REGULARIZATION OF RATIONAL GROUP ACTIONS

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Abstract. We give a modern proof of the Regularization Theorem of André Weil which says that for every rational action of an algebraic group G on a variety X there exist a variety Y with a regular action of G and a G-equivariant birational map $X \dashrightarrow Y$. Moreover, we show that a rational action of G on an affine variety X with the property that each g from a dense subgroup of G induces a regular automorphism of X, is a regular action.

The aim of this note is to give a modern proof of the following *Regularization Theorem* due to André Weil, see [Wei55]. We will follow the approach in [Zai95]. Our base field $\Bbbk$ is algebraically closed. A *variety* is an algebraic $\Bbbk$-variety, and an *algebraic group* is an algebraic $\Bbbk$-group.

Theorem 1. *Let G be an algebraic group and X a variety with a rational action of G. Then there exists a variety Y with a regular action of G and a birational G-equivariant morphism $\phi\colon X \dashrightarrow Y$.*

We do not assume that G is linear or connected, nor that X is irreducible. This creates some complications in the arguments. The reader is advised to start with the case where G is connected and X irreducible in a first reading.

We cannot expect that the birational map ϕ in the theorem is a morphism. Take the standard Cremona involution σ of $\mathbb{P}^2$, given by $(x:y:z) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$, which collapses the coordinate lines to points. This cannot happen if σ is a regular automorphism. However, removing these lines, we get $\Bbbk^* \times \Bbbk^*$ where σ is a well-defined automorphism.

More generally, consider the rational action of $G := \mathrm{PSL}_2 \times \mathrm{PSL}_2$ on $\mathbb{P}^2$ induced by the birational isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. Then neither an open set carries a regular G-action, nor $\mathbb{P}^2$ can be embedded into a variety Y with a regular G-action.

As we will see in the proof below, one first constructs a suitable open set $U \subseteq X$ where the rational action of G has very specific properties, and then one shows that U can be equivariantly embedded into a variety Y with a regular G-action.

1.1. **Rational maps.** We first have to define and explain the different notion used in the theorem above. We refer to [Bla16] for additional material and more details.

Recall that a *rational map* $\phi\colon X \dashrightarrow Y$ between two varieties X, Y is an equivalence class of pairs (U, ϕ_U) where $U \subseteq X$ is an open dense subset and $\phi_U\colon U \to Y$ a morphism. Two such pairs (U, ϕ_U) and (V, ϕ_V) are equivalent if $\phi_U|_{U\cap V} = \phi_V|_{U\cap V}$. We say that ϕ is *defined in* $x \in X$ if there is a (U, ϕ_U) representing ϕ such that $x \in U$. The set of all these points forms an open dense subset $\mathrm{Dom}(\phi) \subseteq X$ called the *domain of definition of* ϕ. We will shortly say that ϕ *is defined in* x if $x \in \mathrm{Dom}(\phi)$.

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For all (U, ϕ_U) representing $\phi\colon X \dashrightarrow Y$ the closure $\overline{\phi_U(U)} \subseteq Y$ is the same closed subvariety of Y. We will call it the *closed image* of ϕ and denote it by $\overline{\phi(X)}$. The rational map ϕ is called *dominant* if $\overline{\phi(X)} = Y$. It follows that the composition $\psi \circ \phi$ of two rational maps $\phi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow Z$ is a well-defined rational map $\psi \circ \phi\colon X \dashrightarrow Z$ in case ϕ is dominant.

A rational map $\phi\colon X \dashrightarrow Y$ is called *birational* if it is dominant and admits an inverse $\psi\colon Y \dashrightarrow X$, $\psi \circ \phi = \mathrm{id}_X$. It then follows that ψ is also dominant and that $\phi \circ \psi = \mathrm{id}_Y$. Clearly, ψ is well-defined by ϕ, and we shortly write $\psi = \phi^{-1}$. It is easy to see that ϕ is birational if and only if there is a (U, ϕ_U) representing ϕ such that $\phi_U \colon U \hookrightarrow Y$ is an open immersion with a dense image. The set of birational maps $\phi\colon X \dashrightarrow X$ is a group under composition which will be denoted by $\mathrm{Bir}(X)$.

A rational map $\phi\colon X \dashrightarrow Y$ is called *biregular in x* if there is an open neighborhood $U \subseteq \mathrm{Dom}(\phi)$ of x such that $\phi|_U \colon U \hookrightarrow Y$ is an open immersion. It follows that the subset $X' := \{x \in X \mid \phi \text{ is biregular in } x\}$ is open in X, and the induced morphism $\phi\colon X' \hookrightarrow Y$ is an open immersion. This implies the following result.

