An introduction to the Cayley-Bacharach theorems

following Eisenbud, Green, Harris, *Cayley-Bacharach theorems and conjectures*

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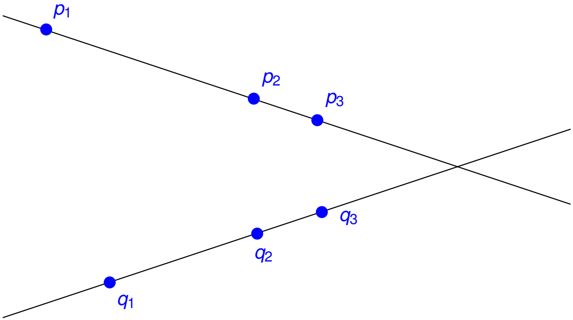
Rémi Bignalet

May 17, 2018

Pappus's Theorem



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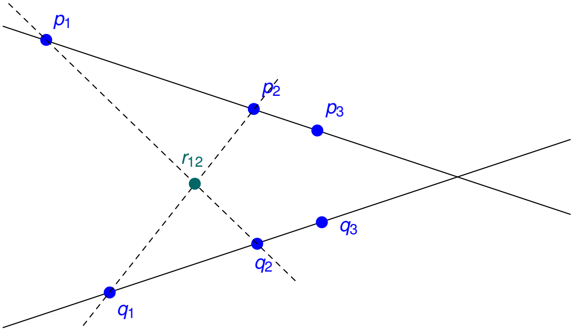

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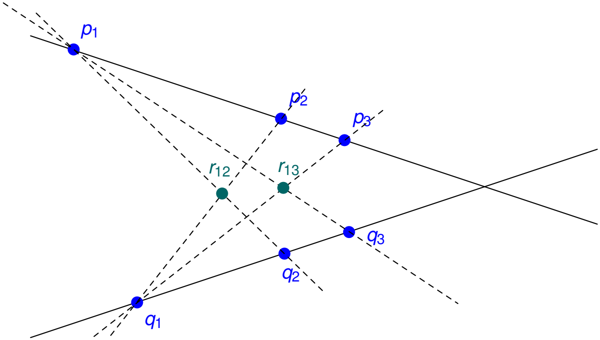

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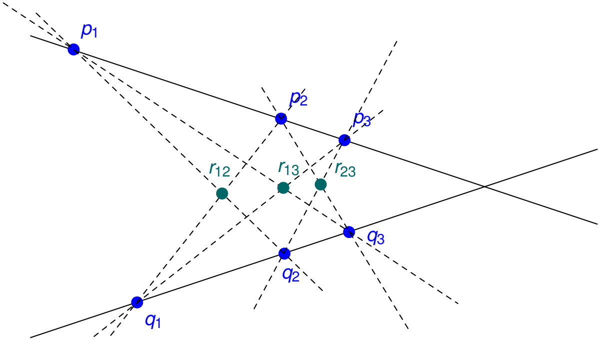

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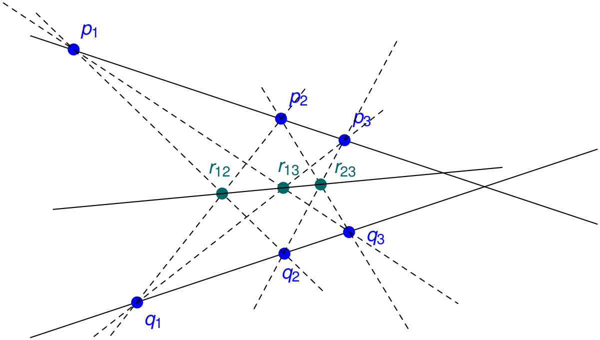
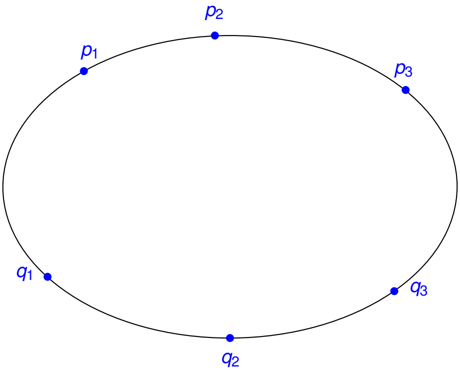


Pappus's Theorem

Theorem (First version of the Cayley-Bacharach theorem, IVth century AC)

Lest L and M be two lines in the plane. Lest p_1, p_2 and p_3 be distinct points of L and let q_1, q_2 and q_3 be distinct points on M all distinct from the point $L \cap M$. If for each $j \neq l \in \{1, 2, 3\}$ we let r_{jk} be the point of intersection of the lines $\overline{p_j q_k}$ and $\overline{p_k q_j}$, then the three points r_{jk} are colinear.

