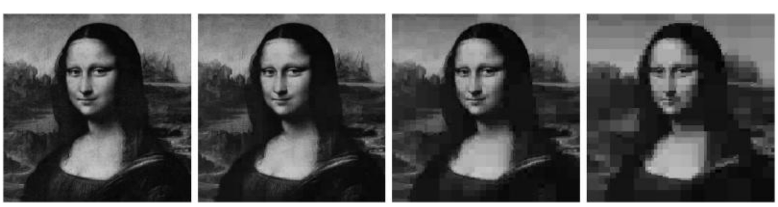
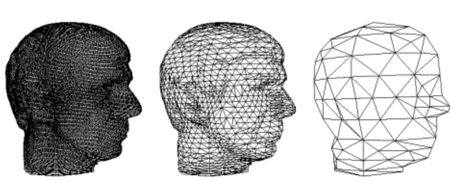
Wavelets

Wavelets: Applications

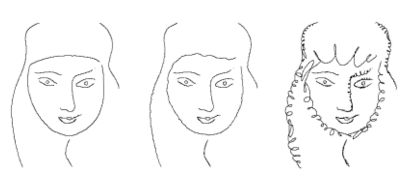
Compression in 2D & 3D



100% 19% 3% 1%



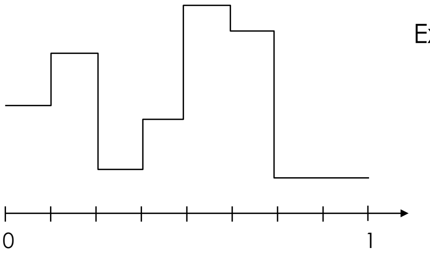
Multiscale Editing:



Wavelets for Computer Graphics : A Primer
Stollniz, DeRose & Salesin

Haar Wavelets

V^j is the vector space of all piece wise constant function on [0,1] dividing the interval in 2^j pieces:



Example V^3 with 8 pieces

Every function with less constant pieces is element of V^3.

=> : $V^0 \subset V^1 \subset V^2 \ldots \subset V^j$

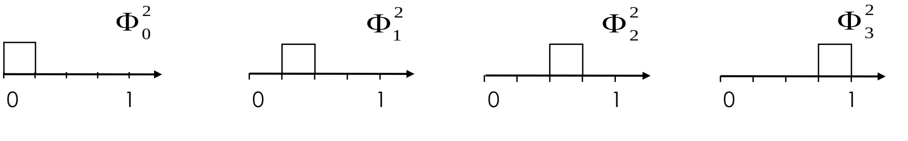
Haar Wavelet Basis Functions

A special type of basis functions of V^j are the "*scaling functions*".

$$\Phi_i^j(x) := \Phi\left(2^j x - i\right) \qquad i = 0,1,\ldots,2^j - 1$$

with $\Phi(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$ are the Haar Wavelet basis functions.

Example: Basis for V^2



$f(x) = c_0\Phi_0^2(x) + c_1\Phi_1^2(x) + c_2\Phi_2^2(x) + c_3\Phi_3^2(x), \quad \forall c_i \in \mathbb{R}$

Any function in V^2 can be represented!

Complementary Space & Wavelets

Definition: The **scalar product** in V^j is

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

Definition: W^j is the **orthogonal complementary space** of V^j in V^{j+1}, → $V^j \oplus W^j = V^{j+1}$

Definition: **Wavelets** are the basis functions $\Psi_i^j(x)$ that span the vector space W^j.

