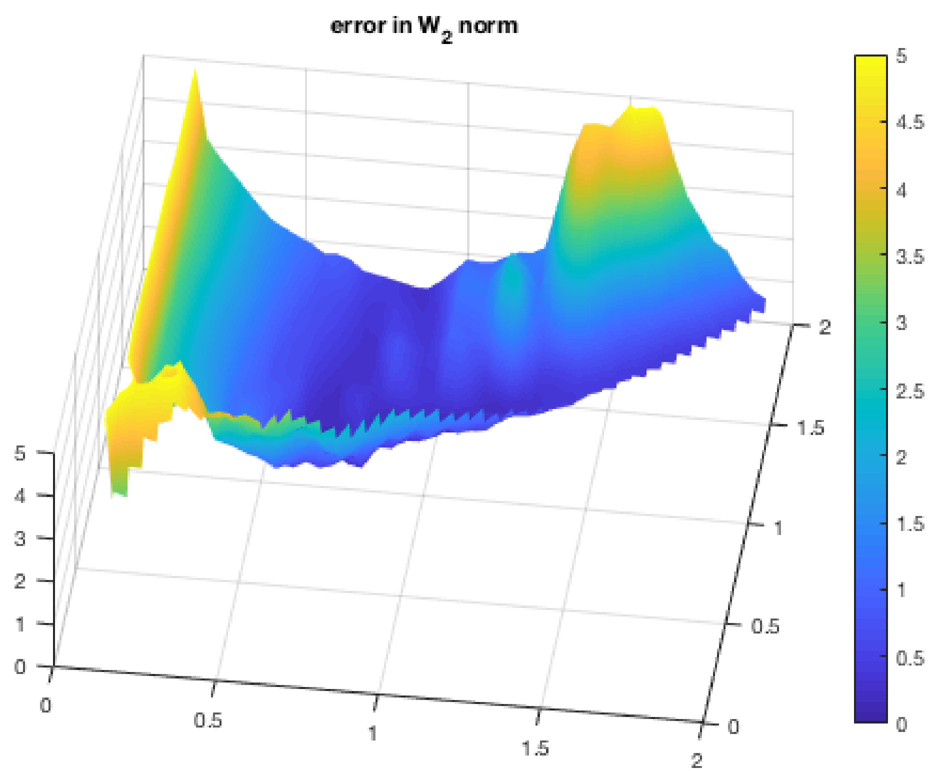
University of Basel

Wasserstein metric based full waveform inversion

Anna Karpova, 20.09.18



Contents

- Introduction

- The forward problem

- Different misfit functionals

- Minimisation methods

- Computational results

- Conclusion

Introduction

- the thesis deals with a simple application of the reflection method to the exploration of oil and gas fields

- full waveform inversion (FWI) is the technique used in reflection seismology to recover the information from observed seismograms

- FWI consists of an efficient solution of the forward problem and iterative improvement of a subsurface model by minimisation a misfit between observed and synthetic seismic waveforms

Introduction

- the least squares (L^2) norm is a classically used misfit functional in FWI, but it usually provides multiple local minima

- Wasserstein metric is another misfit functional we want to use in this work, because it has some desirable properties like convexity and insensitivity to noise

- the goal of this work is to create an example of a two-dimensional seismic problem and to solve it with FWI technique once with L^2 norm and once with Wasserstein metric

Introduction

Elastic waves in two-dimensional space can be modelled by the following wave equation in order to obtain the resulting wavefield *u(x,z,t)* for a given wave velocity *c(x,z)*

$$u_{tt}(x,z,t) - \nabla \cdot (c^2(x,z)\nabla u(x,z,t)) \;=\; s(x,z,t),$$

where *s(x,z,t)* is the source function.