Lemma 1. *Let $\phi\colon X \dashrightarrow Y$ be a birational map. Then the set*

$$\mathrm{Breg}(\phi) := \{x \in X \mid \phi \textit{ is biregular in } x\}$$

is open and dense in X.

Remark 1. If X is irreducible, a rational dominant map $\phi\colon X \dashrightarrow Y$ defines a $\Bbbk$-linear inclusion $\phi^*\colon \Bbbk(Y) \hookrightarrow \Bbbk(X)$ of fields. Conversely, for every inclusion $\alpha\colon \Bbbk(Y) \hookrightarrow \Bbbk(X)$ of fields there is a unique dominant rational map $\phi\colon X \dashrightarrow Y$ such that $\phi^* = \alpha$. In particular, we have an isomorphism $\mathrm{Bir}(X) \xrightarrow{\sim} \mathrm{Aut}_\Bbbk(\Bbbk(X))$ of groups, given by $\phi \mapsto (\phi^*)^{-1}$.

1.2. **Rational group actions.**

Definition 1. Let X, Z be varieties. A map $\phi\colon Z \to \mathrm{Bir}(X)$ is called a *morphism* if there is an open dense set $U \subseteq Z \times X$ with the following properties:

- (i) The induced map $(z, x) \mapsto \phi(z)(x)\colon U \to X$ is a morphism of varieties.
- (ii) For every $z \in Z$ the open set $U_z := \{x \in X \mid (z, x) \in U\}$ is dense in X.
- (iii) For every $z \in Z$ the birational map $\phi(z)\colon X \dashrightarrow X$ is defined in U_z.

Equivalently, we have a rational map $\Phi\colon Z \times X \to X$ such that, for every $z \in Z$,

- (i) the open subset $\mathrm{Dom}(\Phi) \cap (\{z\} \times X) \subseteq \{z\} \times X$ is dense, and
- (ii) the induced rational map $\Phi_z\colon X \dashrightarrow X$, $x \mapsto \Phi(z, x)$, is birational.

This definition allows to define the ZARISKI-topology on $\mathrm{Bir}(X)$ in the following way.

Definition 2. A subset $S \subseteq \mathrm{Bir}(X)$ is *closed* if for every morphism $\rho\colon Z \to \mathrm{Bir}(X)$ the inverse image $\rho^{-1}(S) \subseteq Z$ is closed.

Now we can define rational group actions on varieties. Let G be an algebraic group and let X be a variety.

Definition 3. A *rational action of G on X* is a morphism $\rho\colon G \to \mathrm{Bir}(X)$ which is a homomorphism of groups.

As we have seen above this means that we have a rational map (denoted by the same letter) $\rho\colon G \times X \dashrightarrow X$ such that the following holds:

(a) $\mathrm{Dom}(\rho) \cap (\{g\} \times X)$ is dense in $\{g\} \times X$ for all $g \in G$,
(b) the induced rational map $\rho_g \colon X \dashrightarrow X$, $x \mapsto \rho(g,x)$, is birational,
(c) the map $g \mapsto \rho_g$ is a homomorphism of groups.

If ρ is defined in (g,x) and $\rho(g,x) = y$ we will say that *$g \cdot x$ is defined and $g \cdot x = y$.*

We will also use the birational map

$$\tilde{\rho} \colon G \times X \dashrightarrow G \times X, \quad (g,x) \mapsto (g, \rho(g,x)),$$

see section 1.5 below.

Remark 2. Note that if $\rho \colon G \times X \dashrightarrow X$ is defined in (g,x), then $\rho_g \colon X \dashrightarrow X$ is defined in x, but the reverse implication does not hold. An example is the following. Consider the regular action of the additive group G_a on the plane $\mathring{A}^2 = \Bbbk^2$ by translation along the x-axis: $s \cdot x := x + (s, 0)$ for $s \in G_a$ and $x \in \mathring{A}^2$. Let $\beta \colon X \to \mathring{A}^2$ be the blow-up of $\mathring{A}^2$ in the origin. Then we get a rational G_a-action on X, $\rho \colon G_a \times X \dashrightarrow X$. It is not difficult to see that ρ is defined in (e, x) if and only if $\beta(x) \neq 0$, i.e. x does not belong to the exceptional fiber, but clearly, $\rho_e = \mathrm{id}$ is defined everywhere.

If $\phi \colon Z \to \mathrm{Bir}(X)$ is a morphism such that $\phi(Z) \subseteq \mathrm{Aut}(X)$, the group of regular automorphisms, one might conjecture that the induced map $Z \times X \to X$ is a morphism. I don't know how to prove this, but maybe the following holds.