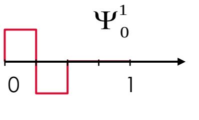
→
- $\Psi_i^j \in W^j$ and $\Phi_i^j \in V^j$ are basis of V_i^{j+1}
- $\forall \Psi_i^j$ and $\forall \Phi_k^j$: $\Psi_i^j \perp \Phi_k^j$

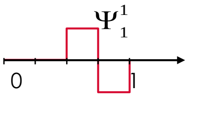
Haar Wavelets

$$\Psi_i^j(x) := \Psi\left(2^j x - i\right) \qquad i = 0, 1, \ldots, 2^j - 1$$

with
$$\Psi(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{else} \end{cases}$$

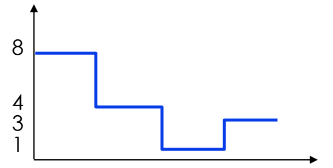
Example: Basis for W^1





Ψ_0^1 and Ψ_1^1 are the basis of W^1

Example



in V^2

$$f(x) = c_0^2 \, \Phi_0^2(x) + c_1^2 \, \Phi_1^2(x) + c_2^2 \, \Phi_2^2(x) + c_3^2 \, \Phi_3^2(x)$$

f(x) = 8x + 4x + 1x + 3x

in $V^1 \oplus W^1$

$$f(x) = c_0^1 \, \Phi_0^1(x) + c_1^1 \, \Phi_1^1(x) + d_0^1 \, \Psi_0^1(x) + d_1^1 \, \Psi_1^1(x)$$

f(x) = 6x + 2x + 2x - 1x

in $V^0 \oplus W^0 \oplus W^1$

$$f(x) = c_0^0 \, \Phi_0^0(x) + d_0^0 \, \Psi_0^0(x) + d_0^1 \, \Psi_0^1(x) + d_1^1 \, \Psi_1^1(x)$$

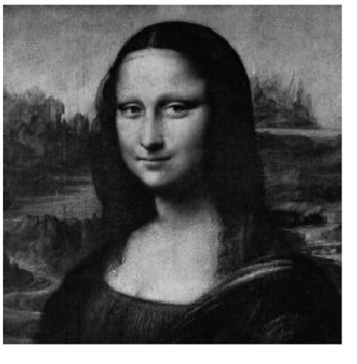
f(x) = 4x + 2x + 2x -1x

2D Application

What is diffent ?

i.e. an image with 256x256 pixels we could convert in a 1D function of V^{16}

possible !!!



But, similarity of neighboring pixels then only would be used in one direction.

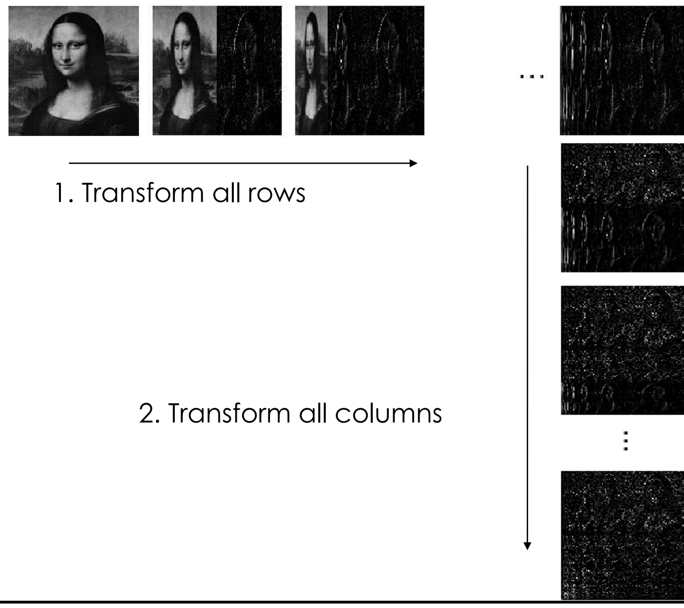
Alternatives:

A) 2D Scaling- and Wavelet functions?? (do not exist)

B) Apply 1D functions separately to columns and rows.

