$ which contain a single 1. Then we delete all other 1’s in the corresponding column. Again we repeat this procedure as often as possible and obtain a matrix $\Omega(M)''$. We call this procedure “row reduction.” It is clear now that a lower bound for the rank of M is given by the number of columns of $\Omega(M)''$ containing a single 1.

Now we choose $v \in V \oplus V$ as above with weights

$$(\alpha', \gamma', \alpha'', \gamma'') = (\varepsilon_3, 1/2(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4), \varepsilon_2, 1/2(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)).$$

We obtain a set $\Omega(v)$ with 45 elements where each element’s weight support has cardinality at most two. (We use Mathematica[®] to perform these and the following calculations.) After applying the “column reduction” we obtain a new set $\Omega(v)'$ that contains 44 elements where 34 of them contain a single weight. For the remaining 10 elements, the “row” reduction produces 10 sets with a single weight. Thus we get $\dim Gv = \dim gv = 44 = 2 \dim V - \dim V//G$, and we are done. \square

Remark A.3. We are grateful to Jan Draisma who did some independent calculations (using GAP) to show that there is a dense orbit in the null cone of $V \oplus V$.

Appendix B Computations for $G_2 \times G_2$

The main result of this appendix is the following proposition. We will give two proofs.

Proposition B.1. *The representation $(\mathbb{C}^7 \otimes \mathbb{C}^7, G_2 \times G_2)$ is not coreduced.*

Proof (First Proof of Proposition B.1). The nontrivial part of the slice representation at the zero weight vector is $G := \mathrm{SL}_3 \times \overline{\mathrm{SL}}_3$ on the four possible versions of $(W := \mathbb{C}^3$ or $W^*)$ tensored with $(\overline{W} := \overline{\mathbb{C}^3}$ or $\overline{W}^*)$. Set $V_1 := W \otimes \overline{W}$, $V_2 := W \otimes \overline{W}^*$, $V_3 := W^* \otimes \overline{W}$ and $V_4 := W^* \otimes \overline{W}^*$.

Lemma B.2. *The G -module*

$$V := V_1 + V_2 + V_3 + V_4$$

is not coreduced.

We have a group N of order 8 that acts on V by interchanging W and W^* , \overline{W} and \overline{W}^* as well as interchanging W, W^* with $\overline{W}, \overline{W}^*$. Then N normalizes the action of G . Here are the steps in the proof of the proposition above.

- (1) We show that there is a minimal generator f of the invariants of (V, G) which is multihomogeneous of degree $(3,3,3,3)$ in the four irreducible subspaces of V .
- (2) We show that, up to the action of the Weyl group and N , there are eight 1-parameter subgroups ρ of G such that the union of the GZ_ρ is $\mathcal{N}(V)$.
- (3) We show that for each such ρ , the differential of f vanishes on Z_ρ .

It then follows from Remark 3.2 that V is not coreduced.

Let $R = \mathbb{C}[a_1, \dots, a_m]$ be a finitely generated \mathbb{N}^d -graded ring where the a_i are homogeneous. Recall that a_1, \dots, a_m are a *regular sequence* in R if a_1 is not a zero divisor and a_{j+1} is not a zero divisor in $R/(Ra_1 + \dots + Ra_j)$, $1 \leq j < m$.

We may write R as a quotient $R = \mathbb{C}[x_1, \dots, x_n]/I$ where the image of x_i in R is a_i , $i = 1, \dots, n$. Let I_s denote the elements of I homogeneous of degree $s := (s_1, \dots, s_d)$, $s_j \in \mathbb{N}$ and let \bar{I} denote $I/(x_1, \dots, x_m)$. We leave the proof of the following lemma to the reader.

Lemma B.3. *Let R , etc. be as above. Then $I_s \rightarrow \bar{I}_s$ is an isomorphism for all $s \in \mathbb{N}^d$.*

The lemma says that we can determine the dimension of the space of relations of the a_i in degree s by first setting a_1, \dots, a_m to zero.