Conjecture. *Let $\rho \colon G \to \mathrm{Bir}(X)$ be a rational action. If $\rho(G) \subseteq \mathrm{Aut}(X)$, then ρ is a regular action.*

We can prove this under additional assumptions.

Theorem 2. *Let $\rho \colon G \to \mathrm{Bir}(X)$ be a rational action where X is affine. Assume that there is a dense subgroup $\Gamma \subseteq G$ such that $\rho(\Gamma) \subseteq \mathrm{Aut}(X)$. Then the G-action on X is regular.*

The proof will be given in the last section 1.9.

Definition 4. Given rational G-actions ρ on X and μ on Y, a dominant rational map $\phi \colon X \to Y$ is called *G-equivariant* if the following holds:

(Equi) For every $(g,x) \in G \times X$ such that (1) ρ is defined in (g,x), (2) ϕ is defined in x and in $\rho(g,x)$, and (3) μ is defined in $(g, \phi(x))$, we have $\phi(\rho(g,x)) = \mu(g, \phi(x))$.

Note that the set of $(g,x) \in G \times X$ satisfying the assumptions of (Equi) is open and dense in $G \times X$ and has the property that it meets all $\{g\} \times X$ in a dense open set.

Remark 3. If G acts rationally on X and if $X' \subseteq X$ is an nonempty open subset, then G acts rationally on X', and the inclusion $X' \hookrightarrow X$ is G-equivariant. Moreover, if G acts rationally on X and if $\phi \colon X \dashrightarrow Y$ is a birational map, then there is uniquely define rational action of G on Y such that ϕ is G-equivariant.

Note that for a rational G-action ρ on X and an open dense set $X' \subseteq X$ with induced rational G-action ρ' we have

$$\mathrm{Dom}(\rho') = \{(g,x) \in \mathrm{Dom}(\rho) \mid x \in X' \text{ and } g \cdot x \in X'\},$$
$$\mathrm{Breg}(\tilde{\rho}') = \{(g,x) \in \mathrm{Breg}(\tilde{\rho}) \mid x \in X' \text{ and } g \cdot x \in X'\}.$$

1.3. **The case of a finite group G.** Assume that G is finite and acts rationally on an irreducible variety X. Then every $g \in G$ defines a birational map $g\colon X \dashrightarrow X$ and thus an automorphism g^* of the field $\Bbbk(X)$ of rational functions on X. In this way we obtain a homomorphism $G \to \operatorname{Aut}_{\Bbbk}(\Bbbk(X))$ given by $g \mapsto (g^*)^{-1}$.

By Remark 3 above we may assume that X is affine. Hence $\Bbbk(X)$ is the field of fractions of the coordinate ring $\mathcal{O}(X)$. Since G is finite we can find a finite-dimensional $\Bbbk$-linear subspace $V \subseteq \Bbbk(X)$ which is G-stable and contains a system of generators of $\mathcal{O}(X)$.

Denote by $R \subseteq \Bbbk(X)$ the subalgebra generated by V. By construction,

(a) R is finitely generated and G-stable, and
(b) R contains $\mathcal{O}(X)$.

In particular, the field of fractions of R is $\Bbbk(X)$. If we denote by Y the affine variety with coordinate ring R, we obtain a regular action of G on Y and a birational morphism $\psi\colon Y \to X$ induced by the inclusion $\mathcal{O}(X) \subseteq R$. Now the Regularization Theorem follows in this case with $\phi := \psi^{-1}\colon X \dashrightarrow Y$.

There is a different way to construct a "model" with a regular G-action, without assuming that X is irreducible. In fact, there is always an open dense set $X_{\text{reg}} \subseteq X$ where the action is regular. It is defined in the following way (cf. Definition 5 below). For $g \in G$ denote by $X_g \subseteq X$ the open dense set where the rational map $\rho_g\colon x \mapsto g \cdot x$ is biregular. Then $X_{\text{reg}} := \bigcap_{g\in G} X_g$ is open and dense in X and the rational G-action on X_{reg} is regular. In fact, ρ_g is biregular on X_{reg}, hence also biregular on $h \cdot X_{\text{reg}}$ for all $h \in G$ which implies that $h \cdot X_{\text{reg}} \subseteq X_{\text{reg}}$.

1.4. **A basic example.** We now give an example which should help to understand the constructions and the proofs below. Let X be a variety with a regular action of an algebraic group G. Choose an open dense subset $U \subseteq X$ and consider the rational G-action on U. Then $\tilde{X} := \bigcup_{g\in G} gU \subseteq X$ is open and dense in X and carries a regular action of G.

