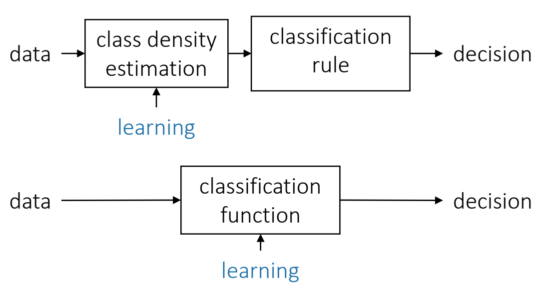
Logistic Regression

Two Worlds: Probabilistic & Algorithmic

We know two conceptual approaches to classification:



Bayes Classifier
Probabilistic classifier with a generative setup based on class density models
Bayes (Gauss), Naïve Bayes

"Direct" Classifiers
Find best parameter (e.g. $\vec{w}$) with respect to a specific loss function measuring misclassification
Perceptron, SVM, Tree, ANN

Can we have a probabilistic classifier with a modelling focus on *classification*?

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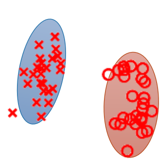
Advantages of Both Worlds

- Posterior distribution has advantages over classification label:
  - Asymmetric risks: need classification probability
  - Classification certainty: Indicator if decision in unsure
- Algorithmic approach with direct learning has advantages:
  - Focus of modelling power on correct classification where it counts
  - Easier decision line interpretation
- Combination?

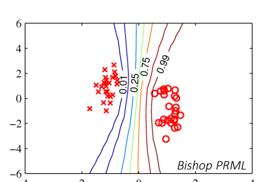
Discriminative Probabilistic Classifier

Bayes Classifier

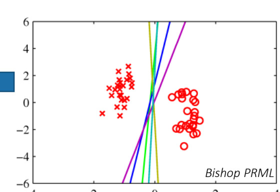


$P(x|C_1)$ $P(x|C_2)$

Discriminative Probabilistic Classifier



Linear Classifier



$g(\vec{x}) = \boldsymbol{w}^T\boldsymbol{x} + w_0$

$P(C_2|x) \propto P(x|C_2)P(C_2)$

Towards a "Direct" Probabilistic Classifier

- Idea 1: Directly learn a posterior distribution

  For classification with the Bayes classifier, the posterior distribution is relevant. We can directly estimate a model of this distribution (we called this as a *discriminative* classifier in Naïve Bayes). We know from Naïve Bayes that we can probably expect a good performance from the posterior model.

- Idea 2: Extend linear classification with probabilistic interpretation

  The linear classifier outputs a distance to the decision plane. We can use this value and interpret it probabilistically: "The further away, the more certain"

Logistic Regression

The *Logistic Regression* will implement both ideas: It is a model of a posterior class distribution for classification and can be interpreted as a probabilistic linear classifier. But it is a fully probabilistic model, not only a "post-processing" of a linear classifier.

It extends the hyperplane decision idea to Bayes world

- Direct model of the posterior for classification
  - Probabilistic model (classification according to a probability distribution)
  - Discriminative model (models posterior rather than likelihood and prior)
- Linear model for classification
  - Simple and accessible (we can understand that)
  - We can study the relation to other linear classifiers, i.e. SVM

History of Logistic Regression

- Logistic Regression is a very “old” method of statistical analysis and in widespread use, especially in the traditional statistical community (not machine learning).
  - 1957/58, Walker, Duncan, Cox
- A method more often used to study and identify explaining factors rather than to do individual prediction.
  - Statistical analysis vs. prediction focus of modern machine learning
  - Many medical studies of risk factors etc. are based on logistic regression

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Statistical Data Models

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We do not know $P(x,y)$ but we can assume a certain form.
---> This is called a data model.