Lemma B.4. *There is a generator f of $R := \mathcal{O}(V)^G$ of multidegree $(3, 3, 3, 3)$.*

We used LiE to compute a partial Poincaré series of R :

$$\begin{aligned} & 1 + (ps + qr) + 2(p^2s^2 + q^2r^2) + (p^3 + q^3 + r^3 + s^3) \\ & \quad + 2(p^3q^3 + p^3r^3 + q^3s^3 + r^3s^3) + 3(p^3s^3 + q^3r^3) \\ & \quad \quad + 2(q^3ps + r^3ps + p^3qr + s^3qr) \\ & \quad \quad + 6(q^3p^2s^2 + r^3p^2s^2 + p^3q^2r^2 + s^3q^2r^2) \\ & \quad \quad + 13(p^3q^3r^3 + p^3q^3s^3 + p^3r^3s^3 + q^3r^3s^3) \\ & \quad + 4pqrs + 10(pq^2r^2s + p^2qrs^2) + 18(pq^3r^3s + p^3qrs^3) \\ & \quad \quad + 37p^2q^2r^2s^2 + 86(p^2q^3r^3s^2 + p^3q^2r^2s^3) + 265p^3q^3r^3s^3. \end{aligned}$$

If there were no relations among the generators of R of degree at most $(3, 3, 3, 3)$, then the Poincaré series would indicate that we have generators in degree (a, b, c, d) of a certain multiplicity which we denote by $\text{gen}(a, b, c, d)$. We list the relevant $\text{gen}(a, b, c, d)$, modulo symmetries given by the group N .

- (1) $\text{gen}(0110) = 1$
- (2) $\text{gen}(0220) = 1$
- (3) $\text{gen}(3000) = 1$
- (4) $\text{gen}(3300) = 1$
- (5) $\text{gen}(0330) = 0$
- (6) $\text{gen}(1111) = 3$
- (7) $\text{gen}(3110) = 1$
- (8) $\text{gen}(3220) = 3$
- (9) $\text{gen}(3330) = 3$
- (10) $\text{gen}(1221) = 5$
- (11) $\text{gen}(1331) = 2$
- (12) $\text{gen}(2222) = 14$
- (13) $\text{gen}(3223) = 13$
- (14) $\text{gen}(3333) = 11$

It is easy to see that the representations $V_i + V_j$, $1 \leq i < j \leq 4$ are cofree. Now $V_2 + V_3$ has generators in degrees (3,0), (0,3), (1,1) and (2,2) while $V_1 + V_2$ has generators in degrees (3,0), (0,3) and (3,3). Thus it is easy to see that we have a regular sequence in R consisting of the (determinant) invariants of degree 3 and those of degree (0110), (0220), (1001) and (2002).

Lemma B.5. *Suppose that we are in one of the cases above, except for (2222), (3223) and (3333). Then R has $\text{gen}(abcd)$ generators in degree $(abcd)$.*

Proof. We set the invariants of our regular sequence equal to zero and see if we have any relations. But then there are no nonlinear polynomials in the remaining generators in the degrees we are worried about. \square

Proof (of Lemma B.4). As usual, we set the elements of our regular sequence equal to zero. This does not change the number of minimal generators of degree (3333). Now how can we have fewer generators than $\text{gen}(3333)$ in degree (3333)? This can only occur if there is a degree $(abcd)$ with an “unexpected” relation such that $(3333) - (abcd)$ is the degree of a generator not in our regular sequence. Thus the only problem could occur because of relations in degree (2222) multiplied by the 3 generators f_1, f_2 and f_3 in degree (1111). Moreover, modulo our regular sequence, the unexpected relations in degree (2222) have the form $(r = \sum_{ij} c_{ij} f_i f_j) = 0$. Thus there are unexpected relations r_1, \dots, r_d , $d \leq 6$. For each relation r_k we add an additional generator y_k in degree (2222) and to get the correct count of non-generators in degree (3333) we have to adjust our formal count by adding $3d$ (from the product of the y_k by the f_i) and subtracting the dimension of the span of the $f_i r_k$ in the polynomial ring $\mathbb{C}[f_1, f_2, f_3]$. But the correction is by less than 11:

Case 1. $d \leq 5$. Then we have a correction of at most $3d - d \leq 10$.