The rational G-action ρ on U is rather special. First of all we see that ρ is defined in (g, u) if and only if $g \cdot u \in U$. This implies that ρ is defined in (g, u) if and only of ρ_g is defined in u. Next we see that if ρ is defined in (g, u), then $\tilde{\rho}\colon G \times U \dashrightarrow G \times U$, $(g, x) \mapsto (g, \rho(g, x))$, is biregular in (g, u). And finally, for any x the (open) set of elements $g \in G$ such that $\tilde{\rho}$ is biregular in (g, x) is dense in G.

A first and major step in the proof is to show (see section 1.5) that for every rational G-action on a variety X there is an open dense subset $X_{\text{reg}} \subseteq X$ with the property that for every $x \in X_{\text{reg}}$ the rational map $\tilde{\rho}\colon G \times X_{\text{reg}} \dashrightarrow G \times X_{\text{reg}}$ is biregular in (g, x) for all g in a dense (open) set of G. Then, in a second step in section 1.6, we construct from X_{reg} a variety Y with a regular G-action together with an open G-equivariant embedding $X_{\text{reg}} \hookrightarrow Y$.

1.5. **G-regular points and their properties.** Let X be a variety with a rational action $\rho\colon G \times X \dashrightarrow X$ of an algebraic group G. Define

$$\tilde{\rho}\colon G \times X \dashrightarrow G \times X, \quad (g, x) \mapsto (g, \rho(g, x)).$$

It is clear that $\operatorname{Dom}(\tilde{\rho}) = \operatorname{Dom}(\rho)$ and that $\tilde{\rho}$ is birational with inverse $\tilde{\rho}^{-1}(g, y) = (g, \rho(g^{-1}, y))$, i.e. $\tilde{\rho}^{-1} = \tau \circ \tilde{\rho} \circ \tau$ where $\tau\colon G \times X \xrightarrow{\sim} G \times X$ is the isomorphism $(g, x) \mapsto (g^{-1}, x)$.

The following definition is crucial.

Definition 5. A point $x \in X$ is called *G-regular* for the rational G-action ρ on X if $\operatorname{Breg}(\tilde{\rho}) \cap (G \times \{x\})$ is dense in $G \times \{x\}$, i.e. $\tilde{\rho}$ is biregular in (g, x) for all g in a dense (open) set of G. We denote by $X_{\text{reg}} \subseteq X$ the set of G-regular points.

Let $\lambda_g \colon G \xrightarrow{\sim} G$ be the left multiplication with $g \in G$. For every $h \in G$ we have the following commutative diagram

$$\begin{array}{ccc} G \times X & \overset{\tilde{\rho}}{\dashrightarrow} & G \times X \\ {\scriptstyle \lambda_h \times \mathrm{id}}\downarrow \simeq & & \vdots\,{\scriptstyle \lambda_h \times \rho_h} \\ G \times X & \overset{\tilde{\rho}}{\dashrightarrow} & G \times X \end{array}$$

This implies the following result.

Lemma 2. *With the notation above we have the following:*

(a) *If ρ is defined in (g, x) and ρ_h defined in $g \cdot x$, then ρ is defined in (hg, x).*
(b) *If $\tilde{\rho}$ is biregular in (g, x) and ρ_h biregular in $g \cdot x$, then $\tilde{\rho}$ is biregular in (hg, x).*

The main proposition is the following.

Proposition 1. (a) *X_{reg} is open and dense in X.*
(b) *If $x \in X_{\text{reg}}$ and if $\tilde{\rho}$ is biregular in (g, x), then $g \cdot x \in X_{\text{reg}}$.*

Proof. (a) Let $G = G_0 \cup G_1 \cup \cdots \cup G_n$ be the decomposition into connected components. Then $D_i := \operatorname{Breg}(\rho) \cap (G_i \times X)$ is open and dense for all i (Lemma 1), and the same holds for the image $\bar{D}_i \subseteq X$ under the projection onto X. Since $X_{\text{reg}} = \bigcap_i \bar{D}_i$, the claim follows.

(b) If $\tilde{\rho}$ is biregular in (g, x), then $\tilde{\rho}^{-1} = \tau \circ \tilde{\rho} \circ \tau$ is biregular in $(g, g \cdot x)$, hence $\tilde{\rho}$ is biregular in $\tau(g, g \cdot x) = (g^{-1}, g \cdot x)$. If x is G-regular, then ρ_h is biregular in x for all h from a dense open set $G' \subseteq G$. Now Lemma 2(b) implies that $\tilde{\rho}$ is biregular in $(hg^{-1}, g \cdot x)$ for all $h \in G'$, hence $g \cdot x \in X_{\text{reg}}$. ☐

Note that for an open dense set $U \subseteq X$ a point $x \in U$ might be G-regular for the rational G-action on X, but not for the rational G-action on U. However, Proposition 1(b) implies the following result.

