Gödel's Incompleteness Theorem in its Historical Context

Seminar: Selbstbezüglichkeit

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Abstract

Gödel's proof that apparently all reasonable formal systems which can model simple arithmetic are either inconsistent or incomplete is both of enormous importance and a masterful display of proof techniques. Since the theorems have such widereaching consequences and appeal, it is a worthwhile undertaking to present a sketch of the essential parts of his paper in a way that is more easily understandable and to put the results in their historical context. This paper is aimed at bachelor or master level students of Computer Science specifically, but we hope it may prove useful to other audiences as well.

1 Introduction

In this paper we will present a moderately detailed account of Gödel's incompleteness theorems. After providing some historical and mathematical context, we will cover much of the proof of the first incompleteness theorem. At the end of the paper we will briefly discuss the impact of Gödel's work.

2 What Lead to Gödel's Paper

In 1878, the mathematician Georg Cantor proposed the *Continuum Hypothesis* (CH). CH states that there is no set whose cardinality is strictly between that of the integers and the real numbers (Dauben, 1979). Cantor would not see his hypothesis resolved within his lifetime. His fruitless attempts to prove his own hypothesis caused him much anxiety (Dauben, 1979, p. 248).

As Gödel (1931) alludes to in his introduction, the decades before this paper were published were marked by a strive towards greater exactness in mathematics. The goal was to discover all important unsolved problems like the Continuum Hypothesis, and solve them. There are three mathematicians who produced seminal work during these few decades which had a considerable direct influence on Gödel's paper.

Each work we will discuss —including Gödel's —used its own notation, many of which have since fallen out of use or will seem foreign to the target audience of this paper. We naturally attempt to unify the notation into a more familiar and modern one. We lean heavily on a translation of Gödel's paper from German to English (Hirzel, 2000) in this regard.

2.1 Peano's Arithmetic

In 1889, Giuseppe Peano published his book *The principles of arithmetic, presented by a new method* (Peano, 1889) in which he detailed an axiomization of mathematics. (Van Heijenoort, 1967, p.83-97) His aim was to create a formal system which models everything of interest to mathematics, ranging from natural numbers to sets, fractions and functions. He also built upon previous attempts to formalize logic and innovatively separated logical and mathematical operators. Much of his work is no longer considered relevant today and was criticized by his contemporaries because his system lacked rules of derivation and his proofs instead relied on an intuitive understanding of arithmetic logical operations. Despite its shortcomings, two things Peano introduced proved to be incredibly influential. The first was much of the notation he used[1]. More importantly, his axiomization of natural numbers, which consisted of nine axioms —or fundamental assumptions —and treated numbers as individuals and the successor relation as an operator whose semantics follow from the axioms, proved especially robust. Four of the axioms are no longer considered to be axioms of natural numbers and rather of the underlying logic and two further

[1] Though, ironically, we will not be using his notation in this paper.

axioms are not necessary for our purposes since —unlike in Peano's book —natural numbers will be the only individuals we deal with. The remaining three put into words are as follows:

1. No natural number has 0 as its successor.[2]
2. If the successors of two numbers are equal, so are the two numbers.
3. If a set contains 0 and the set containing a number implies that the set also contains its successor, this set contains all natural numbers.

The last axiom is sometimes referred to as the axiom of induction since it allows proofs by induction over the natural numbers.

At the time there was somewhat of a consensus that Peano's Axioms successfully modelled all there is to natural numbers. Among the proponents was prominent mathematician and logician Bertrand Russell (Weitz, 1952, p. 137).

2.2 Russell's Principia Mathematica

In 1903, Bertrand Russell published his book *The principles of Mathematics* (Russell, 1903) in which he reported on the advances made in mathematics by Cantor and Peano among others and presented his now famous paradox which he had discovered two years prior. The paradox can be paraphrased as: "Consider the set which contains all sets which do not contain themselves. The inclusion of this set in itself implies its exclusion and vice versa." The implication of this paradox is that any system which allows the definition of such a set (or an equivalent construct) is necessarily *inconsistent* or *incomplete*. That is to say, there exists a statement such that the statement and its negation are both provable or unprovable respectively.

Russell set out to resolve this paradox by developing a *theory of types* which enforces a strict type on sets and only allows sets to contains sets with a type lower than its own. Along with some other newly introduced concepts, this was published in a three volume work, *Principia Mathematica.*(Whitehead and Russell, 1910, 1912, 1913) In it Russell defines the formal system PM, which aims to be a minimal axiomization of all of mathematics. PM was a significant improvement over Peano's axiomization since it had fewer unnecessary axioms and included *rules of derivation* which provided a well defined way to prove statements. The hope was that removing intuition from mathematical proofs would also provide a principled way to deal with paradoxes. The minimal axiomization came at the cost of brevity, explaining the 1998 page count. Infamously, the proof for $1 + 1 = 2$ appears only hundreds of pages into the first volume.