Case 2. $d = 6$. Then the correction is $18 - \dim S^3(\mathbb{C}^3) = 8$. \square

We now have our generator f of degree (3333). Next we need to calculate the irreducible components of the null cone, up to the action of N .

Let ρ be a 1-parameter subgroup of $G := \text{SL}_3 \times \overline{\text{SL}}_3$ whose weights for \mathbb{C}^3 are a, b and c and whose weights for $\overline{\mathbb{C}^3}$ are \bar{a}, \bar{b} and \bar{c} . We have that $a \geq b \geq c$ and similarly for \bar{a} , etc. We also can assume that no weight of V is zero. Of course, many choices of a , etc. will give the same subset Z_ρ in V . We say that a particular choice of a , etc. is a *model* if it gives the correct Z_ρ .

The action of our group N does not change the weights that occur, just in which of the four components they occur. Thus to show that df vanishes on $\mathcal{N}(V)$, we can always reduce to the case that $a > \bar{a}$ and that the other numbers are negative (or zero). For every possibility we will give a model such that df vanishes on Z_ρ .

Lemma B.6. *We have that $c - \bar{b} \leq c - \bar{c} \leq c + \bar{a} \leq b + \bar{a} \geq b - \bar{c} \geq b - \bar{b}$. Moreover, $c - \bar{b} < 0$ and not both $c - \bar{c}$ and $b - \bar{b}$ are positive.*

Proof. The string of inequalities is obvious. If $c - \bar{b} > 0$, then $b - \bar{c} > 0$ and adding we get that $-a + \bar{a} > 0$ which is a contradiction. Similarly, not both $c - \bar{c}$ and $b - \bar{b}$ can be positive. \square

Given the lemma, there are eight possibilities for the signs of $c-\bar{b}, c-\bar{c}, \dots, b-\bar{b}$ which we present in matrix form. In another matrix, we present the values $a, b, c, \bar{a}, \bar{b}$ and \bar{c} of a 1-parameter subgroup ρ which is a model for the signs. Note that the signs tell you exactly which vectors in V are in the positive weight space of ρ .

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 4 & -2 & -2 & 1 & 0 & -1 \\ 8 & -3 & -5 & 4 & -2 & -2 \\ 4 & -1 & -3 & 2 & 0 & -2 \\ 3 & 0 & -3 & 2 & -1 & -1 \\ 6 & -3 & -3 & 4 & -2 & -2 \\ 8 & -3 & -5 & 6 & -2 & -4 \\ 7 & -2 & -5 & 6 & -3 & -3 \\ 4 & -2 & -2 & 3 & 0 & -3 \end{pmatrix}$$

Proof of Lemma B.2. Consider signs which have the 1-parameter subgroup ρ with weights $(8 -3 -5 6 -2 -4)$ as model. Then the largest negative weight occurring in V is -14 while the positive weights occurring in V_1, \dots, V_4 are

$$(1, 3, 4, 6, 14), (1, 2, 10, 12), (1, 1, 3, 9, 11), (7, 7, 5, 9).$$

Now consider a monomial m in the weight vectors which occurs in f . If df does not vanish on Z_ρ then there is a monomial with only one negative weight vector. But the sum of the positive weights occurring in m is at least $3 + 3 + 3 + 2 * 5 = 19$ which is greater than 14. Hence this is impossible and df vanishes on GZ_ρ . One similarly (and more easily) sees that df vanishes on GZ_ρ in the other 7 cases.