Simplest form besides constant (one prototype) is a linear model.

$$Lin_w(x) = \sum_{i=1}^{d} w_i x_i + w_0 = \langle w, x \rangle + w_0 = \vec{w}^T \vec{x} + w_0$$

$$\tilde{x} = \begin{bmatrix} 1 \\ x \end{bmatrix}, \ \tilde{w} = \begin{bmatrix} w_0 \\ w \end{bmatrix}$$

$$\Rightarrow Lin_w(x) = \langle w, x \rangle + w_0 = \langle \tilde{w}, \tilde{x} \rangle$$

▶ Linear Methods:

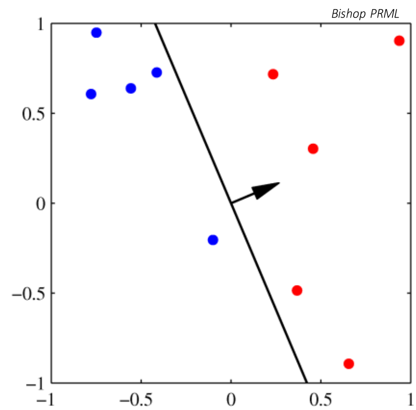
Classification: Logistic Regression (no typo!)
Regression: Linear Regression

Repetition: Linear Classifier

Linear classification rule:

$$g(\boldsymbol{x}) = \boldsymbol{w}^T\boldsymbol{x} + w_0$$

$$g(\boldsymbol{x}) \geq 0 \;\Rightarrow$$
$$g(\boldsymbol{x}) < 0 \;\Rightarrow$$



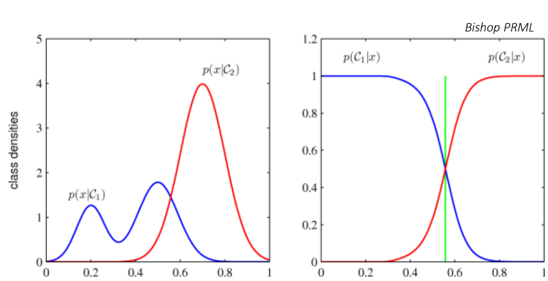
Decision boundary is a a *hyperplane*

Repetition: Posterior Distribution

- Classification with Posterior distribution: Bayes

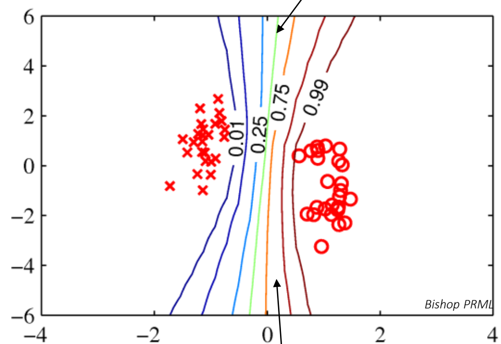
  Based on class densities and a prior

$$P(C_1|\boldsymbol{x}) = \frac{p(\boldsymbol{x}|C_1)P(C_1)}{p(\boldsymbol{x}|C_1)P(C_1) + \; p(\boldsymbol{x}|C_2)P(C_2)} \qquad P(C_2|\boldsymbol{x}) = \frac{p(\boldsymbol{x}|C_2)P(C_2)}{p(\boldsymbol{x}|C_1)P(C_1) + \; p(\boldsymbol{x}|C_2)P(C_2)}$$



Combination: Discriminative Classifier

Decision boundary



Probabilistic interpretation of classification output: ~distance to separation plane

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Notation Changes

- We work with two classes
  - Data with (numerical) feature vectors $\vec{x}$ and labels $\boldsymbol{y} \in \{\mathbf{0}, \mathbf{1}\}$
  - We do not use the notation of Bayes with ω anymore. We will need the explicit label value of $\boldsymbol{y}$ in our models later.