Later in this paper, we will introduce *system P*, which Gödel used in his original paper and is a subset of PM.

[2] Peano used 1 as the first natural number, not 0.

2.3 Hilbert's Problems

With several approaches to alleviate Russell's paradox and related paradoxes having been developed and several promising attempts at formalizing all of mathematics being worked on, there was a sense that mathematics was almost done. All that would be needed is a few more brilliant ideas to identify axioms that need to be added to or removed from systems. After that mathematics would become grunt work, requiring only to methodically apply rule of derivation to prove all interesting statements about mathematics. It was in this spirit that David Hilbert had published 23 open questions in mathematics a few years earlier. (Hilbert, 1902)
The problems ranged from very specific and precise (18.b: What is the densest sphere packing?) to very vague (23: Further development of the calculus of variations.). The first problem on Hilbert's list was unsurprisingly the Continuum Hypothesis. The second is of special interest to us, it simply requests to prove that the axioms of arithmetic are consistent.

3 Gödel's Undecidability Theorem

As a response to Hilbert's second problem, Gödel (1931) showed that a wide class of formal systems, including any reasonable extension of system P and ZF —another popular formal system at the time —are necessarily either incomplete or inconsistent. (Gödel's original claim was actually slightly weaker, but Rosser (1936) improved Gödel's theorem resulting in the form as it is most often expressed today.) Furthermore, he also showed that any such system cannot prove its own consistency or the consistency of any system that is more powerful than it.

In this section, we will provide an overview of Gödel's proof for his first incompleteness theorem for system P and its extensions specifically. We will not aim to achieve the same rigor and completeness as Gödel, but rather to give the reader a feel for the tactics employed in the proof as well as the confidence that they understand how the gaps in this overview could be filled to formulate the entire proof.

3.1 System P

For the purposes of Gödel's proof, he dealt with a subset of PM combined with the Peano axioms, which we will refer to as *system P*. The fundamental, undefined symbols of system P are:

- 0 (zero), S (the successor function), $\neg$ (negation), $\vee$ (disjunction), $\forall$ (for all), (,)
- x_n^m with $m, n \geq 1$ (a variable of type n)

We permit ourselves the following shortcuts for variables: x_i, y_i, $\ldots$ as a stand-in for any unspecified variable of type i and x, y, $\ldots$ as any unspecified

variable of type 1.
The notion of a *sign of type n* is defined as follows. 0 is a sign of type 1 and x_n^m is a sign of type n. If p is a sign of type 1, then so is Sp. An *elementary formula* has the form $p(q)$ (p contains q) where p is a sign of type $n+1$ and q a sign of type n. The set of *well-formed formulas* in system P is the smallest set which contains all elementary formulas and contains $\neg(\phi)$, $(\phi) \vee (\psi)$ and $\forall x(\phi)$ if ϕ and ψ are well-formed formulas and x is a variable. We will leave away parentheses following the common conventions. For quantifiers we shall write $\forall x :$ when we do. We also introduce the following abbreviations:

- $\phi \wedge \psi \coloneqq \neg(\neg\phi \vee \neg\psi)$
- $\phi \longrightarrow \psi \coloneqq \neg\phi \vee \psi$
- $\phi \longleftrightarrow \psi \coloneqq (\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \phi)$
- $\exists x : \phi \coloneqq \neg\forall x : \neg\phi$
- $x_n = y_n \coloneqq \forall x_{n+1}(x_{n+1}(x_n) \longleftrightarrow x_{n+1}(y_n))$

System P has the following axioms:

- The Peano axioms.
  1. $\neg(Sx = 0)$,
  2. $Sx = Sy \longrightarrow x = y$,
  3. $(x_2(0) \wedge \forall x : x_2(x) \longrightarrow x_2(Sx)) \longrightarrow \forall y : x_2(y)$.
- The proposition axioms.
  Given any well-formed formulas ϕ, ψ and χ the following are axioms:
  1. $\phi \vee \phi \longrightarrow \phi$
  2. $\phi \vee \psi \longrightarrow \phi$
  3. $\phi \vee \psi \longrightarrow \psi \vee \phi$
  4. $(\phi \longrightarrow \psi) \longrightarrow (\phi \vee \chi \longrightarrow \psi \vee \chi)$

  The proposition axioms encode semantics of the logical operator $\vee$.
- The quantor axioms
  Given any formula ϕ, variable v_n, formula where v_n does not occur freely ψ and a sign c of type n, the following are axioms:
  1. $\forall v_n(\phi) \longrightarrow \phi_{c \leftarrow v_n}$ where $\phi_{c \leftarrow v_n}$ denotes the formula obtained when every free occurrence of v_n in ϕ is replaced by c
  2. $\forall v_n(\phi) \vee \psi \longrightarrow \forall v_n(\phi) \vee \psi$

  The quantor axioms encode the intuitive meaning of the universal quantifier $forall$.