observed data: $$g = u(x_r, 0, t)$$

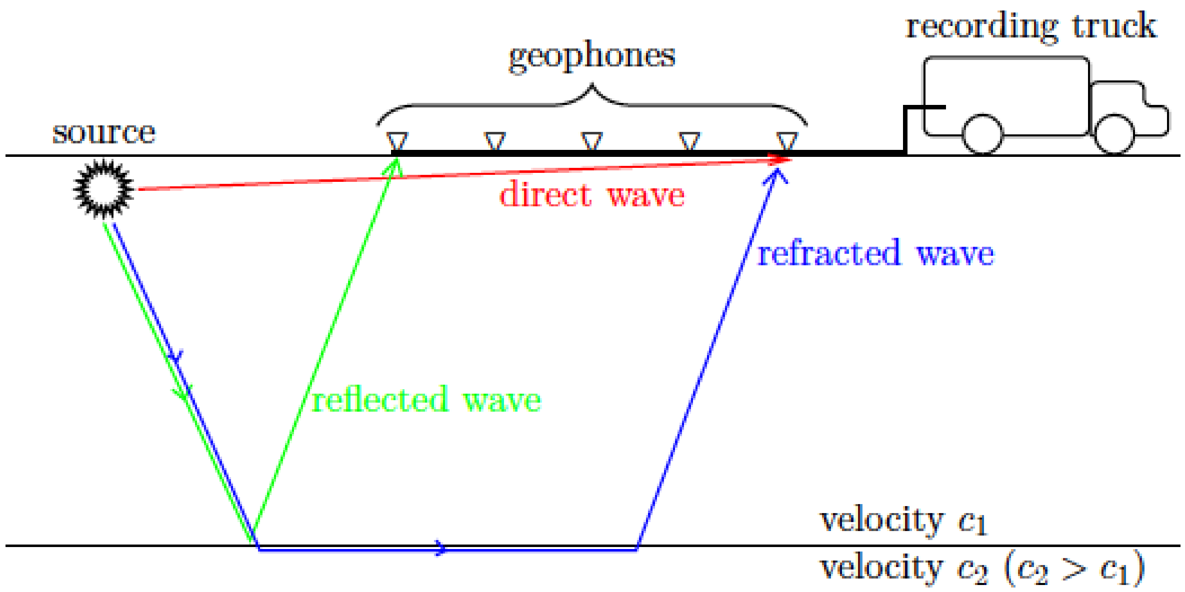
modelled data: $$f(c) = u(x_r, 0, t)$$

minimisation problem:

$$\tilde{c} = \underset{c}{\operatorname{argmin}}\, d(f(c), g),$$

where *d(f,g)* is a misfit functional for two signals

The forward problem
The seismic reflection experiment



Cartoon of the land seismic experiment

The forward problem
The problem statement

$$\Omega = (0,a) \times (0,b),\ a,b > 0 \qquad \Omega_\infty = \{(x,z) \in \mathbb{R}^2 \mid z \geq 0\}$$

$$
\begin{aligned}
u_{tt}(x,z,t) - \nabla \cdot (c^2(x,z)\nabla u(x,z,t)) &= s(x,z,t) && (x,z) \in \Omega,\ t>0, && (2.1a)\\
u_{tt}(x,z,t) - c^2\Delta u(x,z,t) &= 0 && (x,z) \in \Omega_\infty \backslash \Omega,\ t>0, && (2.1b)\\
\frac{\partial}{\partial n} u(x,z,t) &= 0 && (x,z) \in \partial\Omega_\infty,\ t>0, && (2.1c)\\
\lim_{(x,z)\to\infty} u(x,z,t) &= 0 && t>0, && (2.1d)\\
u(x,z,0) &= u_0(x,z), && (x,z) \in \Omega, && (2.1e)\\
u_t(x,z,0) &= v_0(x,z) && (x,z) \in \Omega. && (2.1f)
\end{aligned}
$$

u(x,z,t) – wave field
s(x,z,t) – source
c(x,z) – wave velocity
u_0, v_0 – initial conditions

The forward problem
The problem statement

The PML modified wave equation (2.5) replace the conditions 2.1a, 2.1b and 2.1d.

$$\begin{cases} u_{tt} + (\zeta + \eta)\, u_t + \zeta\eta\, u &= \nabla \cdot (c^2 \nabla u) + s + \nabla \cdot \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \\ \phi_t &= -\zeta\phi + c^2(\eta - \zeta)\dfrac{\partial u}{\partial x}, \\ \psi_t &= -\eta\psi + c^2(\zeta - \eta)\dfrac{\partial u}{\partial z}. \end{cases} \tag{2.5}$$