Pascal's theorem



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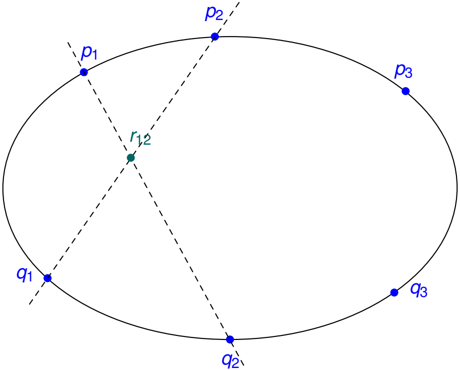

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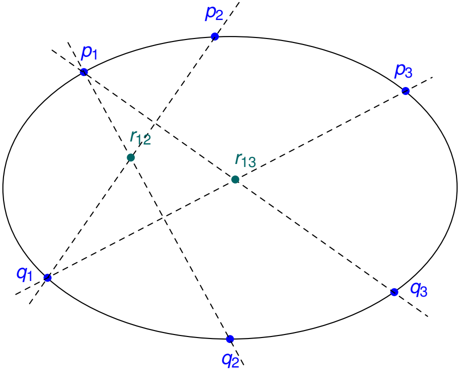

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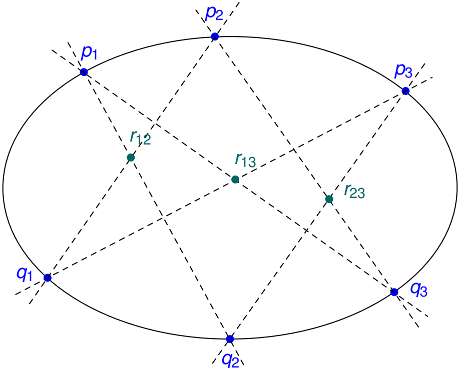

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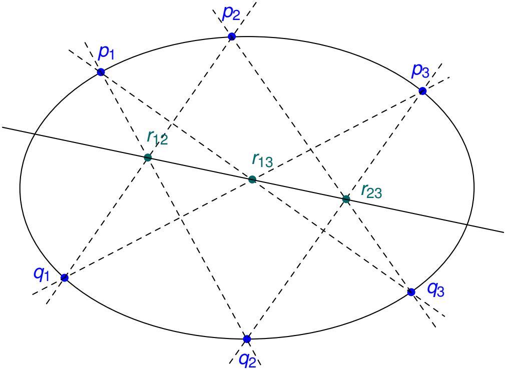

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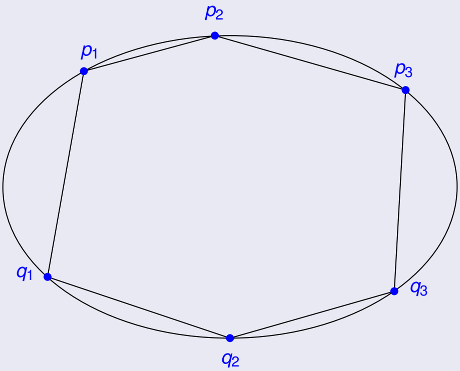
Theorem (Pascal's theorem, 1640)

If a hexagon is inscribed in a conic in the projective plane, the opposite sides of the hexagon meet in three collinear points.



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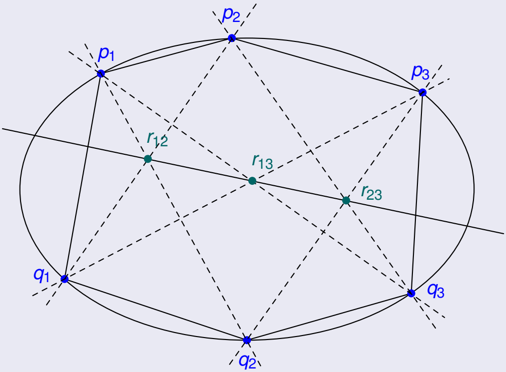