- The reducability axiom
  Given any variables v_n and u_{n+1} and the formula ϕ where u_{n+1} does not occur freely, the following is an axiom:

  1. $\exists u_{n+1}(\forall v_n(u_{n+1}(v_n)) \longleftrightarrow \phi)$

  The reducability axiom can be interpreted as: for each formula with one free variable, there exists a set which contains exactly those elements for which the formula holds.

- The set axiom
  Given any variables x_n, x_{n+1} and y_{n+1}, the following is an axiom:

  1. $(\forall x_n(x_{n+1}(x_n) \longleftrightarrow y_{n+1}(x_n))) \longrightarrow x_{n+1} = y_{n+1}$

  We call a formula which is identical to another formula with the exception of all variables which are all of type n higher a *type lift* of the other formula. The set axiom can be interpreted as: a set is defined entirely by its elements.

In order to now define the set of all *provable formulas* of system P, we introduce the rules of derivation for system P: *immediate consequence*. We say a formula ϕ is the immediate consequence of the formulas ψ and χ if

$$\chi \text{ is } \psi \longrightarrow \phi$$

or

$$\phi \text{ is } \forall v : \psi.$$

The set of *provable formulas* is the smallest set containing every axiom which is closed under the relationship immediate consequence.

3.2 Gödel Numbering

The overall strategy of the proof is to construct as statement within system P which asserts its own unprovability. Any proof of such a statement would simultaneously demonstrate that the assertion that it is unprovable is false. Any proof of its negation would mean that the original statement must be provable too, which would make the formal system inconsistent. Thus, the only options are that the system is inconsistent or incomplete.

In order to construct such a statement, we must enable system P to refer to itself. The key component is a mapping from statements within the system to whole numbers. We call such a mapping a *Gödel numbering*.

Definition 3.1. *The Gödel number of the primitive symbols are*
$\#('0') = 1,\ \#('S') = 3,\ \#('\neg') = 5,\ \#('\vee') = 7,$
$\#('\forall') = 9,\ \#('(') = 11,\ \#(')') = 13$
$\#('x_n^m') = p^n$ *where p is the mth prime larger than 13 and $n > 0$*
and the Gödel number of a sequence as
$\#(\langle a_1, a_2, \ldots, a_n \rangle) = \prod_{k \le n} p_k^{\#(a_k)}$ *where p_k is the kth prime.*

This ensures that every primitive sign, formula and sequence of formulas of system P has a unique Gödel number. For instance, the first Peano axiom $\neg(Sx_1 = 0)$, which is an abbreviation for $\neg\forall x_2^1(\neg(\neg(\neg x_2^1(Sx_1^1) \vee x_2^1(0)) \vee \neg(\neg x_2^1(0) \vee x_2^1(Sx_1^1))))$ has the Gödel number $2^{\#(\text{'}\neg\text{'})} \cdot 3^{\#(\text{'}\forall\text{'})} \cdot \ldots \cdot 149^{\#(\text{')'})} = 2^5 \cdot 3^9 \cdot \ldots \cdot 151^{13}$

3.3 Translating statements about system P into statements within system P

The next step is to construct statements within system P which are equivalent to statements about the system. As we have seen, the canonical formulas for even simple statements are quite lengthy. To prevent having to write out these long formulas, Gödel (1931) used *primitive recursive* functions.

Definition 3.2. *Constant functions and the successor function $succ(x) = x+1$ are primitive recursive functions.*
The function f defined as

$$f(0, x_2, x_3, \ldots, x_n) \coloneqq g(x_2, x_3, \ldots, x_n)$$
$$f(k+1, x_2, x_3, \ldots, n) \coloneqq h(f(k, x_2, x_3, \ldots, x_n), x_2, x_3, \ldots, x_n)$$

is a primitive recursive function if both g and h are primitive recursive functions. Any function $f(x_1, x_2, \ldots, x_n) \coloneqq g(h_1(x_1, x_2, \ldots, x_n), \ldots, h_2(x_1, x_2, \ldots, x_n))$ is primitive recursive if g and h_i are.

We further call an n-ary relation R primitive recursive if and only if there exists a primitive recursive function f such that $(x_1, x_2, \ldots, x_n) \in R$ if and only if $f(x_1, x_2, \ldots, x_n) = 0$. We also write $R(x_1, x_2, \ldots, x_n)$.

