Bogoliubov Theory in the Gross-Pitaevskii Limit

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Outline

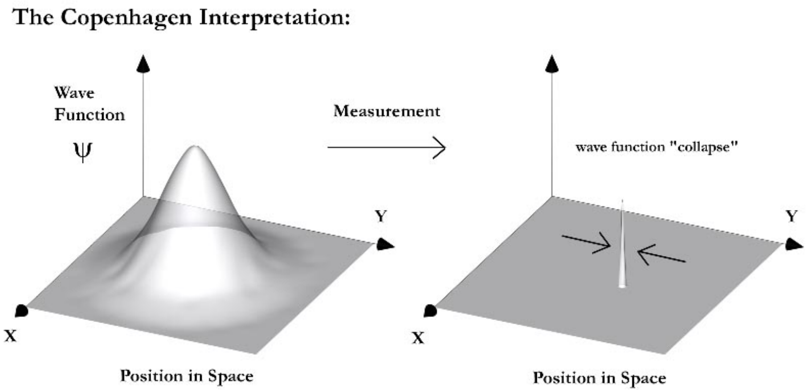
Preliminaries

Goal of this talk:
Understanding the asymptotic of "small" eigenvalues of the operator

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \kappa \sum_{1\leq i<j\leq N} N^2 V(N(x_i - x_j))$$

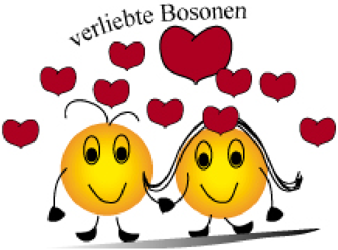
for $N \to \infty$. Based on the Paper: "Bogoliubov Theory in the Gross-Pitaevskii Limit" by C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein.

Physics

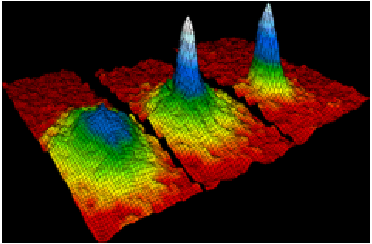


The Copenhagen Interpretation:
Wave
Function
ψ
Measurement
wave function "collapse"
Y
X
Y
X
Position in Space
Position in Space

Physics



Bosons can sit in the same quantum state



Bose-Einstein condensation

Fock space

We always consider $\Lambda = \left[-\frac{1}{2}, \frac{1}{2}\right]^{\times 3}$ with periodic boundary conditions. The **bosonic Fock space** is defined as

$$\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\Lambda^n) = \bigoplus_{n\geq 0} L^2(\Lambda)^{\otimes_s n}$$

with scalar product given by

$$\langle \Phi, \Psi \rangle_{\mathcal{F}} = \sum_{n\geq 0} \langle \Phi^{(n)}, \Psi^{(n)} \rangle_{L^2(\Lambda^n)}.$$

Fock space

For $g \in L^2(\Lambda)$ we introduce the **creation operator** $a^\star(g)$ and the **annihilation operator** $a(g)$

$$(a^\star(g)\Psi)^{(n)}(x_1,\dots,x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} g(x_j)\Psi^{(n-1)}(x_1,\dots,\hat{x}_j,\dots x_n)$$

$$(a(g)\Psi)^{(n)}(x_1,\dots,x_n) = \sqrt{n+1} \int dx\ \overline{g}(x)\Psi^{(n+1)}(x,x_1,\dots,x_n)$$

We have the commutation relation

$$[a(g), a^\star(h)] = \langle g, h\rangle, \quad [a(g), a(h)] = [a^\star(g), a^\star(h)] = 0.$$

Fock space

We define $\Lambda^\star = 2\pi\mathbb{Z}^3$ and $\Lambda^\star_+ = 2\pi\mathbb{Z}^3 \setminus \{0\}$. Furthermore, we set $\varphi_p(x) = e^{-ip\cdot x}$ for $p \in \Lambda^\star$ and

$$a_p^\star = a^\star(\varphi_p), \qquad \text{and} \qquad a_p = a(\varphi_p).$$

The **number of particles operator** is

$$(\mathcal{N}\Psi)^{(n)} = n\Psi^{(n)}.$$

We can write

$$\mathcal{N} = \sum_{p\in\Lambda^\star} a_p^\star a_p$$

Fock space

We set

$$\mathcal{F}_+ = \bigoplus_{n\geq 0} L^2_\perp(\Lambda)^{\otimes_s n}$$

where $L^2_\perp(\Lambda)$ is the orthogonal complement of φ_0. Also we define

$$\mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^{N} L^2_\perp(\Lambda)^{\otimes_s n}$$