Corollary 1. *For the rational G-action on X_{reg} every point is G-regular.*

This allows to reduce to the case of a rational G-action such every point is G-regular.

Lemma 3. *Assume that $X = X_{\text{reg}}$. If ρ_g is defined in x, then ρ_g is biregular in x.*

Proof. Assume that ρ_g is defined in $x \in X$. There is an open dense subset $G' \subseteq G$ such ρ_h is biregular in $g \cdot x$ and ρ_{hg} is biregular in x for all $h \in G'$. Since $\rho_{hg} = \rho_h \circ \rho_g$ we see that ρ_g is biregular in x. ☐

For a rational map $\phi \colon X \dashrightarrow Y$ the *graph* $\Gamma(\phi)$ is defined in the usual way:

$$\Gamma(\phi) := \{(x, y) \in X \times Y \mid \phi \text{ is defined in } x \text{ and } \phi(x) = y\}.$$

In particular, $\operatorname{pr}_X(\Gamma(\phi)) = \operatorname{Dom}(\phi)$ and $\operatorname{pr}_Y(\Gamma(\phi)) = \phi(\operatorname{Dom}(\phi))$.

The next lemma will play a central rôle in the construction of the regularization.

Lemma 4. *Consider a rational G-action ρ on a variety X and assume that every point of X is G-regular. Then, for every $g \in G$, the graph $\Gamma(\rho_g)$ is closed in $X \times X$.*

Proof. Let $\Gamma := \overline{\Gamma(\rho_g)}$ be the closure of the graph of ρ_g in $X \times X$. We have to show that for every $(x_0, y_0) \in \Gamma$ the rational map ρ_g is defined in x_0, or, equivalently, that the morphism $\pi_1 := \mathrm{pr}_1|_\Gamma \colon \Gamma \to X$ induced by the first projection is biregular in (x_0, y_0).

Choose $h \in G$ such that ρ_{hg} is biregular in x_0 and ρ_h is biregular in y_0, and consider the induced birational map $\Phi := (\rho_{hg} \times \rho_h) \colon X \times X \dashrightarrow X \times X$. If Φ is defined in $(x, y) \in \Gamma(\rho_g)$, $y := g \cdot x$, then $\Phi(x, y) = ((hg) \cdot x, (hg) \cdot x) \in \Delta(X)$ where $\Delta(X) := \{(x, x) \in X \times X \mid x \in X\}$ is the diagonal. It follows that $\overline{\Phi(\Gamma)} \subseteq \Delta(X)$.

$$\begin{array}{ccc}
X \times X & \overset{\rho_{hg} \times \rho_h}{\dashrightarrow} & X \times X \\
\uparrow \subseteq & & \uparrow \subseteq \\
\Gamma & \overset{\phi}{\dashrightarrow} & \Delta(X) \\
\pi_1 \downarrow & & \mathrm{pr}_1 \downarrow \simeq \\
X & \overset{\rho_{hg}}{\dashrightarrow} & X
\end{array}$$

Since Φ is biregular in (x_0, y_0), we see that $\phi := \Phi|_\Gamma \colon \Gamma \dashrightarrow \Delta(X)$ is also biregular in (x_0, y_0). By construction, we have $\rho_{hg} \circ \pi_1 = \mathrm{pr}_1 \circ \phi$. Since ρ_{hg} is biregular in $\pi_1(x_0, y_0)$ and ϕ is biregular in (x_0, y_0) (and $\mathrm{pr}_1|_{\Delta(X)}$ is an isomorphism) it follows that π_1 is biregular in (x_0, y_0), hence the claim. □

The last lemma is easy.

Lemma 5. *Consider a rational action ρ of G on a variety X. Assume that there is a dense open set $U \subseteq X$ such that $\tilde{\rho}$ defines an open immersion $\tilde{\rho} \colon G \times U \hookrightarrow G \times X$. Then the open dense subset $Y := \bigcup_g g \cdot U \subseteq X$ carries a regular G-action.*

Proof. It is clear that every ρ_g induces an isomorphism $U \xrightarrow{\sim} g \cdot U$. This implies that Y is stable under all ρ_g. It remains to see that the induced map $G \times Y \to Y$ is a morphism. By assumption, this is clear on $G \times U$, hence also on $G \times g \cdot U$ for all $g \in G$, and we are done. □

1.6. **The construction of a regular model.** In view of Corollary 1 our Theorem 1 will follow from the next result.