In the original paper, 46 relations and functions between numbers which are equivalent to various statements about system P are built up. With the exception of the last relation, they are constructed using rules which are proven to yield primitive recursive relations. For the purpose of this paper, it will suffice to merely list those relations which we consider of particular importance or which serve as an illustrative example of the more interesting methods employed in constructing the full set of 46 relations. We will call attention to any new rules of construction that are used and provide a brief argumentation as to why they produce primitive recursive relations.

3.3.1 Divisibility

$$\text{DIV}(x, y) \equiv \underset{z \le x}{\exists} z : x = y \cdot z$$

This relation holds if and only if x is divisible by y. Since multiplication is repeated addition which, in turn, is repeated application of the successor function, it is obvious that multiplication is primitive recursive. What is more interesting is $\underset{z \le x}{\exists} z$ which should be read as a bounded existential quantifier, and not simply as an existential quantifier and a separate constraint on z. It is possible to define a primitive recursive function χ such that $\chi(x, z_1, \ldots, z_n) = 0$ if for all $y \leq x$ the primitive relation $R(y, z_1, \ldots, z_n)$ does not hold and $\chi(x, z_1, \ldots, z_n) = y$ where y is the smallest natural number for which the relation does hold otherwise. The function is 0 if x is 0. For any larger x, it is $\chi(x-1, z_1, \ldots, z_n)$ if that is larger than 0 or the relation doesn't hold, x if the relation does hold and 0 otherwise. In order to obtain an existential quantifier from this, we simply need to instantiate this function χ with some value which is larger than the lowest y for which the relation holds. This is why we can only construct a bounded existential quantifier, since we can use the bound as an input for χ.

3.3.2 The Gödel Number of an Item in a Sequence

$$\begin{aligned}\textsc{item}(n, x) \coloneqq & \underset{y \leq x}{\textsc{argmin}}\, \textsc{div}(x, \textsc{prime}\textsc{Factor}(n, x)^y) \\ & \wedge \neg\textsc{div}(x, \textsc{prime}\textsc{Factor}(n, x)^{y+1})\end{aligned}$$

This function returns the Gödel number of the nth item in the sequence s such that $\#(s) = x$. Note that this sequence can be both a formula or a sequence of formulas. $\underset{y \leq x}{\textsc{argmin}}$ is taken to mean the smallest y, smaller than or equal to x for which the following relation holds and 0 if no such y exists. $\textsc{prime}\textsc{Factor}(n, x)$ is the nth smallest prime factor of x. This function can be constructed similarly to the bounded existential quantifier.

3.3.3 The Concatenation of Two Sequences

$$\begin{aligned}x \circ y \coloneqq \underset{z \leq \textsc{prime}(|x|+|y|)^{x+y}}{\textsc{argmin}} & \left(\underset{n \leq |x|}{\forall} n : \textsc{item}(n, z) = \textsc{item}(n, x)\right) \\ & \wedge \left(\underset{n \leq |y|}{\forall} n : \textsc{item}(n + |x|, z) = \textsc{item}(n, y) \vee n = 0\right)\end{aligned}$$

Given $\#(s)$ and $\#(t)$ as inputs, this function is $\#(st)$, where st is the concatenation of the sequences s and t. $|x|$ here refers the length of the sequence s such that $\#(s) = x$, which can be defined similarly to $\textsc{item}(n, x)$. $\textsc{prime}(n)$ is the nth prime number. This function provides a great example of how flexible bounded versions of quantifiers can be despite their restriction. Since z is the Gödel number of the concatenation, it will be expressible as

$$\begin{aligned}2^{n_1} \cdot 3^{n_2} \cdot \ldots \cdot p_{|x|+|y|}^{n_{|x|+|y|}} &= \prod_{1 \leq k \leq |x|+|y|} p_k^{n_k} \\ &\leq \prod_{1 \leq k \leq |x|+|y|} p_k^{\max_i n_i} \\ &< p_{|x|+|y|}^{(|x|+|y|) \cdot \max_i n_i} \\ &< p_{|x|+|y|}^{x+y}\end{aligned}$$

Clearly, we could have found a lower bound, but the resulting function would have been harder to express and understand.

3.3.4 The Type of a Variable

$$\text{VTYPE}(n, x) \equiv \underset{z \leq x}{\exists} z : \text{ISPRIME}(z) \wedge x = z^n \wedge z > 13 \wedge n \neq 0$$

This relation holds if x is the Gödel number of a variable of type n, i.e. $x = \#(\text{'}x_n^m\text{'})$ for some m.