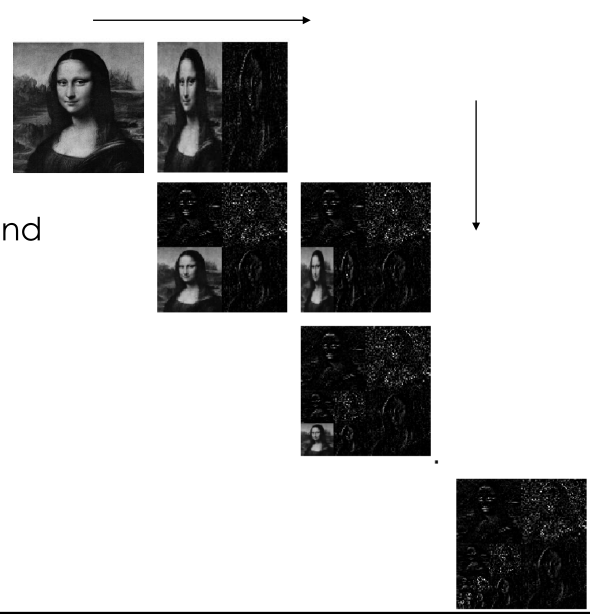
B) is the usual case: →

Standard 2D Haar Wavelet Transformation



Non Standard Transformation

Alternate row and column transformation.



Standard or Non Standard!??

Both methods result in a valid wavelet transformation.
They differ in computation load and in their support.

A) Standard Haar Wavelet basis is a tensor product of 1d basis functions

B) Non standard Haar Wavelet basis defines a

2D scaling function $\Phi\Phi(x, y) := \Phi(x)\Phi(y)$

with wavelet functions

$$\Phi\Psi(x, y) := \Phi(x)\Psi(y)$$

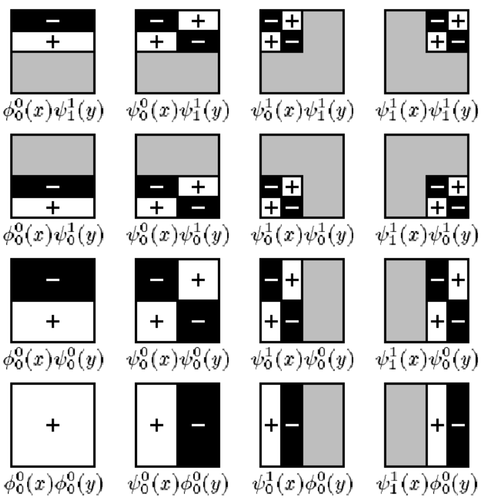
$$\Psi\Phi(x, y) := \Psi(x)\Phi(y)$$

$$\Psi\Psi(x, y) := \Psi(x)\Psi(y)$$

Haar Wavelets in 2D

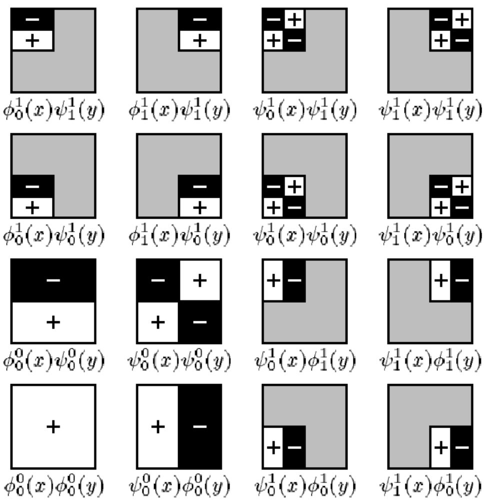
Standard Haar Wavelet basis for ${}_{2D}V^2$

+ = +1, - = -1, grey = 0



Non Standard Haar Wavelet basis ${}_{2D}V^2$

+ = +1, - = -1, grey = 0



Compression

$$f(x) = \sum_{i=0}^{M-1} c_i U_i(x)$$ with an orthonormal basis $\{ U_i \}$

and $\pi(i)$ a permutation of $i = 1....M$ with $\left|c_{\pi(i)}\right| \geq \left|c_{\pi(i+1)}\right|$

Search for $\hat{M}$ with $\hat{f}(x) = \sum_{i=0}^{\hat{M}-1} c_{\pi(i)} U_{\pi(i)}(x)$ and tolerance ε

that $\left\| f(x) - \hat{f}(x) \right\|_{L_2} < \varepsilon$ and $\hat{M} < M$ minimal.