Theorem 3. *Let X be a variety with a rational action of G. Assume that every point of X is G-regular. Then there is a variety Y with a regular G-action and a G-equivariant open immersion $X \hookrightarrow Y$.*

From now on X is a variety with a rational G-action ρ such that $X_{\mathrm{reg}} = X$. Let $S := \{g_0 := e, g_1, g_2, \ldots, g_m\} \subseteq G$ be a finite subset. These g_i's will be carefully chosen in the proof of Theorem 2 below. Let $X^{(0)}, X^{(1)}, \ldots, X^{(m)}$ be copies of the variety X. On the disjoint union $X(S) := X^{(0)} \cup X^{(1)} \cup \cdots \cup X^{(m)}$ we define the following relations between elements $x_i, x_i' \in \Xi$:

(1) For any i: $x_i \sim x_i' \iff x_i = x_i'$;

(2) For $i \neq j$: $x_i \sim x_j \iff \rho_{g_j^{-1} g_i}$ is defined in x_i and sends x_i to x_j.

It is not difficult to see that this defines an equivalence relation. (For the symmetry one has to use Lemma 3.) Denote by $\tilde{X}(S) := X(S)/\sim$ the set of equivalence classes endowed with the induced topology.

Lemma 6. *The maps $\iota_i\colon \Xi \to \tilde{X}(S)$ are open immersions and endow $\tilde{X}(S)$ with the structure of a variety.*

Proof. By definition of the equivalence relation and the quotient topology the natural maps $\iota_i\colon \Xi \to \tilde{X}(S)$ are injective and continuous. Denote the image by $\tilde{X}^{(i)}$. We have to show that $\tilde{X}^{(i)}$ is open in $\tilde{X}(S)$, or, equivalently, that the inverse image of $\tilde{X}^{(i)}$ in $X(S)$ is open. This is clear, because the inverse image in Ξ of the intersection $\tilde{X}^{(i)} \cap \tilde{X}^{(j)}$ is the open set of points where $\rho_{g_j^{-1}g_i}$ is defined.

It follows that $\tilde{X}(S)$ carries a unique structure of a prevariety such that the maps $\iota_i\colon \Xi \hookrightarrow \tilde{X}(S)$ are open immersions. It remains to see that the diagonal $\Delta(\tilde{X}(S)) \subseteq \tilde{X}(S) \times \tilde{X}(S)$ is closed. For this it suffices to show that $\Delta_{ij} := \Delta(\tilde{X}(S)) \cap (\tilde{X}^{(i)} \times \tilde{X}^{(j)})$ is closed in $\tilde{X}^{(i)} \times \tilde{X}^{(j)}$ for all i, j. This follows from Lemma 4, because Δ_{ij} is the image of $\Gamma(\rho_{g_j^{-1}g_i}) \subseteq \Xi \times X^{(j)}$. In fact, for $x_i \in \Xi$ and $x_j \in X^{(j)}$ we have $(\bar{x}_i, \bar{x}_j) \in \Delta_{ij}$ if and only if $x_i \sim x_j$. This means that $\rho_{g_j^{-1}g_i}$ is defined in x_i and $\rho_{g_j^{-1}g_i}(x_i) = x_j$, i.e. $(x_i, x_j) \in \Gamma(\rho_{g_j^{-1}g_i})$. □

Fixing the open immersion $\iota_0\colon X = X^{(0)} \hookrightarrow \tilde{X}(S)$ we obtain a rational G-action $\bar{\rho} = \bar{\rho}_S$ on $\tilde{X}(S)$ such that ι_0 is G-equivariant (Remark 3). If we consider each Ξ as the variety X with the rational G-action $\rho^{(i)}(g, x) := \rho(g_i g g_i^{-1}, x)$, then, by construction of $\tilde{X}(S)$, the open immersions $\iota_i\colon \Xi \hookrightarrow \tilde{X}(S)$ are all G-equivariant.

Lemma 7. *For all i, the rational map $\bar{\rho}_{g_i}$ is defined on $\tilde{X}^{(0)}$ and defines an isomorphism $\bar{\rho}_{g_i}\colon \tilde{X}^{(0)} \xrightarrow{\sim} \tilde{X}^{(i)}$.*

Proof. Consider the open immersion $\tau_i := \iota_i \circ \iota_0^{-1}\colon \tilde{X}^{(0)} \hookrightarrow \tilde{X}(S)$ with image $\tilde{X}^{(i)}$. We claim that $\tau_i(\bar{x}) = g_i \cdot \bar{x}$. It suffices to show that this holds on an open dense set of $\tilde{X}^{(0)}$. Let $U \subseteq X$ be the open dense set where $g_i \cdot x$ is defined. For $x \in U$ and $y := g_i \cdot x \in X$ we get, by definition, $\iota_0(y) = \iota_i(x)$. On the other hand, $\iota_0(y) = \iota_0(g_i \cdot x) = g_i \cdot \iota_0(x)$. Hence, $g_i \cdot \iota_0(x) = \iota_i(x)$, and so $\tau_i(\bar{x}) = g_i \cdot \bar{x}$ for all $\bar{x} \in \iota_0(U)$. □