3.3.5 Numbers Represented in System P

$$\text{SUCC}(0, x) \coloneqq x$$
$$\text{SUCC}(n+1, x) \coloneqq \text{SEQ}\,(\#(\text{'}S\text{'})) \circ \text{SUCC}(n, x)$$

In this translation of the successor function, we see an application of the recursion scheme in the definition of primitive recursive functions, where a function is permitted to recursively depend upon itself provided one of its arguments in the recursive call is decreased. $\text{SEQ}(x)$ is simply 2^x and thus returns the Gödel number of the singleton sequence which contains only s such that $\#(s) = x$.

We can also use this successor function to define a function which transforms a number into the Gödel number of that number's representation in system P:

$$\text{NUM}(n) \coloneqq \text{SUCC}\,(n, \#(\text{'}0\text{'}))$$

3.3.6 The Type of Any Sign

$$\begin{aligned}\text{TYPE}(n, x) \equiv \Big(n = 1 \wedge \underset{m,n \leq x}{\exists} m, n : (m = \#(\text{'}0\text{'}) \vee \text{VTYPE}\,(1, m))\\ \wedge\, x = \text{SUCC}(n, \text{SEQ}(m))\Big)\\ \vee \Big(n > 1 \wedge \underset{v \leq x}{\exists} v : \text{VTYPE}(n, v) \wedge x = \text{SEQ}(v)\Big)\end{aligned}$$

$\text{TYPE}(n, x)$ holds if and only if x is the Gödel number of a segment of a formula which represents an object of type n. $\text{PAREN}(z)$ is the Gödel number of (ϕ) if $\#(\phi) = z$.

3.3.7 Elementary Formulas

$$\text{ELFM}(x) \equiv \underset{y,z,n \leq x}{\exists} y, z, n : \text{TYPE}(n+1, y) \wedge \text{TYPE}(n, z) \wedge x = y \circ \text{PAREN}(z)$$

Holds if and only if x is the Gödel number of an elementary formula like $x_{n+1}(x_n)$ or $x_2(SS0)$.

3.3.8 Inductively Building up Well-Formed Formulas

$$\textsc{fmSeq}(s) \equiv |s| > 0 \wedge \underset{n \le |s|}{\forall} n(n = 0 \vee \textsc{elFm}(\textsc{item}(n,s)) \\ \vee \underset{p,q \le n}{\exists} p, q(\textsc{op}(\textsc{item}(n,s), \textsc{item}(p,s), \textsc{item}(q,s)) \\ \wedge p > 0 \wedge q > 0))$$

This relation holds if and only if s is Gödel number of a sequence of Gödel numbers of formulas which are either elementary formulas or a logical operator applied to at most two formulas that appeared earlier in the sequence. $\textsc{op}(x, y, z)$ holds if and only if x is the Gödel number of a formula obtained by applying $\forall$, $\vee$ or $\neg$ to the formulas y and z are the Gödel numbers of. As a consequence, we know that every formula in the sequence s is well-formed. Note that $\textsc{item}(n, s)$ in this case does not refer to the Gödel number of a primitive symbol, but rather a formula. Likewise, s is the Gödel number of a sequence of formulas, not simply a formula.

3.4 Well-Formed Formulas

$$\textsc{isFm}(x) \equiv \underset{s \le \textsc{prime}(|x|^2)^{x \cdot |x|^2}}{\exists} s : \textsc{fmSeq}(s) \wedge x = \textsc{lastItem}(s)$$

Using $\textsc{fmSeq}(s)$ we can finally define a relation which holds if and only if x is the Gödel number of a well-formed formula by asserting that x is the last item in such a formula sequence.

3.4.1 Bound Variables

$$\textsc{bound}(v, n, x) \equiv \textsc{isVar}(v) \wedge \textsc{isFm}(x) \\ \wedge \underset{a,b,c \le x}{\exists} a, b, c(x = a \circ \textsc{forall}(v, b) \circ c \wedge \textsc{isFm}(b) \\ \wedge |a| + 1 \le n \le |a| + |\textsc{forall}(v, b)|)$$

The $\textsc{bound}$ relation holds if and only if the variable with the Gödel number v is not free at position n in the formula with Gödel number x. It forms the basis for a number of following relations relating to free and bound variables. $\textsc{forall}(v, x)$ is the Gödel number of $\forall y(\phi)$ where v is the Gödel number of the variable y and x that of the formula ϕ. Intuitively, the relation checks if the formula position n falls within a part of the formula where the variable was bound by a universal quantifier.