- Classification goal: infer the best class label $\{\mathbf{0}\ \boldsymbol{or}\ \mathbf{1}\}$ for a given feature point

$$y^* = \arg \max_{y \in \{0,1\}} P(y|\boldsymbol{x})$$

- All our modeling focuses only on the posterior of having class 1:

$$P(y = 1\ |\boldsymbol{x}\,)$$

  - Obtaining the other is trivial: $P(y = 0\ |\boldsymbol{x}\,) = 1 - P(y = 1\ |\boldsymbol{x}\,)$

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Parametric Posterior Model

We need a model for the posterior distribution, depending on the feature vector (of course) and neatly parameterized.

$$P(y = 1 \mid \boldsymbol{x}, \boldsymbol{\theta}) = f(\boldsymbol{x};\ \boldsymbol{\theta})$$

The linear classifier is a good starting point. We know its parametrization very well:

$$g(\boldsymbol{x};\ \boldsymbol{w}, w_0) = \boldsymbol{w}^T\boldsymbol{x} + w_0$$

We thus model the posterior as a function of the linear classifier:

$$P(y = 1 \mid\ \boldsymbol{x}, \boldsymbol{w}, w_0) = f(\boldsymbol{w}^T\boldsymbol{x} + w_0)$$

Posterior from classification result: "scaled distance" to decision plane

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Logistic Function

To use the *unbounded* distance to the decision plane in a probabilistic setup, we need to map it into the interval $[0, 1]$

This is very similar as we did in neural nets: *activation function*

The *logistic function* $\sigma(x)$ squashes a value $x \in \mathbb{R}$ to $[0, 1]$

$$\sigma(x) = \frac{1}{1 + \mathrm{e}^{-x}}$$

The logistic function is a smooth, soft threshold

$$\sigma(x) \to 1 \quad x \to \infty$$
$$\sigma(x) \to 0 \quad x \to -\infty$$
$$\sigma(0) = \tfrac{1}{2}$$

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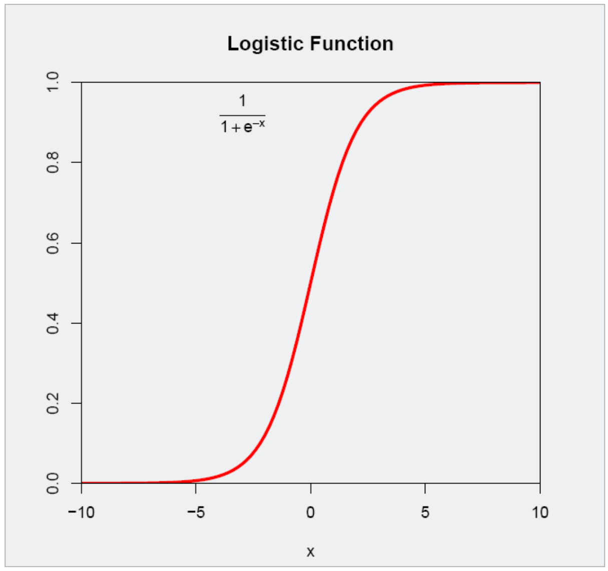
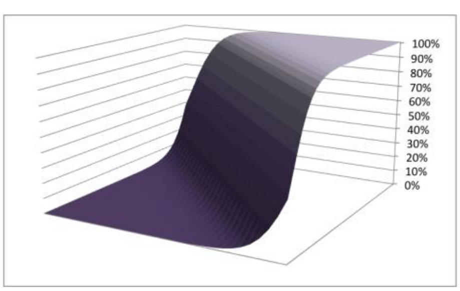
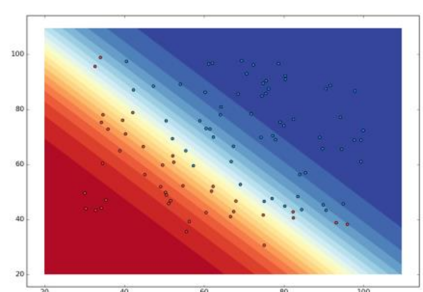
The Logistic Function



The Logistic “Regression”





The Logistic Regression Posterior

We model the posterior distribution for classification in a two-classes-setting by applying the logistic function to the linear classifier:

$$P(y = 1|\ x) = \sigma\big(g(x)\big)$$

$$P(y = 1|\ \boldsymbol{x}, \boldsymbol{w}, w_0) = f(\boldsymbol{w}^T\boldsymbol{x} + w_0) = \frac{1}{1 + e^{-(\boldsymbol{w}^T\boldsymbol{x} + w_0)}}$$

This a location-dependent model of the posterior distribution, parametrized by a linear hyperplane classifier.