Projective Geometry

Definition

$$\mathbb{P}^2(\mathbb{C}) = \mathbb{C}^3 \backslash \{0\} / (x, y, z) \sim (\lambda x, \lambda y, \lambda z)$$

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Remark

Given $f = \sum_{i+j+k=d} \alpha_{ijk} x^i y^j z^k$

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Remark

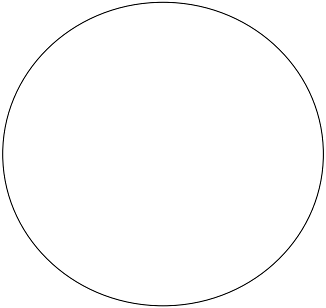
Given $f = \sum_{i+j+k=d} \alpha_{ijk} x^i y^j z^k$ *homogeneous polynomial of degree* d *in three variables:*

$$\{(x_0 : y_0 : z_0) \in \mathbb{P}^2(\mathbb{C}), f(x_0, y_0, z_0) = 0\}$$

makes sense as a subset of $\mathbb{P}^2(\mathbb{C})$.

Pappus' theorem as deformation

$\{y^2 + xz = 0\}$



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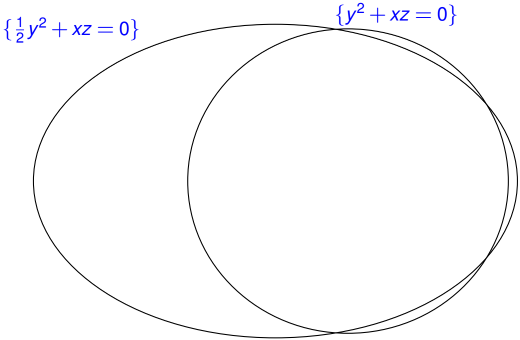

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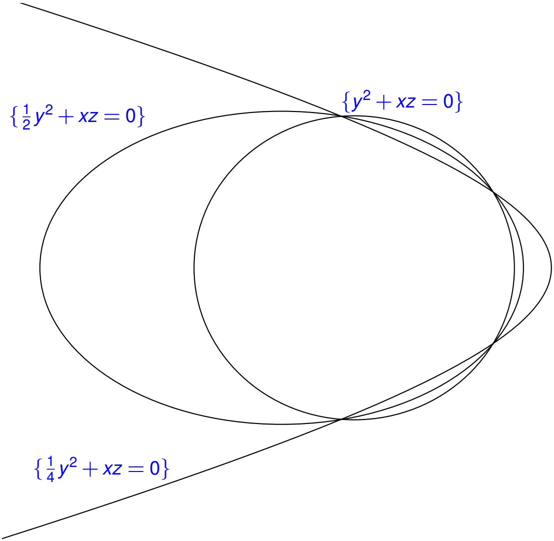

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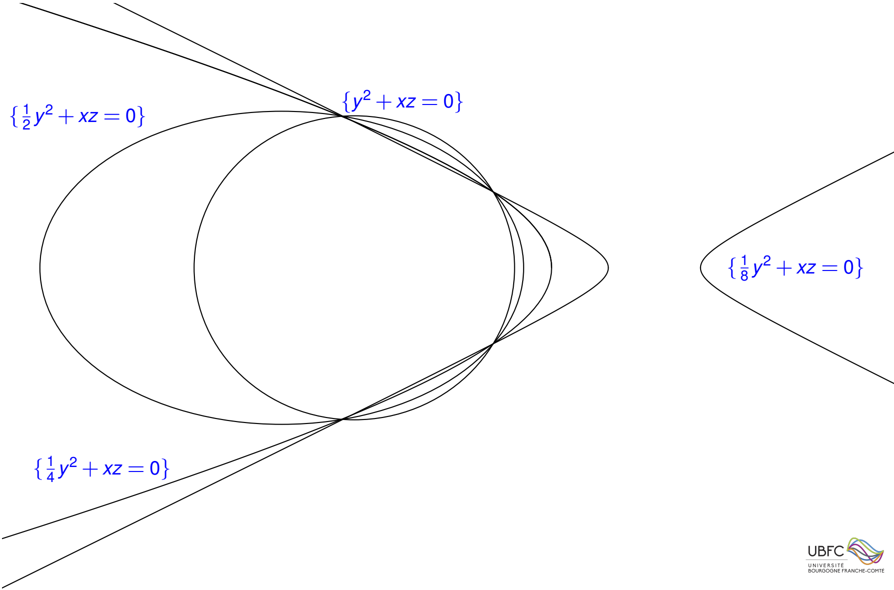