3.4.2 Substituting Formulas into Other Formulas

$$\textsc{subst'}(0, x, v, y) \coloneqq x \\ \textsc{subst'}(k + 1, x, v, y) \coloneqq \textsc{insert}(\textsc{subst'}(k, x, v, y), \textsc{freePlace}(k, v, x), y) \\ \textsc{subst}(x, v, y) \coloneqq \textsc{subst'}(\textsc{freePlaces}(v, x), x, v, y)$$

$\text{SUBST}(x, v, y)$ is the Gödel number of a formula where the formula referred to by y is inserted at every position in the formula referred to by x where the variable referred to by v occurs freely. $\text{SUBST'}(k, x, v, y)$ is an auxiliary function which does the same as SUBST but only inserts the formula for the first k free occurrences of the variable. FREEPLACE and FREEPLACES are the kth free occurrence and the amount of free occurrences of the variable respectively. $\text{INSERT}(x, n, y)$ is the Gödel number of a sequence which is equal to the sequence x refers to with the nth item replaced by the sequence y is the Gödel number of.

3.4.3 The Peano Axioms

$$\text{PEANOAXIOM}(x) \equiv x = pa_1 \vee x = pa_2 \vee x = pa_3$$

The Peano axioms have specific Gödel numbers, so we can simply precalculate them and check if a Gödel number x is equal to any of them.

3.4.4 The Proposition Axioms

$$\text{PROP2AXIOM}(x) \equiv \underset{y,z \leq x}{\exists} y, z : \text{ISFM}(y) \wedge \text{ISFM}(z) \wedge x = \text{IMPL}(y, \text{OR}(y, z))$$

This relation holds if x is the Gödel number of a formula which follows from the second proposition axiom[3]. More precisely, the formula has the form $p \to p \vee q$ with p and q being any two well-formed formulas. IMPL and OR are the Gödel numbers of implications and disjunctions respectively. We can translate the other three proposition axioms analogously.

3.4.5 The Quantor Axioms

$$\begin{aligned}\text{QUANTOR1AXIOM}(x) \equiv \underset{v,y,z \leq n}{\exists} v, y, z : &\text{VTYPE}(n, v) \wedge \text{TYPE}(n, z) \wedge \text{ISFM}(y)\\ &\wedge \text{QUANTOR1AXIOMCONDITION}(z, y, v)\\ &\wedge x = \text{IMPL}(\text{FORALL}(v, y), \text{SUBST}(y, v, z))\end{aligned}$$

In order to define the first quantor axiom, we require an auxiliary relation $\text{QUANTOR1AXIOMCONDITION}(z, y, v)$, which holds if the formula to be inserted does not contain any free variables which would be bound at any position where the variable to be replaced is free. Other than that, it is similar to the Peano axioms in that it simply asserts the existence of Gödel numbers which fulfill some necessary conditions and finally asserts that x is the Gödel number of a specific composition of the formulas or symbols these Gödel numbers refer to. The second quantor axiom and the reducability axiom can be constructed similarly.

[3]We defined the relation for the second axiom only because the original paper used the first axiom as an example. There is nothing special about either of the two axioms that would lead us to pick one over the other as an example.

3.4.6 The Set Axiom

$$\textsc{setAxiom}(x) \equiv \underset{n \le x}{\exists} n : x = \textsc{typeLift}(n, sa)$$

Since the set axiom concerns a specific formula and any type lift of this formula, we can once again precalculate its Gödel number sa. $\textsc{typeLift}(n, x)$ can be implemented using an argmin to find the Gödel number of a formula which is identical to the original formula, but has variables which are n types higher.

This gives us relations which can be used to check if a number is the Gödel number of an axiom of system P; we can move on to proofs.

3.4.7 Immediate Consequence

$$\textsc{consequence}(x, y, z) \equiv y = \textsc{impl}(z, x) \lor \underset{v \le x}{\exists} v : \textsc{isVar}(v) \land x = \textsc{forall}(v, y)$$

This relation holds if and only if x is the Gödel number of an immediate consequence of the formulas y and z refer to.

3.4.8 Inductively Building up Valid Proofs

$$\begin{aligned}\textsc{isProof}(s) \equiv\ & |s| > 0 \\ & \land \underset{n \le |s|}{\forall} n\Big(n = 0 \lor \textsc{isAxiom}(\textsc{item}(n, s)) \\ & \quad \lor \underset{p,q \le n}{\exists} p, q\big(\textsc{consequence}(\textsc{item}(n, s), \textsc{item}(p, s), \textsc{item}(n, s)) \\ & \quad\ \land p > 0 \land q > 0\big)\Big)\end{aligned}$$

$\textsc{isProof}(s)$ recognizes s as a proof if and only if s is a sequence of Gödel numbers of formulas which are either axioms or the immediate consequence of at most two formulas that appeared earlier in the sequence. Compare this definition to that of $\textsc{fmSeq}(s)$. Elementary formulas have been replaced by axioms and applications of operators have been replaced by the immediate consequence rule, but the structure is identical.