We denote by $\mathcal{N}_+$ the restriction of $\mathcal{N}$ to $\mathcal{F}_+$. We can write

$$\mathcal{N}_+ = \sum_{p\in\Lambda_+^\star} a_p^\star a_p$$

For $p \in \Lambda_+^\star$ we define the **modified annihilation and creation operator**

$$b_p^\star = a_p^\star \sqrt{\frac{N-\mathcal{N}_+}{N}}, \qquad b_p = \sqrt{\frac{N-\mathcal{N}_+}{N}} a_p$$

Scattering length

The **scattering length** a_0 of an interaction potential κV is defined via the zero-energy scattering equation

$$\begin{cases} \left[-\Delta + \frac{\kappa}{2} V\right] f & = 0 \\ \lim_{|x| \to \infty} f(x) & = 1. \end{cases}$$

For $|x|$ outside the support of V we have

$$f(x) = 1 - \frac{a_0}{|x|}$$

Alternative definition

$$8\pi a_0 = \kappa \int dx \; V(x) f(x)$$

The scattering length of $V(N\cdot)$ is a_0/N.

Statement of the Theorem

Theorem (Boccato, Brennecke, Cenatiempo, Schlein, 2018)

Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric, compactly supported and assume that the coupling constant $\kappa \geq 0$ is small enough. Then, in the limit $N \to \infty$, the ground state energy E_N of the Hamilton operator

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \kappa \sum_{1 \leq i < j \leq N} N^2 V(N(x_i - x_j))$$

is given by

$$\begin{aligned} E_N &= 4\pi(N-1)a_N \\ &- \frac{1}{2} \sum_{p \in \Lambda_+^\star} \left[p^2 + 8\pi a_0 - \sqrt{|p|^4 + 16\pi a_0 p^2} - \frac{(8\pi a_0)^2}{2p^2} \right] + \mathcal{O}(N^{-\frac{1}{4}}) \end{aligned}$$

Statement of the Theorem continued

continued

where

$$8\pi a_N = \kappa \hat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k \kappa^{k+1}}{(2N)^k} \sum_{p_1,\dots,p_k \in \Lambda_+^\star} \frac{\hat{V}(p_1/N)}{p_1^2} \times \left(\prod_{i=1}^{k-1} \frac{\hat{V}((p_i - p_{i+1})/N)}{p_{i+1}^2} \right) \hat{V}(p_k/N)$$

Moreover, the spectrum of $H_N - E_N$ below a threshold ζ consists of eigenvalues given, in the limit $N \to \infty$, by

$$\sum_{p \in \Lambda_+^\star} n_p \sqrt{|p|^4 + 16\pi a_0 p^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3))$$

Here $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^\star$ and $n_p \neq 0$ for only finitely many p.

Idea of the Proof

Main Strategy

Strategy: Find a unitary transformation U such that

$$UH_NU^\star = D + \mathcal{E},$$

where D is a diagonal operator and such that $\mathcal{E}$ is small on "low-energy states".

Main fact: If U is a unitary transform and A an operator, then

$$\sigma(UAU^\star) = \sigma(A)$$

and ξ is an eigenfunction of A iff $U\xi$ is an eigenfunction of $UAU^\star$.

Plan of the conjugations

We will conjugate several times to achieve our goal:

1. U_N**:**
Factoring out the condensate

2. Quadratic Bogoliubov transformation $e^{B(\eta)}$**:**
Extracting some relevant contributions out of the cubic and quartic terms

3. Cubic transformation e^{A}**:**
Eliminating the cubic term

4. Quadratic Bogoliubov transform $e^{B(\tau)}$**:**
Diagonalizing the quadratic term

A first conjugation: factoring out the condensate

We can rewrite our Hamiltonian using annihilation and creation operators

$$H_N = \sum_{p\in\Lambda^\star} p^2 a_p^\star a_p + \frac{\kappa}{2N} \sum_{p,q,r\in\Lambda^\star} \hat{V}(r/N) a_p^\star a_q^\star a_{q-r} a_{p+r}$$