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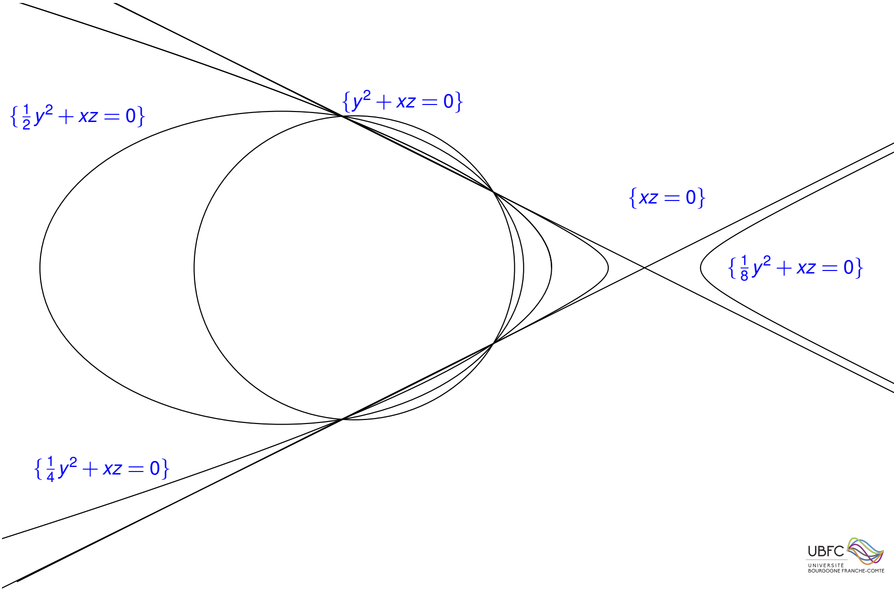

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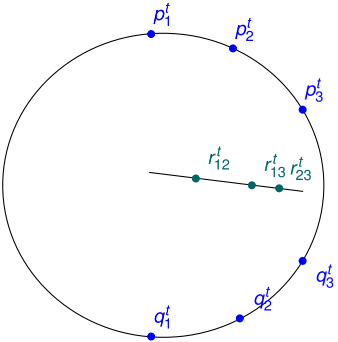

$det(r_{12}^t, r_{13}^t, r_{23}^t) = 0$

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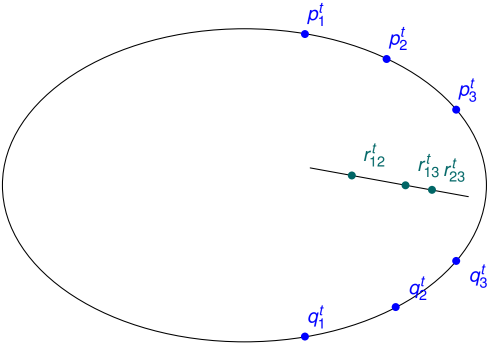

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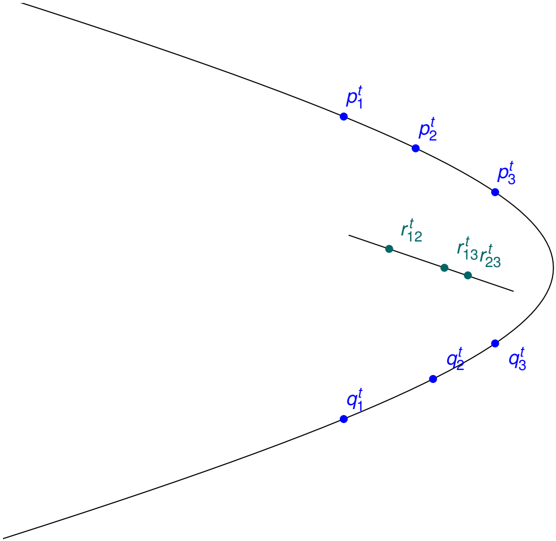