3.4.9 Provability

It would be tempting to define $\textsc{provable}(x)$ analogously to $\textsc{isFm}(x)$, but this is where the restriction to bounded quantifiers becomes relevant. Since no upper bound on the length of the shortest proof for a formula can be derived from the formula, we cannot assert its existence using a primitive recursive relation. We can however define the following primitive recursive relation:

$$\textsc{proves}(s, x) \equiv x = \textsc{lastItem}(s) \land \textsc{isProof}(s)$$

And although it is not primitive recursive, we can likewise define the relation which holds if and only if a formula is provable within system P:

$$\text{PROVABLE}(x) \equiv \exists\, s : \text{PROVES}(s, x)$$

3.5 The first incompleteness Theorem

In order to tie provability to primitive recursive relations, Gödel (1931) proves the following:

Theorem 3.1 (Gödel, 1931)**.** *Given an n-ary primitive recursive relation R, there exists a formula in system P with Gödel number r with at most n free variables with Gödel numbers* $v_1, \cdots, v_n$ *such that for any natural numbers* $x_1, \cdots, x_n$

if $R(x_1, \cdots, x_n)$ *then*
$$\text{PROVABLE}(\text{SUBST}(\cdots \text{SUBST}(r, v_1, \text{NUM}(x_1) \cdots), v_n, \text{NUM}(x_n)))$$
and
if not $R(x_1, \cdots, x_n)$ *then*
$$\text{PROVABLE}(\text{NOT}(\text{SUBST}(\cdots \text{SUBST}(r, v_1, \text{NUM}(x_1) \cdots), v_n, \text{NUM}(x_n)))).$$

Proof sketch. This can be proven with induction over the structure of primitive recursive functions. For the base case of a constant function, the formula can just be any tautology or contradiction depending on whether the function is constant 0 or some other number.
Since system P can model recursion and composition of functions, it's easy to convince oneself that the induction step for both schemes can be done. □

We can now dive into the final steps to the proof of Gödel's first incompleteness theory. To assure that the proof is applicable to more systems besides system P, we define the notion of extensions to system P. We allow system P to be extended by any primitive recursive set of formulas Φ, by which we mean a set of formulas whose Gödel numbers are exactly the range of a primitive recursive function. Note that, while every finite set of formulas is primitive recursive, Φ does not need to be finite to qualify.
For such an extension of system P, we can define the relations ISPROOF_Φ, PROVES_Φ and PROVABLE_Φ analogously to their counterparts for the unextended system P with the exception that an item in a Φ-proof as recognized by ISPROOF_Φ does not have to be an axiom or an immediate consequence of two earlier formulas, but can also be an element of Φ. We call such an extension of system P *ω-consistent* if and only if there is no x and v such that $x = \#(\phi)$, $\phi \in \Phi$, v is a free variable of ϕ and

$$\begin{aligned} &\text{PROVABLE}_\Phi\left((\forall n : \text{SUBST}(x, \#(v), \text{NUM}(n)))\right) \\ \wedge &\text{PROVABLE}_\Phi\left(\text{NOT}\left(\text{FORALL}\left(\#(v), x\right)\right)\right) \end{aligned}$$

The notion of ω-consistency is stronger than consistency, since we cannot derive a contradiction from the formula ϕ if it exists. The apparent contradiction only arises within the system when the infinite set of concrete formulas resulting from substituting all natural numbers into ϕ are taken into account that the contradiction could be proven[4].

Theorem 3.2 (Gödel, 1931)**.** *For each ω-consistent set of formulas Φ which is primitive recursive, there exists a formula with Gödel number r such that neither* $\textsc{provable}_{\Phi}\,(\textsc{not}\,(\textsc{forall}\,(\#(\text{'}x\text{'}), r)))$ *nor* $\textsc{provable}_{\Phi}\,(\textsc{forall}\,(\#(\text{'}x\text{'}), r))$ *hold.*

Proof sketch. Consider the relation

$$\textsc{doesNotProveRecursive}(s, x) \equiv \neg\textsc{proves}_{\Phi}(s, \textsc{subst}(x, \#(\text{'}v\text{'}), \textsc{num}(x))) \tag{1}$$

This can be read as the proof referred to by the Gödel number s does not prove the formula with the Gödel number x with x (the number represented in system P) substituted for every free occurence of the variable v. Since this relation is primitive recursive, per Theorem 3.1, there exists a corresponding Gödel number q with two free variables for which every combination of s and x for which the relation holds, the formula obtained from substituting s and x is provable and the negation is provable for combinations of s and x for which the relation does not hold.
We can now define the Gödel numbers

$$noProofForRecursive \coloneqq \textsc{forall}(\#(\text{'}s\text{'}), q) \tag{2}$$

and

$$r \coloneqq \textsc{subst}(q, \#(\text{'}x\text{'}), \textsc{num}(noProofForRecursive)) \tag{3}$$

r is the Gödel number of a formula with one free variable s which holds if s is not a proof for a formula which asserts no proof exists for it.
When we investigate the formula with Gödel number $\textsc{forall}(\#(\text{'}s\text{'}), r)$ we find

$$\begin{aligned}\textsc{forall}(\#(\text{'}s\text{'}), r) &= \textsc{forall}(\#(\text{'}s\text{'}),\\ &\quad \textsc{subst}(q, \#(\text{'}x\text{'}), \textsc{num}(noProofForRecursive))) \text{ (by 3)}\\ &\equiv \textsc{subst}(\textsc{forall}(\#(\text{'}s\text{'}), q), \#(\text{'}x\text{'}),\\ &\quad \textsc{num}(noProofForRecursive)))\\ &= \textsc{subst}(noProofForRecursive, \#(\text{'}x\text{'}),\\ &\quad \textsc{num}(noProofForRecursive) \text{ (by 2)}\end{aligned} \tag{4}$$

[4]Hence the use of ω, which is also a common name for the smallest infinite ordinal number.

This tells us that the formula with the Gödel number $\text{FORALL}(\#(\text{'}s\text{'}), r)$ is equivalent to the formula which —when the Gödel number of another formula is plugged into it, asserts this formula with its own Gödel number plugged into it cannot be proven —with its own Gödel number plugged into it. i.e. *"I cannot be proven"*.
Taking into account 1 and 4 we get that if there were an s such that $\text{PROVES}_\phi(s, \text{FORALL}(\#(\text{'}s\text{'}), r))$, then it is also provable that $\text{PROVABLE}_\phi(\text{NOT}(\text{SUBST}(r, \#(\text{'}s\text{'}), \text{NUM}(s)))))$ which by the first quantor axiom implies that the negation of the antecedent is also provable, which would be a contradiction in the system which in turn would contradict the assumption that ϕ is ω-consistent.
Similarly, we get that since $\text{FORALL}(\#(\text{'}s\text{'}), r)$ is not provable, there can't be a single s which is a proof for it. Because of the properties of q as discussed above, this means that for every number s which is not the Gödel number of a valid proof, we can prove that the formula with Gödel number r holds. Since this must hold for every natural number, the formula with Gödel number $\text{NOT}(\text{FORALL}(\#(\text{'}s\text{'}), r))$ cannot be provable without resulting in an ω-inconsistent system which contradicts the assumption. □

The consequence of this theorem, as well as other related theorems in Gödel's paper and Rosser's improvement (Rosser, 1936) is that the main goal mathematicians had been striving towards for decades was unattainable. Any interesting system of arithmatic will necessarily contain statements which can neither be proven nor disproven, or every statement —including false ones —can be proven. It still is widely believed that the former is the case. It was ultimately proven that the Continuum Hypothesis which caused Cantor so much anciety is one such statement which can neither be proven nor disproven.

4 The Aftermath

Gödel's incompleteness theorems had tremendous impact. John von Neumann allegedly attended the talk where Gödel presented his results and simply said "it's over" after Gödel was done (de León, 2011). After Einstein's Theories of Relativity and Heisenberg's uncertainty principle had shaken up the field of physics, it seemed that now even something as seemingly simple like arithmetic was not free of paradoxes. Just five years later, the field of Computer Science would see a similar result in the halting problem due to Alan Turing, which arguably had an even bigger impact on its field.

Despite this, most mathematicians continue their work as if Gödel's theorems didn't exist. One important reason for this could be the disconnect between the formal systems Gödel's proof was concerned with and the informal systems most mathematicians work in. (Feferman, 2006) Even among those who care deeply about Hilbert's second problem, there exists no consensus on whether Gödel's proof actually solves the problem. At the time of writing, eight of the problems are generally accepted to be resolved, while nine problems are partially resolved not universally accepted to be resolved. Excluding the two problems which are deemed to vague to ever be resolved, that leaves only 4 problems which remain fully open questions.

Gödel's theorems also have a tantalizing metaphysical allure which arguably played a role in bringing about postmodernism. In his pop philosophy bestseller *Gödel, Escher, Bach: an eternal golden braid*, Hofstadter (1979) used Gödel's work as a basis for his philosophy that all meaning stems from Gödelesque self referentiality which he calls *strange loops*.

Keeping in mind the considerable theoretical relevance of Gödel's findings and the metaphysical interpretations it invites, it is more than worth it to take an effort to understand the theorems, their proofs and the context in which they arose. We hope that this paper may prove useful for some to be a step in that direction.

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