First we will conjugate with the following unitary map

$$U_N : L_s^2(\Lambda^N) \to \mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^{N} L_\perp^2(\Lambda)^{\otimes_s n}$$

where

$$U_N \left(\sum_{n=0}^{N} \alpha^{(n)} \otimes_s \varphi_0^{\otimes(N-n)} \right) = (\alpha_0, \ldots, \alpha_N)$$

Excitation Hamiltonian

One obtains the excitation Hamiltonian

$$\mathcal{L}_N = U_N H_N U_N^\star : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}, \qquad \mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)}$$

where

$$\begin{aligned}
\mathcal{L}_N^{(0)} &= \frac{N-1}{2N}\kappa \hat{V}(0)(N-\mathcal{N}_+) + \frac{\kappa\hat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+) \\
\mathcal{L}_N^{(2)} &= \sum_{p\in\Lambda_+^\star} p^2 a_p^\star a_p + \sum_{p\in\Lambda_+^\star} \kappa\hat{V}(p/N)\left[b_p^\star b_p - \frac{1}{N}a_p^\star a_p\right] \\
&\quad + \frac{\kappa}{2}\sum_{p\in\Lambda_+^\star} \hat{V}(p/N)[b_p^\star b_{-p}^\star + b_p b_{-p}] \\
\mathcal{L}_N^{(3)} &= \frac{\kappa}{\sqrt{N}} \sum_{p,q\in\Lambda^\star : p+q\neq 0} \hat{V}(p/N)[b_{p+q}^\star a_{-p}^\star a_q + a_q^\star a_{-p} b_{p+q}] \\
\mathcal{L}_N^{(4)} &= \frac{\kappa}{2N} \sum_{p,q\in\Lambda^\star, r\in\Lambda^\star :\ r\neq -p,-q} \hat{V}(r/N) a_{p+r}^\star a_q^\star a_p a_{q+r}
\end{aligned}$$

Bogoliubov transform: standard vs generalized

Let $\eta \in l^2(\Lambda_+^\star)$ with $\eta_p = \eta_{-p}$. The standard Bogoliubov transformation associated with η is

$$e^{\tilde{B}(\eta)} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^\star}(\eta_p a_p^\star a_{-p}^\star - \overline{\eta}_p a_p a_{-p})\right]$$

Not suited for our purpose, as it does not leave $\mathcal{F}_+^{\leq N}$ invariant. But the **generalized Bogoliubov transformation**

$$e^{B(\eta)} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^\star}(\eta_p b_p^\star b_{-p}^\star - \overline{\eta}_p b_p b_{-p})\right]$$

does leave $\mathcal{F}_+^{\leq N}$ invariant.

Bogoliubov transform: standard vs generalized

Drawback: we don't get identities for the conjugation. For the standard Bogoliubov transformation we have

$$e^{-\tilde{B}(\eta)} a_p e^{\tilde{B}(\eta)} = \cosh(\eta_p) a_p + \sinh(\eta_p) a^\star_{-p}$$

On the other hand we only get

$$e^{-B(\eta)} b_p e^{B(\eta)} = \cosh(\eta_p) b_p + \sinh(\eta_p) b^\star_{-p} + d_p$$

where d_p is some error. Luckily one gets nice enough estimates for the error terms.

Second conjugation: quadratic Bogoliubov transformation

We introduce the notation

$$\mathcal{K} = \sum_{p \in \Lambda_+^\star} p^2 a_p^\star a_p$$

$$\mathcal{V}_N = \frac{\kappa}{2N} \sum_{p,q \in \Lambda^\star, r \in \Lambda^\star:\ r \neq -p,-q} \hat{V}(p/N) a_{p+r}^\star a_q^\star a_{q+r} a_p$$

$$\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$$

We define the new excitation Hamiltonian to be

$$\mathcal{G}_N = e^{-B(\eta)} \mathcal{L}_N e^{B(\eta)} = e^{-B(\eta)} U_N H_N U_N^\star e^{B(\eta)} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

What is small?