Proof of Theorem 3. (a) Since $X_{\text{reg}} = X$, we see that for any $x \in X$ there is a $g \in G$ such that $(g, x) \in D$, hence $\bigcup_g gD = G \times X$ where G acts on $G \times X$ by left-multiplication on G. As a consequence, we have $\bigcup_i g_i D = G \times X$ for a suitable finite subset $S = \{g_0 = e, g_1, \ldots, g_m\} \subseteq G$. This set S will be used to construct $\tilde{X}(S)$.

(b) Let $D^{(0)} \subseteq G \times \tilde{X}^{(0)}$ be the image of D, and consider the rational map $\tilde{\rho}_S\colon G \times \tilde{X}^{(0)} \dashrightarrow G \times \tilde{X}(S)$, $(g, \bar{x}) \mapsto (g, \bar{\rho}(g, \bar{x}))$. We claim that $\tilde{\rho}_S$ is biregular. In fact, for any i, the map $(g, x) \mapsto (g, g \cdot x)$ is the composition of $(g, x) \mapsto (g, (g_i^{-1}g)x)$ and $(g, y) \mapsto (g, g_i y)$ where the first one is biregular on $g_i D^{(0)}$ with image in $G \times \tilde{X}^{(0)}$, and the second is biregular on $G \times \tilde{X}^{(0)}$, by Lemma 7. Now the claim follows, because $G \times \tilde{X}^{(0)} = \bigcup_i g_i D^{(0)}$, by (a).

(c) It follows from (b) that the rational action $\bar{\rho}$ of G on $\tilde{X}(S)$ has the property, that $\tilde{\rho}_S$ defines an open immersion $G \times \tilde{X}^{(0)} \hookrightarrow G \times \tilde{X}(S)$. Now Theorem 3 follows from Lemma 5, setting $Y := \tilde{X}(S)$. □

1.7. **Normal and smooth models.** If X is an irreducible G-variety, i.e. a variety with a regular action of G, then it is well-known that the normalization $\tilde{X}$ has a unique structure of a G-variety such that the normalization map $\eta\colon \tilde{X} \to X$ is G-equivariant. If X is reducible, $X = \bigcup_i X_i$, we denote by $\tilde{X}$ the disjoint union of the normalizations of the irreducible components X_i, $\tilde{X} = \dot{\bigcup}_i \tilde{X}_i$, and by $\eta\colon \tilde{X} \to X$ the obvious morphism which will be called the *normalization of* X. The proof of the following assertion is not difficult.

Proposition 2. *Let X be a G-variety and $\eta\colon \tilde{X} \to X$ its normalization. Then there is a unique regular G-action on $\tilde{X}$ such that η is G-equivariant.*

It is clear that for any G-variety X the open set X_{smooth} of smooth points is stable under G. Thus smooth models for a rational G-action always exist.

The next result, the *equivariant resolution of singularities*, can be found in Kollár's book [Kol07]. He shows in Theorem 3.36 that in characteristic zero there is a functorial resolution of singularities $\mathcal{BR}(X)\colon X' \to X$ which commutes with surjective smooth morphisms. This implies (see his Proposition 3.9.1) that every action of an algebraic group on X lifts uniquely to an action on X'.

Proposition 3. *Assume $\operatorname{char} \Bbbk = 0$, and let X be a G-variety. Then there is a smooth G-variety Y and a proper birational G-equivariant morphism $\phi\colon Y \to X$.*

1.8. **Projective models.** The next results show that there are always smooth projective models for connected algebraic groups G. More precisely, we have the following propositions.

Proposition 4. *Let G be a connected algebraic group acting on a normal variety X. Then there exists an open cover of X by quasi-projective G-stable varieties.*

Proposition 5. *Let G be a connected algebraic group acting on a normal quasi-projective variety X. Then there exists a G-equivariant embedding into a projective G-variety.*

Outline of Proofs. Both propositions are due to Sumihiro in case of a connected linear algebraic group G [Sum74, Sum75]. They were generalized to a connected algebraic group G by Brion in [Bri10, Theorem 1.1 and Theorem 1.2]. □

In this context let us mention the following *equivariant* Chow-*Lemma*. For a connected linear algebraic group G it was proved by Sumihiro [Sum74] and later generalized to the non-connected case by Reichstein-Youssin [RY02]. It implies that projective models always exist for linear algebraic groups G.