Logistic Regression is a Linear Classifier

The logistic regression posterior leads to a linear classifier:

$$P(y = 1|\ \boldsymbol{x}, \boldsymbol{w}, w_0) = \frac{1}{1 + \exp(-(\boldsymbol{w}^T\boldsymbol{x} + w_0))}$$

$$P(y = 0|\ \boldsymbol{x}, \boldsymbol{w}, w_0) = 1 - P(y = 1|\ \boldsymbol{x}, \boldsymbol{w}, w_0)$$

$P(y = 1|\ \boldsymbol{x}, \boldsymbol{w}, w_0) > \frac{1}{2} \quad \Rightarrow \quad y = 1$ classification; $y = 0$ otherwise

Classification boundary is at: $P(y = 1|\ \boldsymbol{x}, \boldsymbol{w}, w_0) = \frac{1}{2}$

$$\Rightarrow \quad \frac{1}{1 + \exp(-(\boldsymbol{w}^T\boldsymbol{x} + w_0))} = \frac{1}{2} \quad \Rightarrow \quad \boxed{\boldsymbol{w}^T\boldsymbol{x} + w_0 = 0}$$

Classification boundary is a hyperplane

Interpretation: Logit

Is the choice of the logistic function justified?

- Yes, the *logit* is a linear function of our data:

  Logit: log of the *odds ratio:* $\ln \frac{p}{1-p}$

$$\ln \frac{P(y=1|\boldsymbol{x})}{P(y=0|\boldsymbol{x})} = \boldsymbol{w}^T\boldsymbol{x} + w_0$$

The linear function (~distance from decision plane) directly expresses our classification certainty, measured by the "odds ratio":
double distance ↔ squared odds
e.g. $3:2 \rightarrow 9:4$

- But other choices are valid, too

  They lead to other models than logistic regression, e.g. probit regression

  → *Generalized Linear Models (GLM)* $E[y] = f^{-1}(\boldsymbol{w}^T\boldsymbol{x} + w_0)$

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The Logistic Regression

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- So far we have made no assumption on the data!
- We can get $r(x)$ from a generative model or model it directly as function of the data (discriminative)

Logistic Regression:

Model: The logit $r(x) = \log \frac{P(y=1|\boldsymbol{x})}{P(y=0|\boldsymbol{x})}$ **=** $\log \frac{p}{1-p}$
is a linear function of the data

$$r(x) = \log \frac{p}{1-p} = \sum_{i=1}^{d} w_i x_i + w_0 = \langle \tilde{w}, \tilde{x} \rangle$$

$$<=> \qquad P(y=1|\,x) = \sigma(\langle \tilde{w}, \tilde{x} \rangle) = \frac{1}{1+\exp(-\langle \tilde{w}, \tilde{x} \rangle)}$$

Training a Posterior Distribution Model

The posterior model for classification requires training. Logistic regression is not just a post-processing of a linear classifier. Learning of good parameter values needs be done with respect to the probabilistic meaning of the posterior distribution.

- In the probabilistic setting, learning is usually estimation

  We now have a slightly different situation than with Bayes: We do not need class densities but a good *posterior distribution*.

- We will use Maximum Likelihood and Maximum-A-Posteriori estimates of our parameters $\boldsymbol{w}, w_0$

Later: This also corresponds to a cost function of obtaining $\boldsymbol{w}, w_0$

Maximum Likelihood Learning

The Maximum Likelihood principle can be adapted to fit the posterior distribution (discriminative case):