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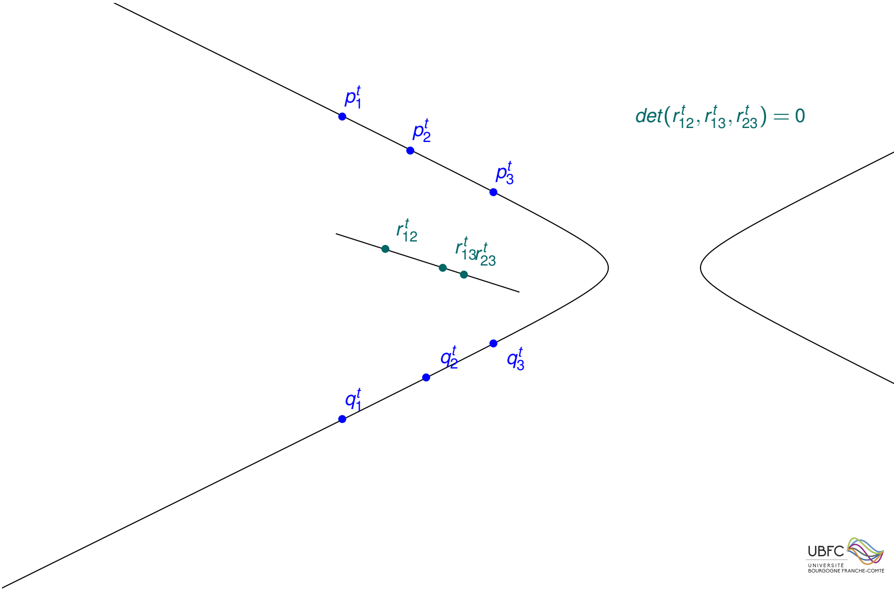

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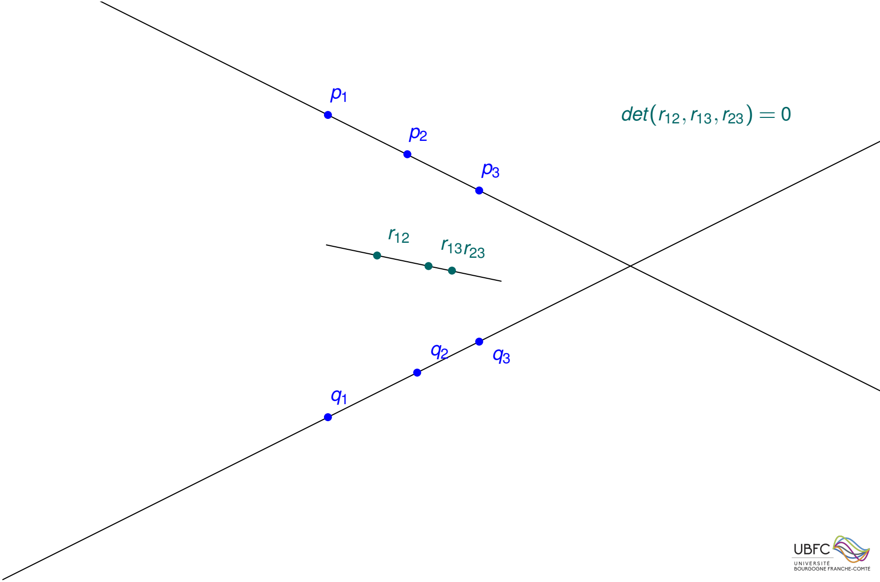

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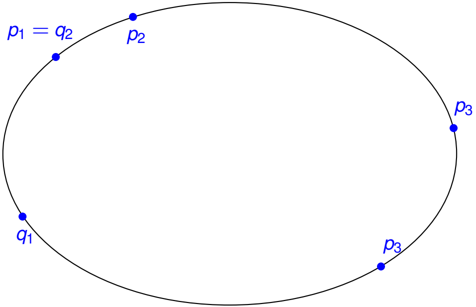
A first limit case



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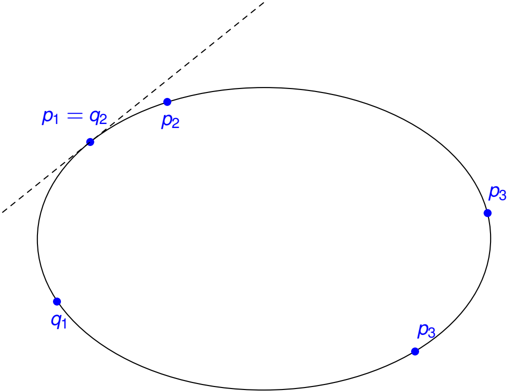