Solution: $\hat{M}$ is the maximum with $\sum_{i=\hat{M}}^{M-1} c_{\pi(i)}^2 \leq \varepsilon^2$

are Haar Wavelets orthonormal ?

Haar Wavelets

$\Phi_0^0(x),\ \Psi_0^0(x),\ \Psi_0^1(x),\ \Psi_1^1(x)\ ...$ Haar wavelets are orthogonal !

z.B. $V^0 \oplus W^0 \oplus W^1$

But not normalized!

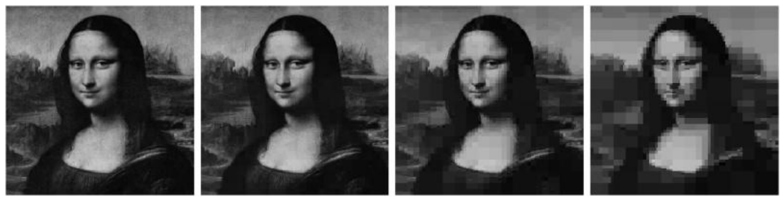
Normalized Haar wavelets:

$$\Phi_i^j(x) := \sqrt{2^j}\ \Phi\left(2^j x - i\right)$$

$$\Psi_i^j(x) := \sqrt{2^j}\ \Psi\left(2^j x - i\right)$$

$\Rightarrow$ the coefficients $c^j \rightarrow \frac{1}{\sqrt{2^j}} c^j$

Haar Wavelet compression



		100%	19%	3%	1%
# coefficients		**100%**	**19%**	**3%**	**1%**
rel. error	L_2	**0%**	**5%**	**10%**	**15%**

In general,
this does not directly translate into the number of bytes required.

Scale analysis `Multi-resolution Analysis´

$$V^0 \subset V^1 \subset V^2 \ldots \subset V^j$$

remark : Haar Wavelets are only one example of wavelets!

Define one-row matrices:

$$\mathbf{\Phi^j}(x) := \left[\Phi_0^j(x),\ \ \Phi_1^j(x), \ldots\ \ \Phi_{M-1}^j(x)\right]$$

$$\mathbf{\Psi^j}(x) := \left[\Psi_0^j(x),\ \ \Psi_1^j(x), \ldots\ \ \Psi_{N-1}^j(x)\right]$$

Refinement:

From $V^0 \subset V^1 \subset V^2 \ldots \subset V^j$ and the linearity results $\exists$ matices $\mathbf{P}, \mathbf{Q}$ with

$$\mathbf{\Phi}^{\mathbf{j}-1}(x) = \mathbf{\Phi^j}(x)\ \ \mathbf{P^j}$$

$$\mathbf{\Psi}^{\mathbf{j-1}}(x) = \mathbf{\Phi^j}(x)\ \ \mathbf{Q^j}$$

Example : Haar Wavelets!

$$\mathbf{P}^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{Q}^2 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Multiresolution Analysis

Notation:

$$\left[\mathbf{\Phi}^{\mathbf{j-1}} \mid \mathbf{\Psi}^{\mathbf{j-1}}\right] = \mathbf{\Phi}^{\mathbf{j}} \quad \left[\mathbf{P}^{\mathbf{j}} \mid \mathbf{Q}^{\mathbf{j}}\right]$$

Example : Haar Wavelets!

$$\left[\Phi_0^1, \Phi_1^1, \Psi_0^1, \Psi_2^1\right] = \left[\Phi_0^2, \Phi_1^2, \Phi_2^2, \Phi_3^2\right] \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Filter Banks

So far we investigated the scaling- and wavelet functions.