- We choose the parameters $\boldsymbol{w}, w_0$ which maximize the *posterior distribution* of the training set $\boldsymbol{X}$ with labels $\boldsymbol{Y}$:

$$\boldsymbol{w}, w_0 = \arg\max_{\boldsymbol{w}, w_0} P(\ \mathrm{Y} \ \mid \ X; \boldsymbol{w}, w_0)$$

$$= \arg\max_{\boldsymbol{w}, w_0} \prod\nolimits_{\boldsymbol{x} \in X} P(y | \ \boldsymbol{x}; \boldsymbol{w}, w_0) \quad \text{(iid)}$$

$$P(y \mid \boldsymbol{x}; \boldsymbol{w}, w_0) = \quad P(\ y = 1 \mid \boldsymbol{x}; \boldsymbol{w}, w_0)^y \ \ P(\ y = 0 \mid \boldsymbol{x}; \boldsymbol{w}, w_0)^{1-y}$$

Logistic Regression: Maximum Likelihood Estimate of w (1)

To simplify the notation we use w, x instead of $\boldsymbol{w}, w_0$

With $P(y=1|x)=\sigma(w^T x)$ and $P(y=0|x)=1-\sigma(w^T x)$

$$\Rightarrow P(y|x)=P(y=1|x)^y\,P(y=0|x)^{1-y} = p^y(1-p)^{1-y}$$

The discriminative (log) likelihood function for our data

$$P(Y|X) = \prod_{i=1}^{N} P(y_i|x_i) = \prod_{i=1}^{N} p_i^{y_i}(1-p_i)^{1-y_i}$$

$$\log P(Y|X) = \sum_{i=1}^{N} y_i \log(p_i)+(1-y_i)\log(1-p_i)$$

"cross-entropy" cost function

$$= \sum_{i=1}^{N} y_i \log\left(\frac{p_i}{1-p_i}\right)+\log(1-p_i)$$

Maximum Likelihood Estimate of $\boldsymbol{w}$ (2)

log-likelihood function continued

$$\log L(Y,X) \equiv \log P(Y|X) = \sum_{i=1}^{N} y_i \log\left(\frac{p_i}{1-p_i}\right)+\log(1-p_i)$$

Remember $p_i=\sigma(w^T x)=\dfrac{1}{1+e^{-w^T x}}$ and linear Logit $log\dfrac{p_i}{1-p_i}=w^T x$

$$\boxed{\log L(Y,X)=\sum_{i=1}^{N} y_i w^T x_i - \log\left(1+e^{w^T x_i}\right)}$$

Maximize the log-likelihood function with respect to w

$$\frac{\partial}{\partial w}\log L(Y,X) \overset{!}{=} 0$$

Maximum Likelihood Estimate of $\boldsymbol{w}$ (3)

$$\frac{\partial}{\partial w}\log L(Y,X) = \frac{\partial}{\partial w}\sum_{i=1}^{N} y_i w^T x_i - \log\left(1+e^{w^T x_i}\right)$$

$$= \sum_{i=1}^{N} y_i x_i^T - \frac{e^{w^T x_i}}{1+e^{w^T x_i}} x_i^T$$

Derivative of a Dot Product

Gradient operator $$\frac{\partial}{\partial \boldsymbol{w}} = \nabla_{\mathbf{w}} = \left[\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \dots, \frac{\partial}{\partial w_d}\right]$$

$$\frac{\partial}{\partial \boldsymbol{w}}\boldsymbol{w}^T\boldsymbol{x} = \left[\frac{\partial}{\partial w_1}\boldsymbol{w}^T\boldsymbol{x}, \frac{\partial}{\partial w_2}\boldsymbol{w}^T\boldsymbol{x}, \dots, \frac{\partial}{\partial w_d}\boldsymbol{w}^T\boldsymbol{x}\right]$$

Per component $$\frac{\partial}{\partial w_i}\boldsymbol{w}^T\boldsymbol{x} = \frac{\partial}{\partial w_i}\sum_{k=0}^{d} w_k x_k = x_i$$

Final derivative $$\boxed{\frac{\partial}{\partial \boldsymbol{w}}\boldsymbol{w}^T\boldsymbol{x} = [x_1, x_2, \dots, x_d] = \boldsymbol{x}^T}$$