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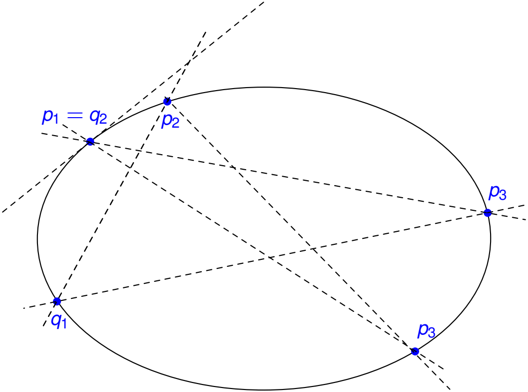

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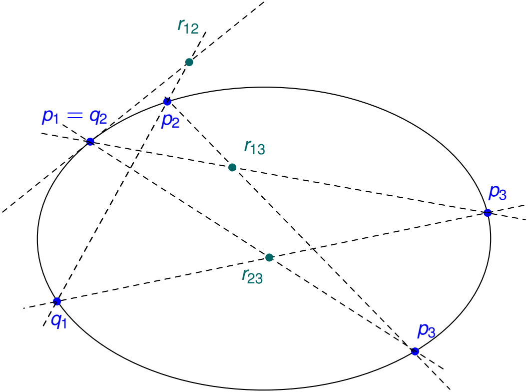

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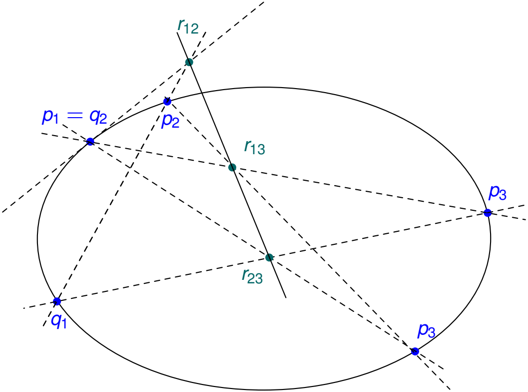


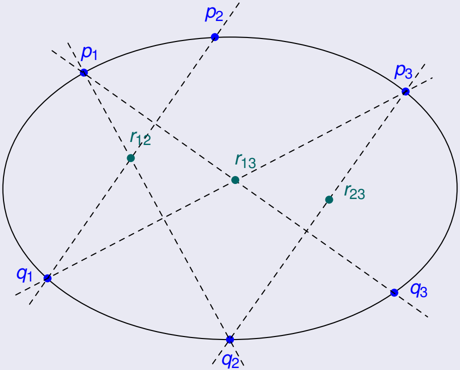
Chasles' Theorem

Theorem (Chasles' Theorem)

Let X_1 and $X_2 \subset \mathbb{P}^n(\mathbb{C})$ be two cubic plane curve meeting in nine points $p_1, \ldots, p_9$. If $X \subset \mathbb{P}^n(\mathbb{C})$ is any cubic containing a priori $p_1, \ldots, p_8$, then X contains p_9 as well.

Application

Application (Pascal's theorem)



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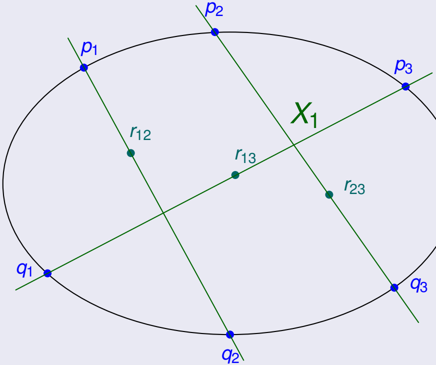

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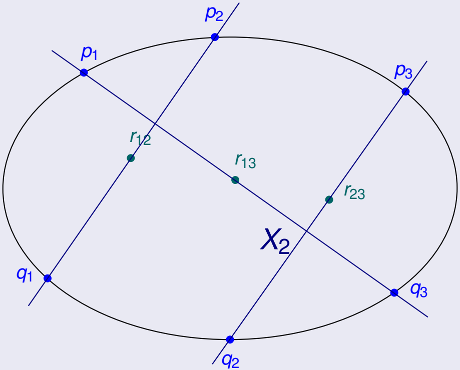