However, the coefficients of the Wavelet transformation are more important .

$$f(x) \in V^j, \quad f = \mathbf{\Phi}^j C^j \quad \text{with} \quad C^j = \left[c_0^j, c_1^j, \ldots. c_{M-1}^j\right]$$

In lower resolution: $f = \mathbf{\Phi}^j C^j = \mathbf{\Phi}^{j-1} C^{j-1} + \mathbf{\Psi}^{j\text{-}1} D^{j\text{-}1}$

$$C^{j\text{-}1} = A^j C^j$$
$$D^{j\text{-}1} = B^j C^j$$

$$\rightarrow \quad f = \mathbf{\Phi}^j C^j = \left[\mathbf{\Phi}^{j\text{-}1} \mid \mathbf{\Psi}^{j-1}\right] \begin{bmatrix} C^{j-1} \\ D^{j-1} \end{bmatrix}$$

A^j, B^j are analysis filter

Filter banks (2)

=> $f = \mathbf{\Phi}^j C^j = \left[\mathbf{\Phi}^{j-1} \,|\, \mathbf{\Psi}^{j-1} \right] \begin{bmatrix} C^{j-1} \\ D^{j-1} \end{bmatrix}$

with

a.) $\left[\mathbf{\Phi}^{j-1} \,|\, \mathbf{\Psi}^{j-1} \right] = \mathbf{\Phi}^j \left[\mathbf{P}^j \,|\, \mathbf{Q}^j \right]$

b.) $C^{j-1} = A^j C^j$

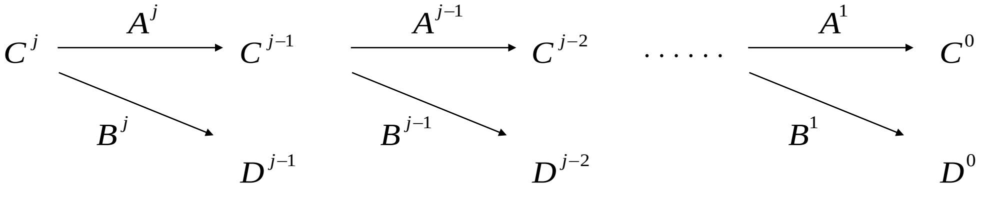
$D^{j-1} = B^j C^j$

$$\Rightarrow \quad f = \mathbf{\Phi}^j C^j = \Phi^j \left[P^j \,|\, Q^j \right] \begin{bmatrix} A^j \\ B^j \end{bmatrix} C^j$$

$$\Rightarrow \quad \left[P^j \,|\, Q^j \right] \begin{bmatrix} A^j \\ B^j \end{bmatrix} = I$$

P^j, Q^j Are synthesis filter

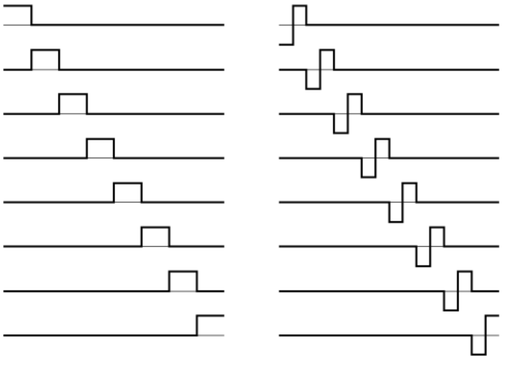
a filter Bank !



Either we know A^j, B^j or P^j, Q^j
-> then we can transform the signal.

Old friends!

$$P^j = \begin{bmatrix} 1 & & & \\ 1 & & & \\ & 1 & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & \\ & & 1 & \\ & & & 1 \\ & & & 1 \end{bmatrix} \qquad Q^j = \begin{bmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ & & \ddots & \\ & & 1 & \\ & & -1 & \\ & & & 1 \\ & & & -1 \end{bmatrix}$$