Maximum Likelihood Estimate of $\boldsymbol{w}$ (3)

$$\frac{\partial}{\partial w}\log L(Y,X) = \frac{\partial}{\partial w}\sum_{i=1}^{N} y_i w^T x_i - \log\left(1+e^{w^T x_i}\right)$$

$$= \sum_{i=1}^{N} y_i x_i^T - \frac{e^{w^T x_i}}{1+e^{w^T x_i}} x_i^T$$

$$\boxed{\frac{e^{w^T x_i}}{1+e^{w^T x_i}} = \frac{1}{1+e^{-w^T x_i}}}$$

$$= \sum_{i=1}^{N} \left(y_i - \sigma\left(w^T x_i\right)\right) x_i^T \overset{!}{=} 0$$

- Non-linear equation in w : no closed form solution.
- The function $\operatorname{Log} L$ is concave therefore a unique maximum exists.

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Iterative Reweighted Least Squares

The concave $\log P(\boldsymbol{Y}|\boldsymbol{X})$ can be maximized iteratively with the Newton-Raphson algorithm: *Iterative Reweighted Least Squares*

$$\boldsymbol{w}^{n+1} \leftarrow \boldsymbol{w}^{n} - \boldsymbol{H}^{-1}\frac{\partial}{\partial \boldsymbol{w}}(\ln P(\boldsymbol{Y}|\boldsymbol{X};\boldsymbol{w}^{n}))$$

Derivatives and evaluation always with respect to $\boldsymbol{w}^n$

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Hessian: Concave Likelihood

$$\boldsymbol{H} = \frac{\partial^2}{\partial \boldsymbol{w} \partial \boldsymbol{w}^T} \ln P(Y|X)$$

We use an old trick to keep it simple:
$\boldsymbol{w} := \begin{bmatrix} w_0 \\ \boldsymbol{w} \end{bmatrix}, \quad \boldsymbol{x} := \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix}$

$$\frac{\partial}{\partial \boldsymbol{w}} \left(\frac{\partial}{\partial \boldsymbol{w}^T} \ln P(Y|X) \right) = -\sum_i \boldsymbol{x}_i \boldsymbol{x}_i^T \sigma\left(\boldsymbol{w}^T \boldsymbol{x}_i\right) \left(1 - \sigma\left(\boldsymbol{w}^T \boldsymbol{x}_i\right)\right) = -\boldsymbol{X}^T \boldsymbol{S} \boldsymbol{X}$$

The Hessian is negative definite:

- The sample covariance matrix $\sum_i \boldsymbol{x}_i \boldsymbol{x}_i^T$ is positive definite
- $\sigma\left(\boldsymbol{w}^T \boldsymbol{x}_i\right) \left(1 - \sigma\left(\boldsymbol{w}^T \boldsymbol{x}_i\right)\right)$ is always positive

The optimization problem is said to be *convex* and has thus a optimal solution which can be iteratively calculated.

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Iterative Reweighted Least Squares

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$$\boldsymbol{w}^{n+1} \leftarrow \boldsymbol{w}^n - \boldsymbol{H}^{-1} \frac{\partial}{\partial \boldsymbol{w}} (\ln P(\boldsymbol{Y}|\boldsymbol{X}; \boldsymbol{w}^n))$$

Derivatives and evaluation always with respect to $\boldsymbol{w}^n$

Method results in an iteration of *reweighted* least-squares steps

$$w^{n+1} = \left(X^T S X\right)^{-1} X^T S \ z$$
$$z = X w^n + S^{-1}\left(Y - P(w^n)\right)$$

- Weighted least-squares with $\boldsymbol{z}$ as target: $\left(X^T S X\right)^{-1} X^T S \ z$
- $\boldsymbol{z}$: adjusted responses (updated every iteration)
- $\boldsymbol{P}(\boldsymbol{w}^n)$: vector of responses $[p_{1,} p_2, \dots, p_N]^T$