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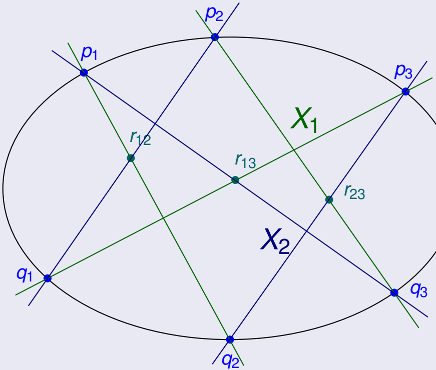

Terminology

Definition

If $\Gamma \subset \mathbb{P}^2(\mathbb{C})$ is a set of distinct points, we say that Γ imposes l conditions on the polynomial of degree d if the subspace $\mathbb{C}[x, y, z]_d^h$ vanishing at $p_1, \ldots, p_m$ has codimension l.

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Example

The set Γ *of* 3 *collinear points imposes two conditions on polynomials of degree* 1 *i.e.* $h_\Gamma(1) = 2$*.*

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Theorem

Given $\Gamma = \{p_1, \ldots, p_9\} = X_1 \cap X_2$ *where* X_1 *and* X_2 *are plane cubics,*

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Remark

Proof is actually showing that $h_\Gamma(3) = h_{\Gamma'}(3) = 8$.

A lemma

Lemma

Let $\Omega = \{p_1, \ldots, p_n\} \subset \mathbb{P}^2$ *be a set of* n *distinct points and let an integer* d *such that* $n \leq 2d + 2$.

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Ω *fail to impose independent conditions on curves of degree* d *if and only if either* $d + 2$ *points of* Ω *are collinear or* $n = 2d + 2$ *and* Ω *is contained in a conic.*

One idea

Theorem (Bézout's theorem)

Let X_1 and $X_2 \subset \mathbb{P}^2(\mathbb{C})$ be plane curves of degree d and e respectively.
If X_1 and X_2 have no common component then they meet in $d \times e$ points.

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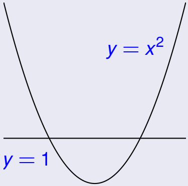

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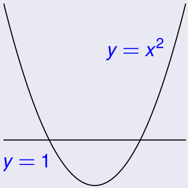

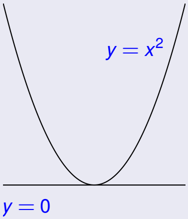

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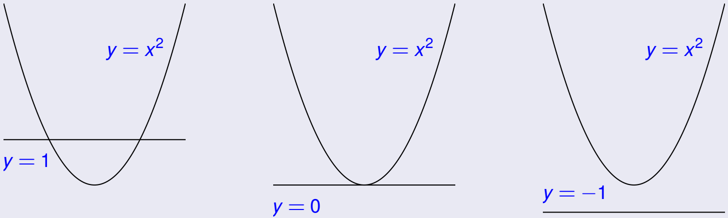

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The theory of curves (XIXth century)

Theorem (Cayley-Bacharach theorem, version 4)

Let X_1 and $X_2 \subset \mathbb{P}^2(\mathbb{C})$ be plane curves of degree d and e respectively, meeting in a collection of $d \times e$ distinct points $\Gamma = X_1 \cap X_2 = \{p_1, \ldots, p_{de}\}$. If $C \subset \mathbb{P}^n(\mathbb{C})$ is any plane curve of degree $d + e - 3$ containing all but one point of Γ, then C contains all of Γ.

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If $k \leq s$ is a nonnegative integer, then the dimension of the vector space of polynomials of degree k, vanishing on Γ' (modulo those containing all of Γ) is equal to the failure of Γ'' to impose independent conditions on polynomials of degree $s - k$.

A last XIXth century version

Theorem (Cayley-Bacharach theorem, version 6)

Let $X_1, \ldots, X_n$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degrees $d_1, \ldots, d_n$ respectively, meeting transversely,

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The XXth century

Definition

Let A be an Artinian ring with residue field $\mathbb{C}$. The ring A is Gorenstein if there exists a $\mathbb{C}$-linear map $A \to \mathbb{C}$ such that the composition

$$Q : A \times A \to A \to \mathbb{C}$$

where the first map is multiplication in A, is a non degenerate pairing on the $\mathbb{C}$-vector space A.

A last version

Theorem (Cayley-Bacharach theorem, version 7)

Let $X_1, \ldots, X_n$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degrees $d_1, \ldots, d_n$ and suppose that the intersection $\Gamma = X_1 \cap \ldots \cap X_n$ is zero-dimensional.

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Thanks for your attention!