All the estimates are in terms of powers of the number of particles operator and the energy operator $\mathcal{H}_N$. This will eventually be small by the following proposition:

Proposition

Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa \geq 0$ is small enough. Let E_N be the ground state energy of H_N. Let $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ belong to the spectral subspace of H_N with energies below $E_N + \zeta$, for some $\zeta > 0$, i.e.

$$\psi_N = 1_{(-\infty; E_N+\zeta]}(H_N)\psi_N.$$

For any $k \in \mathbb{N}$ there exists $C = C(k) > 0$ such that for all N

$$\langle e^{-B(\eta)} U_N \psi_N, (\mathcal{N}_+ + 1)^k (\mathcal{H}_N + 1) e^{-B(\eta)} U_N \psi_N \rangle \leq C(1 + \zeta^{k+1})$$

Second conjugation: quadratic Bogoliubov transformation

Proposition

Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa \geq 0$ is small enough. Then we have

$$\mathcal{G}_N = C_{\mathcal{G}_N} + \mathcal{Q}_{\mathcal{G}_N} + \mathcal{H}_N + C_N + \mathcal{E}_{\mathcal{G}_N}$$

and there exists $C > 0$ such that

$$\pm \mathcal{E}_{\mathcal{G}_N} \leq \frac{C}{\sqrt{N}} (\mathcal{H}_N + \mathcal{N}_+^2 + 1)(\mathcal{N}_+ + 1)$$

as quadratic forms on $\mathcal{F}_+^{\leq N} \times \mathcal{F}_+^{\leq N}$.

Third conjugation: cubic transformation

Let $P_L = \{p \in \Lambda_+^\star \ : \ |p| \leq N^{1/2}\}$ and $P_H = \Lambda_+^\star \setminus P_L$ and define

$$A = \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r [\sinh(\eta_v) b_{r+v}^\star b_{-r}^\star b_{-v}^\star + \cosh(\eta_v) b_{r+v}^\star b_{-r}^\star b_v - h.c.]$$

Proposition

Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Then we have for $\mathcal{I}_N = e^{-A} \mathcal{G}_N e^A$

$$\mathcal{I}_N = C_{\mathcal{I}_N} + \mathcal{Q}_{\mathcal{I}_N} + \mathcal{V}_N + \mathcal{E}_{\mathcal{I}_N}$$

and there exists $C > 0$ such that

$$\pm \mathcal{E}_{\mathcal{I}_N} \leq \frac{C}{N^{1/4}} [(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3]$$

as quadratic forms on $\mathcal{F}_+^{\leq N} \times \mathcal{F}_+^{\leq N}$.

Diagonalize $\mathcal{Q}_{\mathcal{I}_N}$ via quadratic Bogoliubov transformation

Proposition

Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric and assume that the coupling constant $\kappa > 0$ is small enough. Then we have for $\mathcal{M}_N = e^{-B(\tau)}\mathcal{I}_N e^{B(\tau)}$

$$\begin{aligned}\mathcal{M}_N = {} & 4\pi(N-1)a_N \\ & + \frac{1}{2}\sum_{p\in\Lambda_+^\star}\left[-p^2 - 8\pi a_0 + \sqrt{p^4 + 16\pi a_0 p^2} + \frac{(8\pi a_0)^2}{2p^2}\right] \\ & + \sum_{p\in\Lambda_+^\star}\sqrt{p^4 + 16\pi a_0 p^2}\, a_p^\star a_p + \mathcal{V}_N + \mathcal{E}_{\mathcal{I}_N}\end{aligned}$$

and there exists $C > 0$ such that as quadratic forms on $\mathcal{F}_+^{\leq N} \times \mathcal{F}_+^{\leq N}$ we have

$$\pm\mathcal{E}_{\mathcal{I}_N} \leq \frac{C}{N^{1/4}}[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3]$$

Further Discussion

Large κ

One uses the smallness of κ at different spots in the proof. Mainly to obtain $\|\eta\| \sim \kappa$ small. In "Optimal Rate for Bose-Einstein Condensation in the Gross-Pitaevskii Regime" by C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein got rid of this assumption.

Main idea:
Exploiting some other parameter to make $\|\eta\|$ small and applying localization techniques.

Extension to $\mathbb{R}^3$

To ensure existence of a ground state we need to add a "confining external potential", i.e. we look at

$$H_N = \sum_{j=1}^{N}(-\Delta_{x_j} + V_{ext}(x_j)) + \kappa \sum_{1\le i<j\le N} N^2 V(N(x_i - x_j))$$

where $V_{ext} \in L^\infty_{loc}(\mathbb{R}^3, \mathbb{R}_{\geq 0})$ such that

$$\lim_{|x|\to\infty} |V_{ext}(x)| = \infty.$$

Difficulties:

• Our functions are no longer periodic (external potential breaks translation-invariance), hence, cannot use discrete Fourier transform
• Ground state φ_0 cannot be determined explicitly

Thank you