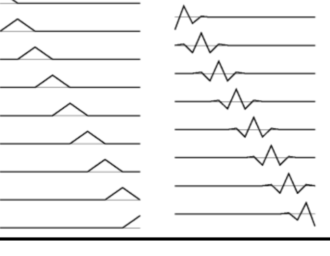


Linear B-spline Wavelets

$$P^1 = \frac{1}{2}\begin{bmatrix} 2 & \\ 1 & 1 \\ & 2 \end{bmatrix} \qquad Q^1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$P^2 = \frac{1}{2}\begin{bmatrix} 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & 1 \\ & & 2 \end{bmatrix} \qquad Q^2 = \begin{bmatrix} -12 & \\ 11 & 1 \\ -6 & -6 \\ 1 & 11 \\ & -12 \end{bmatrix}$$

$$P^{j\geq 3} = \frac{1}{2}\begin{bmatrix} 2 & & & & \\ 1 & 1 & & & \\ & 2 & & & \\ & 1 & 1 & & \\ & & 2 & & \\ & & 1 & \ddots & \\ & & & \ddots & 1 \\ & & & & 2 \\ & & & & 1 & 1 \\ & & & & & 2 \end{bmatrix} \qquad Q^{j\geq 3} = \begin{bmatrix} -12 & & & & & \\ 11 & 1 & & & & \\ -6 & -6 & & & & \\ 1 & 10 & 1 & & & \\ & -6 & -6 & & & \\ & 1 & 10 & & & \\ & & -6 & \ddots & & \\ & & 1 & \ddots & 1 & \\ & & & \ddots & -6 & \\ & & & & 10 & 1 \\ & & & & -6 & -6 \\ & & & & 1 & 11 \\ & & & & & -12 \end{bmatrix}$$



Kubic B-spline Wavelets

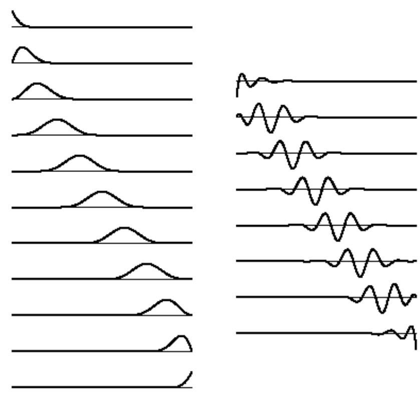
$$P^1 = \frac{1}{2}\begin{bmatrix} 2 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 2 \end{bmatrix} \qquad Q^1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

$$P^2 = \frac{1}{16}\begin{bmatrix} 16 & & & & \\ 8 & 8 & & & \\ & 12 & 4 & & \\ & 3 & 10 & 3 & \\ & & 4 & 12 & \\ & & & 8 & 8 \\ & & & & 16 \end{bmatrix} \qquad Q^2 = \begin{bmatrix} -1368 & \\ 2064 & 240 \\ -1793 & -691 \\ 1053 & 1053 \\ -691 & -1793 \\ 240 & 2064 \\ & -1368 \end{bmatrix}$$

$$P^{j\geq 3} = \frac{1}{16}\begin{bmatrix} 16 & & & & & & & & & \\ 8 & 8 & & & & & & & & \\ & 12 & 4 & & & & & & & \\ & 3 & 11 & 2 & & & & & & \\ & & 8 & 8 & & & & & & \\ & & 2 & 12 & 2 & & & & & \\ & & & 8 & 8 & & & & & \\ & & & 2 & 12 & \ddots & & & & \\ & & & & 8 & \ddots & & & & \\ & & & & 2 & \ddots & 2 & & & \\ & & & & & \ddots & 8 & & & \\ & & & & & & 12 & 2 & & \\ & & & & & & 8 & 8 & & \\ & & & & & & 2 & 11 & 3 & \\ & & & & & & & 4 & 12 & \\ & & & & & & & & 8 & 8 \\ & & & & & & & & & 16 \end{bmatrix}$$

