SPDEs and regularity structures

Harprit Singh

ETH Zürich
hasingh@student.ethz.ch

12.04.2018

Content

Section 1

Space time white noise

Gaussian white noise

Intuitively one would like to construct a random function ξ on (let's say) $\mathbb{R}^d$ s.t.

$$(\xi(x))_{x \in \mathbb{R}^d} \text{ are i.i.d. Gaussian random variables.}$$

Gaussian white noise

Intuitively one would like to construct a random function ξ on (let's say) $\mathbb{R}^d$ s.t.

$$(\xi(x))_{x \in \mathbb{R}^d} \text{ are i.i.d. Gaussian random variables.}$$

This turns out to be useless!
Space time white noise captures this intuition (up to one caveat).

Gaussian white noise

Intuitively one would like to construct a random function ξ on (let's say) $\mathbb{R}^d$ s.t.

$$(\xi(x))_{x\in\mathbb{R}^d} \text{ are i.i.d. Gaussian random variables.}$$

This turns out to be useless!
Space time white noise captures this intuition (up to one caveat).

1. We give a probabilistic definition.
2. A more analytic definition.
3. An intuitive construction and a useful analytic one.

Definitions of space time white noise

Definitions of space time white noise

A probabilistic definition:

Definition

Space-time white noise on $\mathbb{R}^d$ is the centred Gaussian random field ξ on $L^2(\mathbb{R}^d)$ with covariance:

$$E[\xi(f)\xi(g)] = \langle f, g\rangle_{L^2(\mathbb{R}^d)}.$$

Definitions of space time white noise

A probabilistic definition:

Definition

Space-time white noise on $\mathbb{R}^d$ is the centred Gaussian random field ξ on $L^2(\mathbb{R}^d)$ with covariance:

$$E[\xi(f)\xi(g)] = \langle f, g \rangle_{L^2(\mathbb{R}^d)}.$$

A more analytic definition:

Definition

Space time white noise on $\mathbb{R}^d$ is the $\mathcal{S}'(\mathbb{R}^d)$-valued random variable satisfying for any $\phi \in \mathcal{S}(\mathbb{R}^d)$:

$$E[\exp(i\langle \xi, \phi \rangle)] = e^{-\frac{1}{2}\|\phi\|_{L^2}^2}.$$

More intuitive construction of White noise

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k=1]=\mathbb{P}[X_k=-1]=\frac{1}{2}$).

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = -1] = \frac{1}{2}$). Set

$$\xi^1(\cdot) := \sum_{k\in\mathbb{Z}^d} X_k \mathbf{1}_{[0,1)^d}(\cdot - k).$$

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = -1] = \frac{1}{2}$). Set

$$\xi^1(\cdot) := \sum_{k\in\mathbb{Z}^d} X_k \mathbf{1}_{[0,1)^d}(\cdot - k).$$

- Denote the rescaled functions

$$\xi^n(\cdot) := \frac{1}{2^{-n\frac{d}{2}}}\xi^1(\frac{\cdot}{2^{-n}}).$$

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = -1] = \frac{1}{2}$). Set
$$\xi^1(\cdot) := \sum_{k\in\mathbb{Z}^d} X_k \mathbf{1}_{[0,1)^d}(\cdot - k).$$
- Denote the rescaled functions
$$\xi^n(\cdot) := \frac{1}{2^{-n\frac{d}{2}}}\xi^1(\frac{\cdot}{2^{-n}}).$$
- Then $\xi^n \to \xi$ in distribution as $n \to \infty$.

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = -1] = \frac{1}{2}$). Set

$$\xi^1(\cdot) := \sum_{k\in\mathbb{Z}^d} X_k \mathbf{1}_{[0,1)^d}(\cdot - k).$$

- Denote the rescaled functions

$$\xi^n(\cdot) := \frac{1}{2^{-n\frac{d}{2}}}\xi^1(\frac{\cdot}{2^{-n}}).$$

- Then $\xi^n \to \xi$ in distribution as $n \to \infty$.

An alternative:

- Alternatively, let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. $\mathcal{N}(0,1)$ random variables,

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = -1] = \frac{1}{2}$). Set

$$\xi^1(\cdot) := \sum_{k\in\mathbb{Z}^d} X_k \mathbf{1}_{[0,1)^d}(\cdot - k).$$

- Denote the rescaled functions

$$\xi^n(\cdot) := \frac{1}{2^{-n\frac{d}{2}}}\xi^1(\frac{\cdot}{2^{-n}}).$$

- Then $\xi^n \to \xi$ in distribution as $n \to \infty$.

An alternative:

- Alternatively, let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. $\mathcal{N}(0,1)$ random variables,let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$.

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = -1] = \frac{1}{2}$). Set

$$\xi^1(\cdot) := \sum_{k\in\mathbb{Z}^d} X_k \mathbf{1}_{[0,1)^d}(\cdot - k).$$

- Denote the rescaled functions

$$\xi^n(\cdot) := \frac{1}{2^{-n\frac{d}{2}}}\xi^1\left(\frac{\cdot}{2^{-n}}\right).$$

- Then $\xi^n \to \xi$ in distribution as $n \to \infty$.

An alternative:

- Alternatively, let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. $\mathcal{N}(0,1)$ random variables,let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$.Then we can define:

$$\xi = \sum_{n\in\mathbb{N}} X_n \cdot f_n$$

More intuitive construction of White noise

- Let $(X_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. family of coin tosses (i.e $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = -1] = \frac{1}{2}$). Set
$$\xi^1(\cdot) := \sum_{k\in\mathbb{Z}^d} X_k \mathbf{1}_{[0,1)^d}(\cdot - k).$$
- Denote the rescaled functions
$$\xi^n(\cdot) := \frac{1}{2^{-n\frac{d}{2}}}\xi^1(\frac{\cdot}{2^{-n}}).$$
- Then $\xi^n \to \xi$ in distribution as $n \to \infty$.

An alternative:

- Alternatively, let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. $\mathcal{N}(0,1)$ random variables,let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$.Then we can define:
$$\xi = \sum_{n\in\mathbb{N}} X_n \cdot f_n \in \mathcal{S}'(\mathbb{R}^d).$$

Section 2

Examples of SDEs and SPDEs

Motivation and examples

Examples of Stochastic ordinary and partial differential equations (SDEs and SPDEs).

- Physical Brownian motion
- Black-Scholes model from mathematical finance
- Stochastic heat equation
- KPZ-equation
- Φ_2^4 and Φ_3^4-equation

Space time white noise is present in all of these equations.

Physical Brownian motion

Model of a small particle

Small particle of mass m has position $x(t) \in \mathbb{R}^3$ determined by Newton's equations:

$$m\ddot{x} = -M\dot{x} + \xi,$$

where $M \in \mathbb{R}^{3\times 3}$, $M > 0$, symmetric models friction, each $(\xi)_i$ is 1-d white noise.

Physical Brownian motion

Model of a small particle

Small particle of mass m has position $x(t) \in \mathbb{R}^3$ determined by Newton's equations:

$$m\ddot{x} = -M\dot{x} + \xi,$$

where $M \in \mathbb{R}^{3\times 3}$, $M > 0$, symmetric models friction, each $(\xi)_i$ is 1-d white noise.

Remarks:

Physical Brownian motion

Model of a small particle

Small particle of mass m has position $x(t) \in \mathbb{R}^3$ determined by Newton's equations:

$$m\ddot{x} = -M\dot{x} + \xi,$$

where $M \in \mathbb{R}^{3\times 3}$, $M > 0$, symmetric models friction, each $(\xi)_i$ is 1-d white noise.

Remarks:

- For $m = 0$ and $M = id$, we recover mathematical Brownian motion.

Physical Brownian motion

Model of a small particle

Small particle of mass m has position $x(t) \in \mathbb{R}^3$ determined by Newton's equations:

$$m\ddot{x} = -M\dot{x} + \xi,$$

where $M \in \mathbb{R}^{3\times 3}$, $M > 0$, symmetric models friction, each $(\xi)_i$ is 1-d white noise.

Remarks:

- For $m = 0$ and $M = id$, we recover mathematical Brownian motion.
- In general M could depend on $x(t)$ (incorporating inhomogeneity of the underlying space).

Models in mathematical finance

Black-Scholes model

The market is modelled by (let's say) two assets (S_0, S_1), satisfying:

$$\frac{dS_0}{S_0} = rdt, \frac{dS_1}{S_1(t)} = \mu(t)dt + \sigma(t)dB$$

S_0 is called *Numéraire* (e.g. a bank account with interest rate r) and S_1 is a risky asset with *drift* μ and volatility σ.

A linear SPDE

Stochastic heat equation

The stochastic heat equation on ($\mathbb{T}^d$ let's say) is given by:

$$\partial_t u = (\Delta - 1)u + \xi,$$

where $u = u(x,t)$ is a function of space-time and ξ is space time white noise.

A linear SPDE

Stochastic heat equation

The stochastic heat equation on ($\mathbb{T}^d$ let's say) is given by:

$$\partial_t u = (\Delta - 1)u + \xi,$$

where $u = u(x,t)$ is a function of space-time and ξ is space time white noise.

- $d = 1$: It arises as scaling limit of symmetric interface models (e.g. the SOS model). It's stationary is solution is (essentially) BM. It has Hölder regularity $\frac{1}{2} - \epsilon$ in space and $\frac{1}{4} - \epsilon$ in time.

A linear SPDE

Stochastic heat equation

The stochastic heat equation on ($\mathbb{T}^d$ let's say) is given by:

$$\partial_t u = (\Delta - 1)u + \xi,$$

where $u = u(x,t)$ is a function of space-time and ξ is space time white noise.

- $d = 1$: It arises as scaling limit of symmetric interface models (e.g. the SOS model). It's stationary is solution is (essentially) BM. It has Hölder regularity $\frac{1}{2} - \epsilon$ in space and $\frac{1}{4} - \epsilon$ in time.
- $d = 2$: this is related to the Gaussian free field. The solution can be seen as a continuous map $[0, T] \to \mathcal{C}^{0-\epsilon}(\mathbb{T}^2)$

A linear SPDE

Stochastic heat equation

The stochastic heat equation on ($\mathbb{T}^d$ let's say) is given by:

$$\partial_t u = (\Delta - 1)u + \xi,$$

where $u = u(x, t)$ is a function of space-time and ξ is space time white noise.

- $d = 1$: It arises as scaling limit of symmetric interface models (e.g. the SOS model). It's stationary is solution is (essentially) BM. It has Hölder regularity $\frac{1}{2} - \epsilon$ in space and $\frac{1}{4} - \epsilon$ in time.
- $d = 2$: this is related to the Gaussian free field. The solution can be seen as a continuous map $[0, T] \to \mathcal{C}^{0-\epsilon}(\mathbb{T}^2)$
- Inhomogeneous scaling $\mathfrak{s} = (1, .., 1, 2)$, i.e. time counts double. Then ξ has regularity $-\frac{d+2}{2} - \epsilon$ and the heat-kernel regularises by 2. Thus u is $\frac{2-d}{2} - \epsilon$ Hölder regular.

A semi-linear SPDE

The KPZ-equation

The KPZ-equation on $\mathbb{T}$ is formally given by

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi.$$

A semi-linear SPDE

The KPZ-equation

The KPZ-equation on $\mathbb{T}$ is formally given by

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi.$$

- It arises as scaling limit of (barely) asymmetric interface models.

A semi-linear SPDE

The KPZ-equation

The KPZ-equation on $\mathbb{T}$ is formally given by

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi.$$

- It arises as scaling limit of (barely) asymmetric interface models.
- Lies in the "crossover regime" between the "Gaussian universality class" and the "KPZ-fixed point".

A semi-linear SPDE

The KPZ-equation

The KPZ-equation on $\mathbb{T}$ is formally given by

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi.$$

- It arises as scaling limit of (barely) asymmetric interface models.
- Lies in the "crossover regime" between the "Gaussian universality class" and the "KPZ-fixed point".
- Problem: $(\partial_x u)^2$ not canonically definable.

A semi-linear SPDE

The KPZ-equation

The KPZ-equation on $\mathbb{T}$ is formally given by

$$\partial_t u = \partial_x^2 u + (\partial_x u)^2 + \xi.$$

- It arises as scaling limit of (barely) asymmetric interface models.
- Lies in the "crossover regime" between the "Gaussian universality class" and the "KPZ-fixed point".
- Problem: $(\partial_x u)^2$ not canonically definable.
- Hopf Cole solution: $Z = e^u$ formally solves

$$\partial_t Z = \partial_x^2 Z + \underbrace{Z \cdot \xi}_{\text{makes sense by "Ito"}} .$$

Set

$$u = \log Z.$$

Φ^4_2 and Φ^4_3-equation

The Φ^4_2 equation

This equation reads:

$$\partial_t u = \triangle u - u^3 + \xi,$$

where u is a generalised function on $[0, T) \times \mathbb{T}^2$ and ξ is again space time white noise.

Φ^4_2 and Φ^4_3-equation

The Φ^4_2 equation

This equation reads:

$$\partial_t u = \triangle u - u^3 + \xi,$$

where u is a generalised function on $[0, T) \times \mathbb{T}^2$ and ξ is again space time white noise.

- Interest arising from from quantum field theory: The invariant measure for this equation is related to bosonic euclidean quantum field theory.

Φ^4_2 and Φ^4_3-equation

The Φ^4_2 equation

This equation reads:

$$\partial_t u = \triangle u - u^3 + \xi,$$

where u is a generalised function on $[0, T) \times \mathbb{T}^2$ and ξ is again space time white noise.

- Interest arising from from quantum field theory: The invariant measure for this equation is related to bosonic euclidean quantum field theory.
- It is also related to the Ising model.

Φ^4_2 and Φ^4_3-equation

The Φ^4_2 equation

This equation reads:

$$\partial_t u = \triangle u - u^3 + \xi,$$

where u is a generalised function on $[0, T) \times \mathbb{T}^2$ and ξ is again space time white noise.

- Interest arising from from quantum field theory: The invariant measure for this equation is related to bosonic euclidean quantum field theory.
- It is also related to the Ising model.
- In dimension 3 it is called Φ^4_3, two new prominent solution theories: the "Theory of regularity structures" and the "Theory of paracontrolled distributions".

Section 3

Regularity structures: Definitions and a theorem

What's a regularity structure and a model?

A vast generalisation of the relationship between Hölder (and Besov) functions and Taylor Polynomials.

What's a regularity structure and a model?

A vast generalisation of the relationship between Hölder (and Besov) functions and Taylor Polynomials.

- Recall: Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function. Fix $\gamma \in \mathbb{R} \setminus \mathbb{N}$. Then the Taylor polynomial $\mathcal{P}_x^{[\gamma]}(f)$ of f at $x \in \mathbb{R}^d$ of order $[\gamma] \in \mathbb{N}$ is the unique polynomial, such that

$$|\langle f - \mathcal{P}_x^{[\gamma]}(f), \phi_x^\lambda \rangle| \lesssim \lambda^\gamma.$$

We write $\phi_x^\lambda(\cdot) := \frac{1}{\lambda^d}\phi(\frac{\cdot - x}{\lambda})$.

What's a regularity structure and a model?

A vast generalisation of the relationship between Hölder (and Besov) functions and Taylor Polynomials.

- Recall: Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function. Fix $\gamma \in \mathbb{R} \setminus \mathbb{N}$. Then the Taylor polynomial $\mathcal{P}_x^{[\gamma]}(f)$ of f at $x \in \mathbb{R}^d$ of order $[\gamma] \in \mathbb{N}$ is the unique polynomial, such that

$$|\langle f - \mathcal{P}_x^{[\gamma]}(f), \phi_x^\lambda \rangle| \lesssim \lambda^\gamma.$$

  We write $\phi_x^\lambda(\cdot) := \frac{1}{\lambda^d}\phi(\frac{\cdot - x}{\lambda})$.
- A regularity structure $\mathcal{T}$ together with a model Z give a way to make the same kind of approximation for certain distributions.

What's a regularity structure and a model?

A vast generalisation of the relationship between Hölder (and Besov) functions and Taylor Polynomials.

- Recall: Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function. Fix $\gamma \in \mathbb{R} \setminus \mathbb{N}$. Then the Taylor polynomial $\mathcal{P}_x^{[\gamma]}(f)$ of f at $x \in \mathbb{R}^d$ of order $[\gamma] \in \mathbb{N}$ is the unique polynomial, such that

$$|\langle f - \mathcal{P}_x^{[\gamma]}(f), \phi_x^\lambda \rangle| \lesssim \lambda^\gamma.$$

  We write $\phi_x^\lambda(\cdot) := \frac{1}{\lambda^d}\phi(\frac{\cdot - x}{\lambda})$.
- A regularity structure $\mathcal{T}$ together with a model Z give a way to make the same kind of approximation for certain distributions.
- Of course not possible for any distribution (as with Taylor polynomials).

Interplay between regularity structure and model

Interplay between regularity structure and model

- A regularity structure $\mathcal{T}$ plays the role of the abstract Polynomials.

Interplay between regularity structure and model

- A regularity structure $\mathcal{T}$ plays the role of the abstract Polynomials.
- The model gives the abstract polynomials $\tau \in \mathcal{T}$ analytic meaning, generalising the map:

$$\textit{Abstract polynomials} \to \textit{(generalised) functions on } \mathbb{R}^d.$$

Interplay between regularity structure and model

- A regularity structure $\mathcal{T}$ plays the role of the abstract Polynomials.
- The model gives the abstract polynomials $\tau \in \mathcal{T}$ analytic meaning, generalising the map:

$$\textit{Abstract polynomials} \rightarrow \textit{(generalised) functions on } \mathbb{R}^d.$$

- One could say, a regularity structure is an algebraic structure, and a model is its analytic 'flesh'.

Interplay between regularity structure and model

- A regularity structure $\mathcal{T}$ plays the role of the abstract Polynomials.
- The model gives the abstract polynomials $\tau \in \mathcal{T}$ analytic meaning, generalising the map:

Abstract polynomials $\rightarrow$ *(generalised) functions on* $\mathbb{R}^d$.

- One could say, a regularity structure is an algebraic structure, and a model is its analytic 'flesh'.

Let us now give some rigorous definitions!

Definition: Regularity structure

Definition

A regularity structure is a triple $\mathcal{T} = (A, T, G)$ consisting of the following elements:

Definition: Regularity structure

Definition

A regularity structure is a triple $\mathcal{T} = (A, T, G)$ consisting of the following elements:

- A discrete index set $A \subset \mathbb{R}$ which is bounded from below and contains zero.

Definition: Regularity structure

Definition

A regularity structure is a triple $\mathcal{T} = (A, T, G)$ consisting of the following elements:

- A discrete index set $A \subset \mathbb{R}$ which is bounded from below and contains zero.
- A graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, where $T_0 = \langle 1 \rangle \approx \mathbb{R}$ and all T_α are finite dimensional. We call T the model space.

Definition: Regularity structure

Definition

A regularity structure is a triple $\mathcal{T} = (A, T, G)$ consisting of the following elements:

- A discrete index set $A \subset \mathbb{R}$ which is bounded from below and contains zero.
- A graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, where $T_0 = \langle 1 \rangle \approx \mathbb{R}$ and all T_α are finite dimensional. We call T the model space.
- A group G of linear operators acting on T, such that for every $\Gamma \in G$ the following holds: The restriction $\Gamma|_{T_0}$ is the identity map and for all $\tau \in T_\alpha$:

$$\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\alpha.$$

  This group G is called the structure group of $\mathcal{T}$.

The example of Taylor Polynomials:

The example of Taylor Polynomials:

- We take $A = \mathbb{N}$, corresponding to the "homogeneities" of polynomials.

The example of Taylor Polynomials:

- We take $A = \mathbb{N}$, corresponding to the "homogeneities" of polynomials.
- We set $T = \bigoplus_{n\in\mathbb{N}} T_n$, where

$$T_n = \langle \{X_1^{n_1} \cdot \ldots \cdot X_d^{n_d} | n_i \in \mathbb{N}, \sum_{i=1}^{d} n_i = n\} \rangle$$

The example of Taylor Polynomials:

- We take $A = \mathbb{N}$, corresponding to the "homogeneities" of polynomials.
- We set $T = \bigoplus_{n \in \mathbb{N}} T_n$, where

$$T_n = \langle \{ X_1^{n_1} \cdot \ldots \cdot X_d^{n_d} | n_i \in \mathbb{N}, \sum_{i=1}^{d} n_i = n \} \rangle$$

- What's the meaning of the group G?

The example of Taylor Polynomials:

- We take $A = \mathbb{N}$, corresponding to the "homogeneities" of polynomials.
- We set $T = \bigoplus_{n \in \mathbb{N}} T_n$, where

$$T_n = \langle \{ X_1^{n_1} \cdot \ldots \cdot X_d^{n_d} | n_i \in \mathbb{N}, \sum_{i=1}^{d} n_i = n \} \rangle$$

- What's the meaning of the group G?
  It corresponds to the natural action of translating polynomials.

The example of Taylor Polynomials:

- We take $A = \mathbb{N}$, corresponding to the "homogeneities" of polynomials.
- We set $T = \bigoplus_{n \in \mathbb{N}} T_n$, where

$$T_n = \langle \{ X_1^{n_1} \cdot \ldots \cdot X_d^{n_d} | n_i \in \mathbb{N}, \sum_{i=1}^{d} n_i = n \} \rangle$$

- What's the meaning of the group G?
  It corresponds to the natural action of translating polynomials.
  In this example it is given by the maps

$$\underbrace{X^k}_{:=\prod_{i=1}^{d}(X_i)^{k_i}} \mapsto \underbrace{(X+h)^k}_{:=\prod_{i=1}^{d}(X_i+h_i)^{k_i}} ,$$

  where $h \in \mathbb{R}^d$. Thus it is isomorphic to $\mathbb{R}^d$.

Definition: Regularity structure

Definition

A regularity structure is a triple $\mathcal{T} = (A, T, G)$ consisting of the following elements:

- A discrete index set $A \subset \mathbb{R}$ which is bounded from below and contains zero.
- A graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, where $T_0 = \langle 1 \rangle \approx \mathbb{R}$ and all T_α are finite dimensional. We call T the model space.
- A group G of linear operators acting on T, such that for every $\Gamma \in G$ the following holds: The restriction $\Gamma|_{T_0}$ is the identity map and for all $\tau \in T_\alpha$:

$$\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\alpha.$$

This group G is called the structure group of $\mathcal{T}$.

Definition: Model

Definition: Model

Definition

Given a regularity structure $\mathcal{T} = (A, T, G)$ and $r > |\min A|$, a model for $\mathcal{T}$ is a pair $Z = (\Pi, \Gamma)$, consisting of

Definition: Model

Definition

Given a regularity structure $\mathcal{T} = (A, T, G)$ and $r > |\min A|$, a model for $\mathcal{T}$ is a pair $Z = (\Pi, \Gamma)$, consisting of

- a map $\Pi : \mathbb{R}^d \to L(T, \mathcal{S}'),\ x \mapsto \Pi_x$, such that

$$|\langle \Pi_x \tau, \phi_x^\lambda \rangle| \lesssim \lambda^\alpha,$$

  for $\tau \in T_\alpha$ and uniformly over $\lambda \in [0, 1]$ and $\{\phi \in C_c^\infty |\ \|\phi\|_{\mathcal{C}^r} \lesssim 1, \mathrm{supp}\phi \subset B_1\}$.

Definition: Model

Definition

Given a regularity structure $\mathcal{T} = (A, T, G)$ and $r > |\min A|$, a model for $\mathcal{T}$ is a pair $Z = (\Pi, \Gamma)$, consisting of

- a map $\Pi : \mathbb{R}^d \to L(T, \mathcal{S}'), \; x \mapsto \Pi_x$, such that

  $$|\langle \Pi_x \tau, \phi_x^\lambda \rangle| \lesssim \lambda^\alpha,$$

  for $\tau \in T_\alpha$ and uniformly over $\lambda \in [0, 1]$ and $\{\phi \in C_c^\infty | \; \|\phi\|_{\mathcal{C}^r} \lesssim 1, \text{supp}\phi \subset B_1\}$.
- and $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G, \; (x, y) \mapsto \Gamma_{x,y}$ satisfying the conditions:

Definition: Model

Definition

Given a regularity structure $\mathcal{T} = (A, T, G)$ and $r > |\min A|$, a model for $\mathcal{T}$ is a pair $Z = (\Pi, \Gamma)$, consisting of

- a map $\Pi : \mathbb{R}^d \to L(T, \mathcal{S}'),\ x \mapsto \Pi_x$, such that

$$|\langle \Pi_x \tau, \phi_x^\lambda \rangle| \lesssim \lambda^\alpha,$$

  for $\tau \in T_\alpha$ and uniformly over $\lambda \in [0, 1]$ and $\{\phi \in C_c^\infty |\ \|\phi\|_{\mathcal{C}^r} \lesssim 1, \mathrm{supp}\phi \subset B_1\}$.

- and $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G,\ (x, y) \mapsto \Gamma_{x,y}$ satisfying the conditions:

$$\Pi_x \Gamma_{x,y} = \Pi_y$$

  and for $\tau \in T_\alpha$

Definition: Model

Definition

Given a regularity structure $\mathcal{T} = (A, T, G)$ and $r > |\min A|$, a model for $\mathcal{T}$ is a pair $Z = (\Pi, \Gamma)$, consisting of

- a map $\Pi : \mathbb{R}^d \to L(T, \mathcal{S}'),\ x \mapsto \Pi_x$, such that

$$|\langle \Pi_x \tau, \phi_x^\lambda \rangle| \lesssim \lambda^\alpha,$$

  for $\tau \in T_\alpha$ and uniformly over $\lambda \in [0, 1]$ and $\{\phi \in C_c^\infty |\ \|\phi\|_{\mathcal{C}^r} \lesssim 1, \mathrm{supp}\phi \subset B_1\}$.
- and $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G,\ (x, y) \mapsto \Gamma_{x,y}$ satisfying the conditions:

$$\Pi_x \Gamma_{x,y} = \Pi_y$$

  and for $\tau \in T_\alpha$

$$|\Gamma_{x,y}\tau|_\beta \lesssim |x - y|^{\alpha - \beta}.$$

Back to the polynomial example

In the case of the polynomial model, the maps (Π, Γ) are given as follows:

Back to the polynomial example

In the case of the polynomial model, the maps (Π, Γ) are given as follows:

- The map Π_x realises an abstract monomial X^k as:

$$(\Pi_x X^k)(y) = (y - x)^k.$$

Back to the polynomial example

In the case of the polynomial model, the maps (Π, Γ) are given as follows:

- The map Π_x realises an abstract monomial X^k as:

$$(\Pi_x X^k)(y) = (y-x)^k.$$

Note that the analytic bound $|\langle \Pi_x X^k, \phi_x^\lambda \rangle| \lesssim \lambda^{|k|}$ holds.

Back to the polynomial example

In the case of the polynomial model, the maps (Π, Γ) are given as follows:

- The map Π_x realises an abstract monomial X^k as:

$$(\Pi_x X^k)(y) = (y - x)^k.$$

  Note that the analytic bound $|\langle \Pi_x X^k, \phi_x^\lambda \rangle| \lesssim \lambda^{|k|}$ holds.
- And the map $\Gamma_{x,y}$ is by:

$$\Gamma_{x,y} X^k = (X + (x - y))^k.$$

Back to the polynomial example

In the case of the polynomial model, the maps (Π, Γ) are given as follows:

- The map Π_x realises an abstract monomial X^k as:

$$(\Pi_x X^k)(y) = (y - x)^k.$$

  Note that the analytic bound $|\langle \Pi_x X^k, \phi_x^\lambda \rangle| \lesssim \lambda^{|k|}$ holds.

- And the map $\Gamma_{x,y}$ is by:

$$\Gamma_{x,y} X^k = (X + (x - y))^k.$$

  Clearly $\Pi_x \Gamma_{x,y} = \Pi_y$ holds. The bound $|\Gamma_{x,y} X^k|_m \lesssim |x - y|^{|k|-m}$ follows from the formula:

$$(X + (x - y))^k = \sum_{l \leq k} \binom{k}{l} (x - y)^{k-l} X^l.$$

Modelled distributions

How can we describe distributions locally using regularity structures?
This is again done in analogy to Hölder functions.

Modelled distributions

A slightly more convoluted looking definition of $\mathcal{C}^\gamma$ functions goes as follows:

Modelled distributions

A slightly more convoluted looking definition of $\mathcal{C}^\gamma$ functions goes as follows:

$$\mathcal{C}^\gamma(\mathbb{R}^d) = \{f \in \mathcal{S}' | \forall x \in \mathbb{R}^d \; \exists P_x^{[\gamma]} \text{ a Polynomial of order}[\gamma] : \\ |\langle f - P_x^{[\gamma]}, \phi_x^\lambda \rangle| \lesssim \lambda^\gamma\}.$$

Modelled distributions

A slightly more convoluted looking definition of $\mathcal{C}^\gamma$ functions goes as follows:

$$\mathcal{C}^\gamma(\mathbb{R}^d) = \{f \in \mathcal{S}' | \forall x \in \mathbb{R}^d \; \exists P_x^{[\gamma]} \text{ a Polynomial of order}[\gamma] : \\ |\langle f - P_x^{[\gamma]}, \phi_x^\lambda \rangle| \lesssim \lambda^\gamma\}.$$

When defining modelled distributions, one defines the generalised "polynomials" first:

Modelled distributions

A slightly more convoluted looking definition of $\mathcal{C}^\gamma$ functions goes as follows:

$$\mathcal{C}^\gamma(\mathbb{R}^d) = \{f \in \mathcal{S}' | \forall x \in \mathbb{R}^d \; \exists P_x^{[\gamma]} \text{ a Polynomial of order} [\gamma] : \\ |\langle f - P_x^{[\gamma]}, \phi_x^\lambda \rangle| \lesssim \lambda^\gamma\}.$$

When defining modelled distributions, one defines the generalised "polynomials" first:

Definition

Given a regularity structure $\mathcal{T} = (A, T, G)$ and a model $Z = (\Pi, \Gamma)$ for it, we define $\mathcal{D}^\gamma$ as the space of all maps $f : \mathbb{R}^d \to T_{<\gamma}$ such that the following bound holds:

$$|f(x+h) - \Gamma_{x+h,x} f(x)|_\alpha \lesssim \|h\|^{\gamma-\alpha},$$

for all $\alpha \in A \cap (-\infty, \gamma)$.

Reconstruction theorem

The following theorem is a lynch pin in this theory:

Theorem (Hairer 14)

Let $\mathcal{T} = (A, T, G)$ be a regularity structure and $Z = (\Pi, \Gamma)$ a model for it. Set $\alpha = \min A$. Then, for $\gamma > 0$, there exists a unique continuous linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}^\alpha$, such that:

$$|\langle \mathcal{R}f - \Pi_x f(x), \phi_x^\lambda \rangle| \lesssim \lambda^\gamma \tag{1}$$

for all $f \in \mathcal{D}^\gamma$ and all models $Z = (\Pi, \Gamma)$. Furthermore, the map $Z \mapsto \mathcal{R}$ is continuous (in an appropriate sense).

Reconstruction theorem

The following theorem is a lynch pin in this theory:

Theorem (Hairer 14)

Let $\mathcal{T} = (A, T, G)$ be a regularity structure and $Z = (\Pi, \Gamma)$ a model for it. Set $\alpha = \min A$. Then, for $\gamma > 0$, there exists a unique continuous linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}^\alpha$, such that:

$$|\langle \mathcal{R}f - \Pi_x f(x), \phi_x^\lambda \rangle| \lesssim \lambda^\gamma \tag{1}$$

for all $f \in \mathcal{D}^\gamma$ and all models $Z = (\Pi, \Gamma)$. Furthermore, the map $Z \mapsto \mathcal{R}$ is continuous (in an appropriate sense).

- For the polynomial regularity structure and $\gamma \notin \mathbb{N}$ the reconstruction map $\mathcal{R}$ is an isomorphism between $\mathcal{D}^\gamma$ and $\mathcal{C}^\gamma$.

Section 4

How to solve subcritical SPDEs using regulartiy structures

A very rough outline!

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

- It turns out this equation is *subcritical*! (This is essential!)

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

- It turns out this equation is *subcritical*! (This is essential!)
- Interpret the equation in mild form: $u = K \star (\xi - u^3)$, started with 0 initial condition. (Which still doesn't make sense!)

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

- It turns out this equation is *subcritical*! (This is essential!)
- Interpret the equation in mild form: $u = K \star (\xi - u^3)$, started with 0 initial condition. (Which still doesn't make sense!)
- Build a regularity structure adapted to the equation. It consist of formal expressions arising in the Picard iteration (roughly speaking).

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

- It turns out this equation is *subcritical*! (This is essential!)
- Interpret the equation in mild form: $u = K \star (\xi - u^3)$, started with 0 initial condition. (Which still doesn't make sense!)
- Build a regularity structure adapted to the equation. It consist of formal expressions arising in the Picard iteration (roughly speaking).
- Solve the equation on the level of modelled distributions, the solution map is continuous. (One can lift the heat Kernel and there is a notion of product on regularity structures.)

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

- It turns out this equation is *subcritical*! (This is essential!)
- Interpret the equation in mild form: $u = K \star (\xi - u^3)$, started with 0 initial condition. (Which still doesn't make sense!)
- Build a regularity structure adapted to the equation. It consist of formal expressions arising in the Picard iteration (roughly speaking).
- Solve the equation on the level of modelled distributions, the solution map is continuous. (One can lift the heat Kernel and there is a notion of product on regularity structures.)
- For each mollified noise ξ_ϵ there is a canonical model. Unfortunately, these models don't converge as $\epsilon \to 0$. Take the BPHZ-lift instead! (Renormalisation!), then there is convergence.

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

- It turns out this equation is *subcritical*! (This is essential!)
- Interpret the equation in mild form: $u = K \star (\xi - u^3)$, started with 0 initial condition. (Which still doesn't make sense!)
- Build a regularity structure adapted to the equation. It consist of formal expressions arising in the Picard iteration (roughly speaking).
- Solve the equation on the level of modelled distributions, the solution map is continuous. (One can lift the heat Kernel and there is a notion of product on regularity structures.)
- For each mollified noise ξ_ϵ there is a canonical model. Unfortunately, these models don't converge as $\epsilon \to 0$. Take the BPHZ-lift instead! (Renormalisation!), then there is convergence.
- Renormalisation corresponds to a change of non linearity in the equation.

How to solve $\partial_t u = \Delta u - u^3 + \xi$ on $\mathbb{T}^3$

- It turns out this equation is *subcritical*! (This is essential!)
- Interpret the equation in mild form: $u = K \star (\xi - u^3)$, started with 0 initial condition. (Which still doesn't make sense!)
- Build a regularity structure adapted to the equation. It consist of formal expressions arising in the Picard iteration (roughly speaking).
- Solve the equation on the level of modelled distributions, the solution map is continuous. (One can lift the heat Kernel and there is a notion of product on regularity structures.)
- For each mollified noise ξ_ϵ there is a canonical model. Unfortunately, these models don't converge as $\epsilon \to 0$. Take the BPHZ-lift instead! (Renormalisation!), then there is convergence.
- Renormalisation corresponds to a change of non linearity in the equation.

Type of result from regularity structures

One obtains the following type of result:

Type of result from regularity structures

One obtains the following type of result:

Theorem (Hairer 14)

Let $\xi_\epsilon = \rho_\epsilon \star \xi$ denote the regularisation of space-time white noise with a compactly supported smooth mollifier ρ_ϵ. Denote by u_ϵ the solutions to

$$\partial_t u_\epsilon = \Delta u_\epsilon + C_\epsilon u_\epsilon - u_\epsilon^3 + \xi_\epsilon.$$

Then, there exist choices of constants C_ϵ diverging as $\epsilon \to 0$, as well as a processes u such that $u_\epsilon \to u$ in probability. Furthermore, while the constants C_ϵ do depend crucially on the choice of mollifiers ρ_ϵ, the limit u does not.

Some comments on the existence theorem

In several ways this result is stated incompletely.

Some comments on the existence theorem

In several ways this result is stated incompletely.

- Some indeterminacy: For example replace C_ϵ by $C_\epsilon + 1$.

Some comments on the existence theorem

In several ways this result is stated incompletely.

- Some indeterminacy: For example replace C_ϵ by $C_\epsilon + 1$.
- What initial conditions make sense?

Some comments on the existence theorem

In several ways this result is stated incompletely.

- Some indeterminacy: For example replace C_ϵ by $C_\epsilon + 1$.
- What initial conditions make sense?
- The convergence in probability takes place in a Besov space C^α where $\alpha < 0$.

Some comments on the existence theorem

In several ways this result is stated incompletely.

- Some indeterminacy: For example replace C_ϵ by $C_\epsilon + 1$.
- What initial conditions make sense?
- The convergence in probability takes place in a Besov space C^α where $\alpha < 0$.
- This result is only finite in time. But there exist a priory "energy estimates" guaranteeing global in time existence.

Thank you!

Thank you!

Before I end:

- Questions?

Thank you!

Before I end:

- Questions?
- Reference: Martin Hairer's website.

Thank you!

Before I end:

- Questions?
- Reference: Martin Hairer's website.
- Thank you again!

Thank you!

Before I end:

- Questions?
- Reference: Martin Hairer's website.
- Thank you again!