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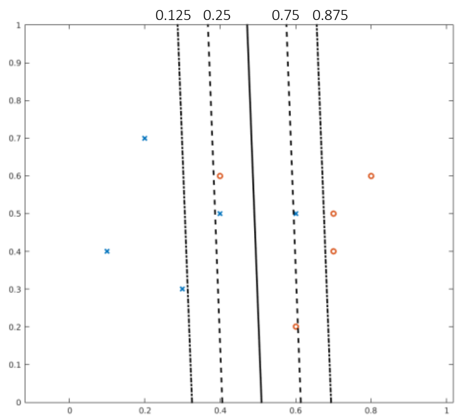
Example: Logistic Regression



Solid line: classification ($p = 0.5$)

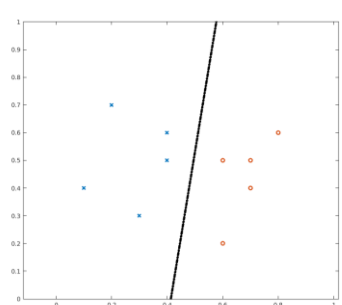
Dashed lines: $p = 0.25$, $p = 0.75$ lines

Probabilistic result: posterior of classification *everywhere*

The posterior probability decays/increases with distance to the decision boundary

Linearly Separable

- Maximum Likelihood learning is problematic in the linearly separable case: w diverges in length
  → leads to classification with *infinite* certainty
- Classification is still right but posterior estimate is not



Prior Assumptions

- Infinitely certain classification is likely an estimation artefact:
  We do not have enough training samples
  → maximum likelihood estimation leads to problematic results
- Solution: MAP estimate with prior assumptions on w

$$P(w) = N\left(w|0, \sigma^2 I\right)$$

Smaller w are preferred (*shrinkage*)

$$P(y|\boldsymbol{x}, \boldsymbol{w}, w_0) = p^y (1-p)^{1-y}$$

Likelihood model is unchanged

$$\boldsymbol{w}, w_0 = \arg\max_{\boldsymbol{w}, w_0} P(\mathrm{Y}|X; \boldsymbol{w}, w_0\,) P(\boldsymbol{w})$$
$$= \arg\max_{\boldsymbol{w}, w_0} P(\boldsymbol{w}) \prod_{\boldsymbol{x} \in X} P(y|\boldsymbol{x}, \boldsymbol{w}, w_0\,)$$

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MAP Learning

$$\ln P(\boldsymbol{w}) \prod_{\boldsymbol{x} \in X} P(y|\boldsymbol{x}, \boldsymbol{w}, w_0\,) =$$
$$\sum \left(y_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0\right) - \ln\left(1 + \exp\left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0\right)\right) \right) - \frac{1}{2\sigma^2} \|\boldsymbol{w}\|^2$$

We need: $\frac{\partial}{\partial \boldsymbol{w}} \|\boldsymbol{w}\|^2 = 2\boldsymbol{w}^T$

$$\frac{\partial}{\partial \boldsymbol{w}} \ln P(Y|X) = \sum_i \left(\left(y_i - \sigma\left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0\right)\right) \boldsymbol{x}_i^T \right) - \frac{1}{\sigma^2} \boldsymbol{w}^T \overset{!}{=} 0$$

- Iterative solution: Newton-Raphson
- Prior enforces a *regularization*

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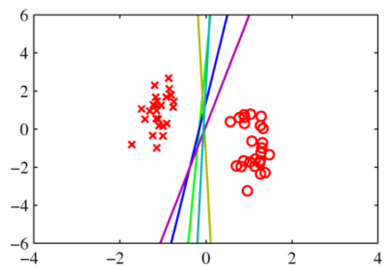
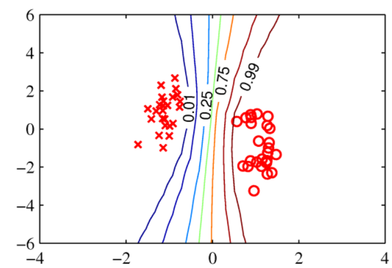
Bayesian Logistic Regression

Idea: In the separable case, there are *many perfect* linear classifiers which all separate the data. *Average* the classification result and accuracy using *all* of these classifiers.

- Optimal way to deal with missing knowledge in *Bayes sense*



Bishop PRML



Bishop PRML

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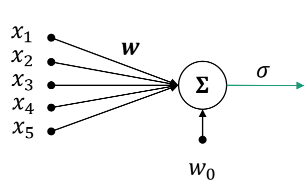
Logistic Regression and Neural Nets

- The standard single neuron with the logistic activation is logistic regression if trained with the same cost function (cross-entropy)
  - But training with least-squares results in a different classifier
- Multiclass logistic regression with soft-max corresponds to what is called a *soft-max layer* in ANN. It is the standard multiclass output in most ANN architectures.



$$P(y = 1|\boldsymbol{x}, \boldsymbol{w}, w_0) = \sigma(\boldsymbol{w}^T \boldsymbol{x} + w_0)$$

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Non-Linear Extension

- Logistic regression is often extended to non-linear cases: $\boldsymbol{x} := \begin{bmatrix} \boldsymbol{x} \\ x_1 x_2 \\ x_2^2 \end{bmatrix}$

  Extension through adding additional *transformed* features

  - Combination terms: $x_i x_j$
  - Monomial terms: x_i^2

  Standard procedure in medicine: inspect resulting $\boldsymbol{w}$ to find important factors and interactions $x_i x_j$ (comes with statistical information).

- Usage of *kernels* is possible: training and classification can be formulated with dot products of data points. The scalar products can be *"replaced"* by kernel expansions with the *kernel trick*.

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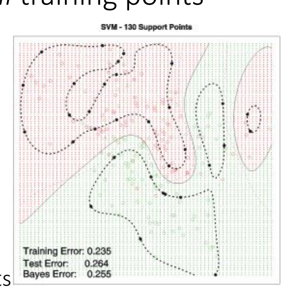
Kernel Logistic Regression

- Equations of logistic regression can be reformulated with dot products:

$$\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{x} = \sum_{i=1}^{N} \alpha_i \boldsymbol{x}_i^T \boldsymbol{x} \to \sum_{i=1}^{N} \alpha_i k(\boldsymbol{x}_i, \boldsymbol{x})$$

- No Support Vectors: kernel evaluations with *all* training points

$$P(y=1|\ \boldsymbol{x}) = \sigma\left(\sum_{i=1}^{N} \alpha_i k(\boldsymbol{x}_i, \boldsymbol{x})\right)$$



IVM (import vector machine):
Extension with only sparse support points

Ji Zhu & Trevor Hastie (2005) Kernel Logistic Regression and the Import Vector Machine, Journal of Computational and Graphical Statistics, 14:1, 185-205, DOI: 10.1198/106186005X25619

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Discriminative vs. Generative

Comparison of logistic regression to naïve Bayes

Ng, Andrew Y., and Michael I. Jordan. "On Discriminative vs. Generative classifiers: A comparison of logistic regression and naive Bayes." Advances in NIPS 14, 2001.

Conclusion:

- Logistic regression has a *lower* asymptotic error
- Naïve Bayes can reach its (higher) asymptotic error *faster*

General over-simplification (dangerous!): use a generative model with few data (more knowledge) and a discriminative model with a lot of training data (more learning)

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Logistic Regression: Summary

- A probabilistic, linear method for classification!
- Discriminative method (Model for posterior)
- Linear model for the Logit

$$\log \frac{p}{1-p} = \langle \tilde{w}, \tilde{x} \rangle$$

- The posterior probability is given by the logistic function of the Logit:

$$P\left(y=1|x\right) = \sigma\left(\langle \tilde{w}, \tilde{x} \rangle\right) = \frac{1}{1+\exp\left(-\langle \tilde{w}, \tilde{x} \rangle\right)}$$

- ML-estimation of $\tilde{w}$ is unique but non-linear
- Logistic regression is a very often used method
- Extendable to multiclass
- General Purpose method, included in every standard software, e.g. glm in R, glmfit/glmval in Matlab – its easy to apply!