$$Q^3 \approx \begin{bmatrix} -686823 & & & \\ 1000000 & -254803 & & \\ -798167 & 697530 & 14449 & \\ 382460 & -921763 & -78158 & 17 \\ -138342 & 1000000 & 314093 & -2088 \\ 28284 & -720755 & -720755 & 28284 \\ -2088 & 314093 & 1000000 & -138342 \\ 17 & -78158 & -921763 & 382460 \\ & 14449 & 697530 & -798167 \\ & & -254803 & 1000000 \\ & & & -686823 \end{bmatrix}$$

$$Q^{j\geq 4} \approx \begin{bmatrix} -686823 & & & & & & & & & \\ 1000000 & 256326 & & & & & & & & \\ -798167 & -701422 & -15882 & & & & & & & \\ 382460 & 925781 & 85895 & 41 & & & & & & \\ -138342 & -1000000 & -343712 & -5110 & & & & & & \\ 28284 & 711467 & 771597 & 69115 & 41 & & & & & \\ -2088 & -297591 & -1000000 & -325750 & -5110 & & & & & \\ 17 & 62856 & 758322 & 761705 & 69115 & & & & & \\ & -4647 & -323843 & -1000000 & -325750 & & & & & \\ & 37 & 68691 & 761705 & 761705 & & 41 & & & \\ & & -5079 & -325750 & -1000000 & & -5110 & & & \\ & & 41 & 69115 & 761705 & \ddots & 69115 & 41 & & \\ & & & -5110 & -325750 & \ddots & -325750 & -5079 & & \\ & & & 41 & 69115 & \ddots & 761705 & 68691 & 37 & \\ & & & & -5110 & & -1000000 & -323843 & -4647 & \\ & & & & 41 & & 761705 & 758322 & 62856 & 17 \\ & & & & & & -325750 & -1000000 & -297591 & -2088 \\ & & & & & & 69115 & 771597 & 711467 & 28284 \\ & & & & & & -5110 & -343712 & -1000000 & -138342 \\ & & & & & & 41 & 85895 & 925781 & 382460 \\ & & & & & & & -15882 & -701422 & -798167 \\ & & & & & & & & 256326 & 1000000 \\ & & & & & & & & & -686823 \end{bmatrix}$$



Computing the c's and d's from P and Q

with
$$C^{j\text{-}1} = A^j C^j, \qquad D^{j\text{-}1} = B^j C^j$$
we an compute the transformation.

However,
given the sparse synthesis filter $\boldsymbol{P}^j$ and $\boldsymbol{Q}^j$,
in general this does not guarantee that $\boldsymbol{A}^j$ and $\boldsymbol{B}^j$ are sparse too.
=> Could be computational expensive!

$$\left[P^j \mid Q^j \right] \begin{bmatrix} A^j \\ B^j \end{bmatrix} = I$$

Computationally more efficient:

Solve the sparse system $\left[P^j \mid Q^j \right] \begin{bmatrix} c^{j-1} \\ d^{j-1} \end{bmatrix} = c^j$ for $\boldsymbol{c}^{j\text{-}1}$ and $\boldsymbol{d}^{j\text{-}1}$ using the LU decomposition (O(n)).

So the transform can be efficiently computed without knowing $\boldsymbol{A}^j$ and $\boldsymbol{B}^j$

Daubechies Wavlets:

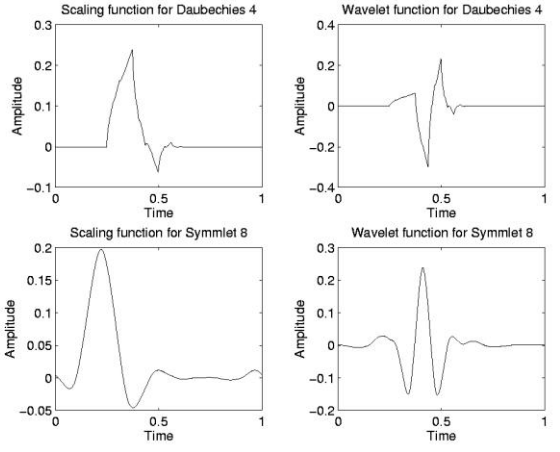


Figure 19: