Configurations of lines on del Pezzo surfaces

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Cubic surfaces

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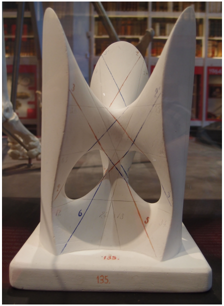
Example

$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$ (Clebsch surface)



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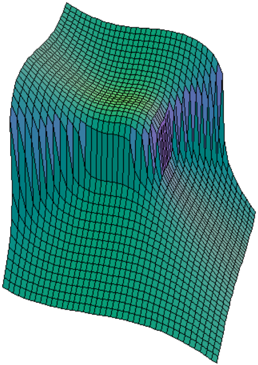
$x^3 + y^3 + z^3 = 1$ (Fermat cubic)



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Theorem (Cayley-Salmon, 1849)

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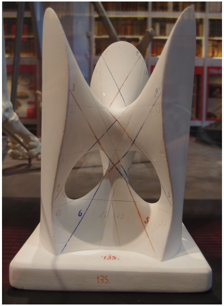
Clebsch surface



Cubic surfaces

A point on a smooth cubic surface in $\mathbb{P}^3$ that is contained in three lines is called an *Eckardt point*.

Lemma (Hirschfeld, 1967)
There are at most 45 Eckardt points on a cubic surface.

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Example
The Clebsch surface has 10 Eckardt points; the Fermat cubic has 18 Eckardt points.

More general: del Pezzo surfaces

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Definition

A del Pezzo surface X is a 'nice' surface over a field k that has an embedding in some $\mathbb{P}^n_k$, such that $-aK_X$ is linearly equivalent to a hyperplane section for some a. The *degree* is the self intersection $(-K_X)^2$ of the anticanonical divisor.

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Question:
What do we know about lines on del Pezzo surfaces of other degrees? Generalizations of Eckardt points?

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Let P be a point in the plane. The construction blowing up replaces P by a line E, called the exceptional curve above P; each point on this line E is identified with a direction through P.

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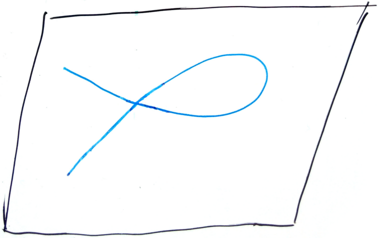
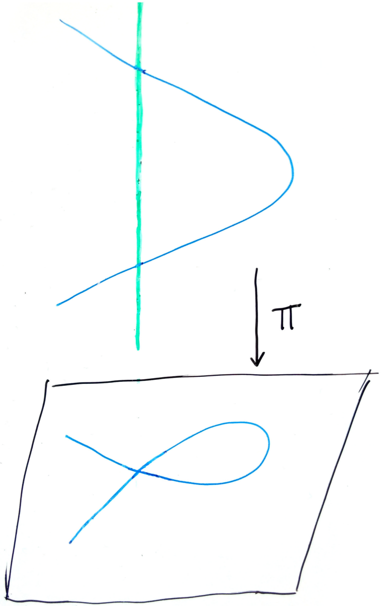
We often do this to *resolve a singularity*.

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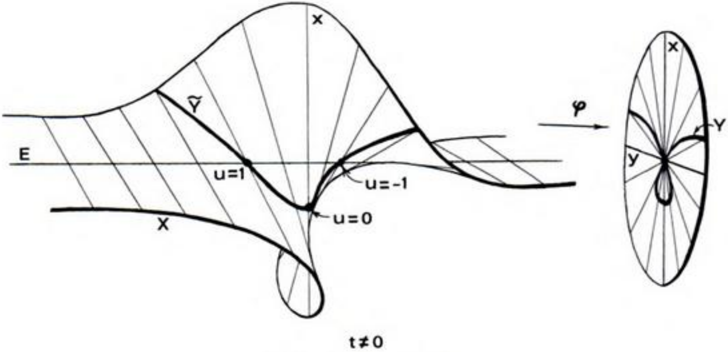

From: Robin Hartshorne, Algebraic Geometry.

Some facts about blow-ups of points

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- Two lines that intersect in the plane in P do not intersect in X! They both intersect the exceptional curve E, but in different points.
- Outside P, everything stays the same.

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Instead of blowing up singular points, we can also blow up 'normal' points in the plane. Doing this in a specific way gives us exactly the del Pezzo surfaces!

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<u>Theorem</u>

Let X be a del Pezzo surface of degree d over an algebraically closed field. Then X is isomorphic to either the product of two lines (only for degree 8), or $\mathbb{P}^2$ blown up in $9 - d$ points in general position.

where *general position* means

- no three points on a line;
- no six points on a conic;
- no eight points on a cubic that is singular at one of them.

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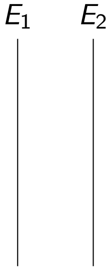
  the strict transform of
- lines through two of the points;
- conics through five of the points;
- cubics through seven of the points, singular at one of them;
- quartics through eight of the points, singular at three of them;
- quintics through eight of the points, singular at six of them;
- sextics through eight of the points, singular at all of them, containing one of them as a triple point.

Degree 7

Blow up 2 points



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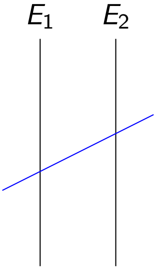

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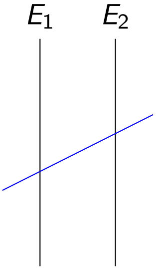
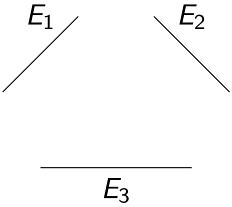


Degree 6

Blow up 3 points



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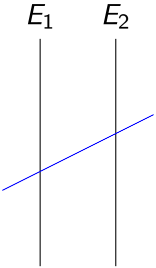

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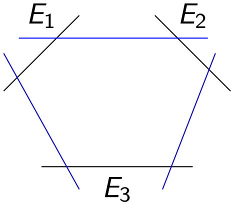

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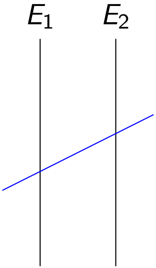

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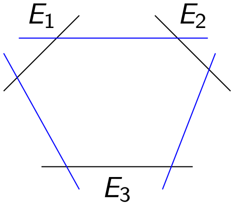
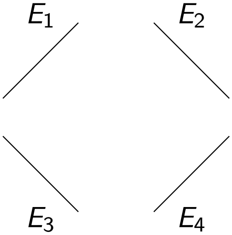


Degree 5

Blow up 4 points



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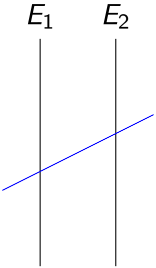

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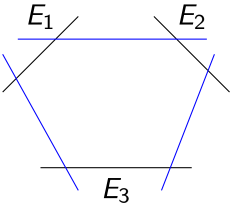

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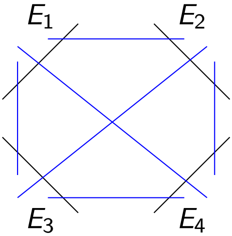

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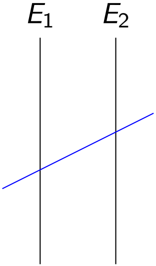

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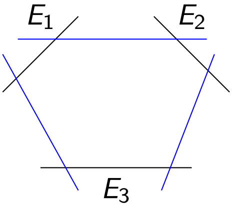

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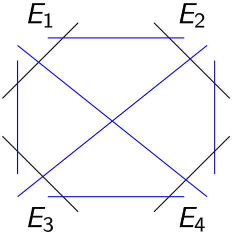

d	1	2	3	4	5	6	7	8
lines on X	240	56	27	16	10	6	3	1

Back to degree three

We blow up 6 points. So the 27 lines are:
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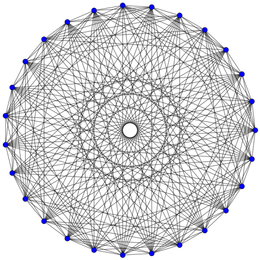

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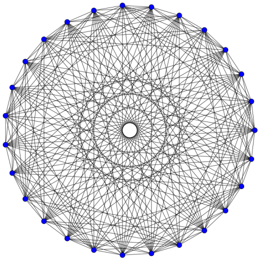

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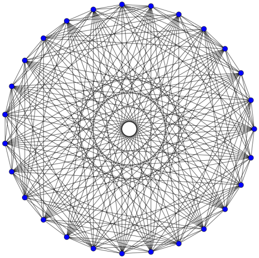

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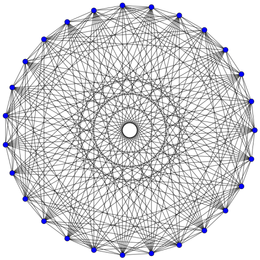
If a point is contained in *n* lines, then the lines form a full subgraph (*clique*) of size *n*.



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$\Longrightarrow$ maximal size of cliques gives an upper bound for the number of lines through one point.

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We also saw that there are at most 45 Eckardt points on a cubic surface; we can see this from the graph as well. $\frac{27 \cdot 5}{3} = 45$.

Degree two

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<u>Fact</u>

Any line l intersects exactly one other line l' with multiplicity two, and 27 other lines with multiplicity one. These 27 lines do not intersect l', and they form again the complement of the Schläfli graph.

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Point in four lines: *generalized Eckardt point*.

Degree one

To get a del Pezzo surface X of degree one we blow up the plane in 8 points $P_1, \ldots, P_8$ in general position. We obtain the following 'lines' (exceptional curves):

- 8 lines above the P_i
- $\binom{8}{2} = 28$ lines through 2 of the P_i
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- $\ldots$

We find a total of 240 lines on X!

How can we study the configurations of these 240 lines?

The root system E_8

Consider the lattice in $\mathbb{R}^8$ given by

$$\Lambda = \left\{ (x_i) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \tfrac{1}{2}\mathbb{Z}\right)^8 \mid \sum x_i \in 2\mathbb{Z} \right\}.$$

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Fact

The 240 lines on a del Pezzo surface of degree one are isomorphic to the root system E_8.

$(\{\text{exceptional curves on } X\} \longrightarrow K_X^{\perp},\ e \longmapsto e + K_X)$

Symmetries in the graph on the 240 lines

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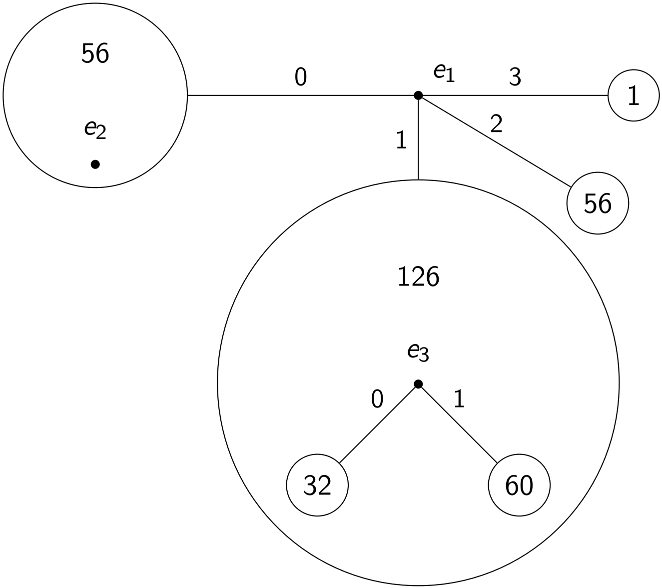
- To study the different cliques in G we use this symmetry.

The graph G on the 240 lines



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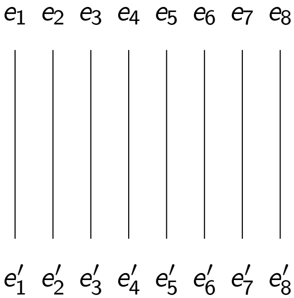
For geometric reasons, it is interesting to distinguish between cliques that have edges of weight 3 in them, and cliques that do not.

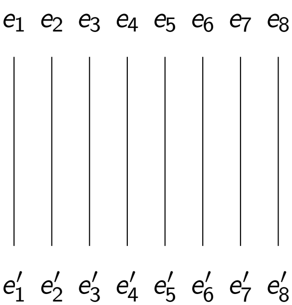
Maximal cliques in G



$$e_i \cdot e_i' = 3$$

Cliques with edges of weight 3: maximal size 16. There are 2025 such cliques.

Maximal cliques in G



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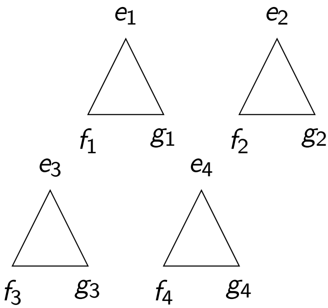

$e_i \cdot f_i = e_i \cdot g_i = f_i \cdot g_i = 2$

Cliques without edges of weight 3: maximal size 12. There are 179200 such cliques.

Sharp upperbound?

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I turns out that for a DP1, the upper bound given by the graph is (almost) never sharp, making this case different from all other degrees.

Classical geometry - the case of cliques of size 16

Proposition

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Let $Q_1, \ldots, Q_8$ be eight points in the plane (over a field with char $\neq 2$) in general position. Let L_i be the line through Q_{2i} and Q_{2i-1} for $i \in \{1, 2, 3, 4\}$,

Classical geometry - the case of cliques of size 16

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Let $Q_1, \ldots, Q_8$ be eight points in the plane (over a field with char $\neq 2$) in general position. Let L_i be the line through Q_{2i} and Q_{2i-1} for $i \in \{1, 2, 3, 4\}$, and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in Q_j.

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Assume that the four lines L_1, L_2, L_3 and L_4 all intersect in one point P. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through P.

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Corollary

No six pairs of 'lines' intersecting with multiplicity three go through one point, hence a point on a del Pezzo surface of degree 1 lies on at most ten lines in characteristic $\neq 2$ (in the case that we consider lines intersecting with multiplicity 3).

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Del Pezzo surfaces of degree one are double covers of a cone in $\mathbb{P}^3$, ramified over a smooth sextic curve.

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Theorem (Van Luijk, W.)
Let X be a del Pezzo surface of degree one over an algebraically closed field k.
Any point on the ramification curve is contained in at most 16 lines for $\operatorname{char} k = 2$, and in at most 10 lines for $\operatorname{char} k \neq 2$.
Any point outside the ramification curve is contained in at most 12 lines for $\operatorname{char} k = 3$, and in at most 10 lines for $\operatorname{char} k \neq 3$.

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The upper bounds are sharp in all characteristics, except possibly in characteristic 5 outside the ramification curve.

Thank you!