Proposition 6 ([Sum74, Theorem 2], [RY02, Proposition 2])**.** *Let G be a linear algebraic group. For every G-variety X there exists a quasi-projective G-variety Y and a proper birational G-equivariant morphism $Y \to X$ which is an isomorphism on a G-stable open dense subset $U \subseteq Y$.*

1.9. **Proof of Theorem 2.** We start with a rational action $\rho\colon G \to \operatorname{Bir}(X)$ of an algebraic group G on a variety X, and we assume that there is a dense subgroup $\Gamma \subseteq G$ such that $\rho(\Gamma) \subseteq \operatorname{Aut}(X)$.

(a) We first claim that the rational G-action on the open dense set $X_{\text{reg}} \subseteq X$ is regular. For every $x \in X_{\text{reg}}$ there is a $g \in \Gamma$ such that $\tilde{\rho}$ is biregular in (g, x). Since, by assumption, the ρ_h are biregular on X for all $h \in \Gamma$ it follows from Lemma 2(b) that $\tilde{\rho}$ is biregular in (g', x) for any $g' \in \Gamma$. Moreover, by Proposition 1(b), we have $g' \cdot x \in X_{\text{reg}}$, hence X_{reg} is stable under Γ.

By Theorem 3 we have a G-equivariant open immersion $X_{\text{reg}} \hookrightarrow Y$ where Y is a variety with a regular G-action. Since the complement $C := Y \setminus X_{\text{reg}}$ is closed and Γ-stable we see that C is stable under $\bar{\Gamma} = G$, hence the claim.

(b) From (a) we see that the rational map $\rho\colon G \times X \dashrightarrow X$ has the following properties:

(i) There is a dense open set $X_{\text{reg}} \subseteq X$ such that ρ is regular on $G \times X_{\text{reg}}$.
(ii) For every $g \in \Gamma$ the rational map $\rho_g\colon X \to X$, $x \mapsto \rho(g, x)$, is a regular isomorphism.

Now the following lemma implies that ρ is a regular action in case X is affine, proving Theorem 2. □

Lemma 8. *Let X, Y, Z be varieties and let $\phi\colon X \times Y \dashrightarrow Z$ be a rational map where Z is affine. Assume the following:*

(a) *There is an open dense set $U \subseteq Y$ such that ϕ is defined on $X \times U$;*
(b) *There is a dense set $X' \subseteq X$ such that the induced maps $\phi_x\colon \{x\} \times Y \to Z$ are morphisms for all $x \in X'$*

Then ϕ is a regular morphism.

Proof. We can assume that $Z = \mathring{A}^1$, so that $\phi = F$ is a rational function on $X \times Y$. We can also assume that X, Y are affine and that $U = Y_f$ with a non-zero divisor $f \in \mathcal{O}(Y)$. This implies that $f^k F \in \mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$ for some $k \geq 0$. Write $f^k F = \sum_{i=1}^n h_i \otimes f_i$ with $\Bbbk$-linearly independent $h_1, \ldots, h_n \in \mathcal{O}(X)$. Setting $F_x(y) := f(x, y)$ for $x \in X$, the assumption implies that $F_x = \sum_{i=1}^n h_i(x) \frac{f_i}{f^k}$ is a regular function on Y for all $x \in X'$.

We claim that there exist $x_1, \ldots, x_n \in X'$ such that the $n \times n$-matrix $(h_i(x_j))_{i,j=1}^n$ is invertible. This implies that the rational functions $\frac{f_i}{f^k}$ are $\Bbbk$-linear combinations of the $F_{x_i} = f(x_i, y) \in \mathcal{O}(Y)$. Hence they are regular, and thus F is regular. The lemma follows.

It remains to prove the claim. Assume that we have found $x_1, \ldots, x_m \in X'$ ($m < n$) such that the $m \times m$-matrix $(h_i(x_j))_{i,j=1}^m$ is invertible. Then there are uniquely defined $\lambda_1, \ldots, \lambda_m \in \Bbbk$ such that $h_{m+1}(x_i) = \sum_{j=1}^m \lambda_j h_j(x_i)$ for $i = 1, \ldots, m$. Since $h_1, \ldots, h_m, h_{m+1}$ are linearly independent, it follows that $h_{m+1} \neq \sum_{j=1}^m \lambda_j h_j$. This implies that there exists $x_{m+1} \in X'$ such that $h_{m+1}(x_{m+1}) \neq \sum_{j=1}^m \lambda_j h_j(x_{m+1})$, and so the matrix $(h_i(x_j))_{i,j=1}^{m+1}$ is invertible. Now the claim follows by induction. □

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