This finishes the first proof of Proposition B.1. □

Second Proof of Proposition B.1. The weights of $V = \mathbb{C}^7$ are the short roots of $G := \mathbf{G}_2$ together with 0, and all weight spaces are 1-dimensional. We use the notation $\Lambda := \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), 0\}$ where $\alpha + \beta$ is the highest weight. Thus the weight spaces of $V \otimes V$ are given by the tensor products $V_\mu \otimes V_\nu, (\mu, \nu) \in \Lambda \times \Lambda$.

We first determine the maximal positive subspaces of $W := V \otimes V$, up to the action of the Weyl group. If ρ is a one-parameter subgroup of $G \times G$ we denote by W_ρ the sum of the ρ -positive weight spaces, i.e.,

$$V_\rho := \bigoplus_{\rho(\mu, \nu) > 0} W_{(\mu, \nu)}.$$

The 1-PSG ρ is defined by the values $a := (\rho, (\alpha, 0)), b := (\rho, (\beta, 0)), a' := (\rho, (0, \alpha)), b' := (\rho, (0, \beta))$. Using the action of the Weyl group, we can assume that

$$a, b, a', b' > 0, \quad a \geq b, \quad a' \geq b', \quad \{a, b, a + b\} \cap \{a', b', a' + b'\} = \emptyset.$$

We can also assume that $a > a'$; we will then get the other maximal positive subspaces by the symmetry $(\mu, \nu) \mapsto (\nu, \mu)$. Now V_ρ depends only on the relative

position of the values $a + b > a \geq b$ and the values $a' + b' > a' \geq b'$. It is not difficult to see that there are eight cases.

- (1) $a + b > a > b > a' + b' > a' \geq b'$ represented by $\rho = (5, 4, 2, 1)$;
- (2) $a + b > a > a' + b' > b > a' \geq b'$ represented by $\rho = (6, 4, 3, 2)$;
- (3) $a + b > a' + b' > a \geq b > a' \geq b'$ represented by $\rho = (6, 5, 4, 3)$;
- (4) $a + b > a > a' + b' > a' > b > b'$ represented by $\rho = (5, 2, 3, 1)$;
- (5) $a + b > a' + b' > a > a' > b > b'$ represented by $\rho = (6, 4, 5, 3)$;
- (6) $a + b > a > a' + b' \geq a' > b' \geq b$ represented by $\rho = (7, 1, 4, 2)$;
- (7) $a + b > a' + b' > a > a' > b > b'$ represented by $\rho = (6, 2, 4, 3)$;
- (8) $a' + b' > a + b > a > a' \geq b' > b$ represented by $\rho = (6, 2, 5, 4)$.

To get the full set of maximal positive subspaces we have to add the 8 ρ 's obtained from the list above by replacing (a, b, a', b') with (a', b', a, b) .

Now we used LiE to look at the covariants of type $\theta_1 \otimes V$. The multiplicities of this covariant in degrees 1 to 9 are $(0, 0, 1, 1, 3, 5, 12, 18, 41)$, and the dimensions of the invariants in these degrees are $(0, 1, 1, 3, 2, 8, 7, 17, 19)$. It follows that at most $37 = 1 \cdot 12 + 1 \cdot 5 + 3 \cdot 3 + 2 \cdot 1 + 8 \cdot 1$ covariants of degree 9 are in the ideal generated by the invariants, hence there are generating covariants of this type in degree 9. Now we have to show that for every positive weight space V_ρ the highest weight $(0, \alpha + \beta)$ of $\theta \otimes V$ cannot be expressed as a sum of 9 weights from V_ρ . Because of duality, each V_ρ has dimension $24 = (7 * 7 - 1)/2$. If we denote by Λ_ρ the set of weights of V_ρ , this amounts to prove that the system

$$\sum_{\lambda \in \Lambda_\rho} x_\lambda \lambda = (0, \alpha + \beta), \quad \sum_{\lambda \in \Lambda_\rho} x_\lambda = 9$$

has no solution in nonnegative integers x_λ . Note that the first condition consists in 4 linear equations in 24 variables. Now we used Mathematica to show that there are no solutions for each one of the sixteen maximal positive weight spaces Z_ρ given above.

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