ϕ, ψ – auxiliary functions
ζ, η – damping profiles

$$\zeta(x) = 0 \quad \text{in } \Omega \text{ and } \zeta(x) > 0 \text{ in } \Omega^{\mathrm{C}},$$

$$\eta(z) = 0 \quad \text{in } \Omega \text{ and } \eta(z) > 0 \text{ in } \Omega^{\mathrm{C}}.$$

The forward problem
Discretisation

$$x_i = i \cdot \Delta x \text{ and } z_j = j \cdot \Delta z,$$

$$\text{where } \Delta x = \frac{1}{k+1}, i = 0, 1, \ldots k+1 \text{ and } \Delta z = \frac{1}{l+1}, j = 0, 1, \ldots l+1$$

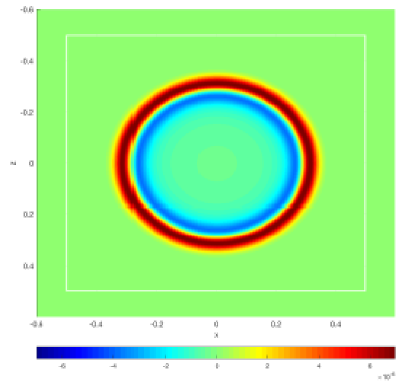
$$w_{ij}^m \simeq \ u(x_i, z_j, t_m)$$

$$\frac{w_{ij}^{m+1} - 2w_{ij}^m + w_{ij}^{m-1}}{(\Delta t)^2} = c_{ij}^2 \left(\frac{w_{i+1,j}^m - 2w_{ij}^m + w_{i-1,j}^m}{(\Delta x)^2} + \frac{w_{i,j+1}^m - 2w_{ij}^m + w_{i,j-1}^m}{(\Delta z)^2} \right)$$

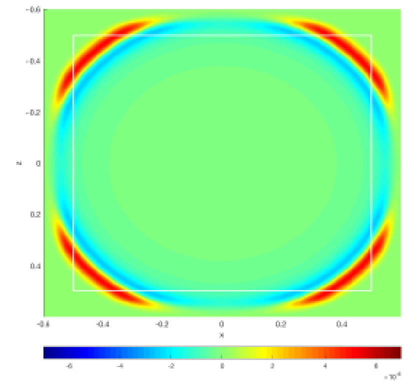
CFL stability condition for equal mesh size $h = \Delta x = \Delta z$:

$$c \left(\frac{\Delta t}{h} \right) \leq \frac{1}{\sqrt{2}}.$$

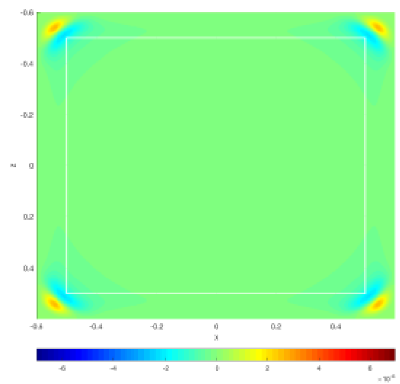
The forward problem
Numerical experiments



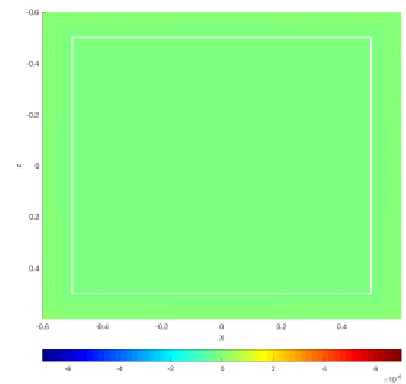
(a) t = 0.4



(b) t = 0.7



(c) t = 0.86



(d) t = 10

Snapshots of the numerical solutions at different times in $\Omega = [-0.5, 0.5]^2$ enclosed by a PML of width $L = 0.1$

The forward problem
Numerical experiments

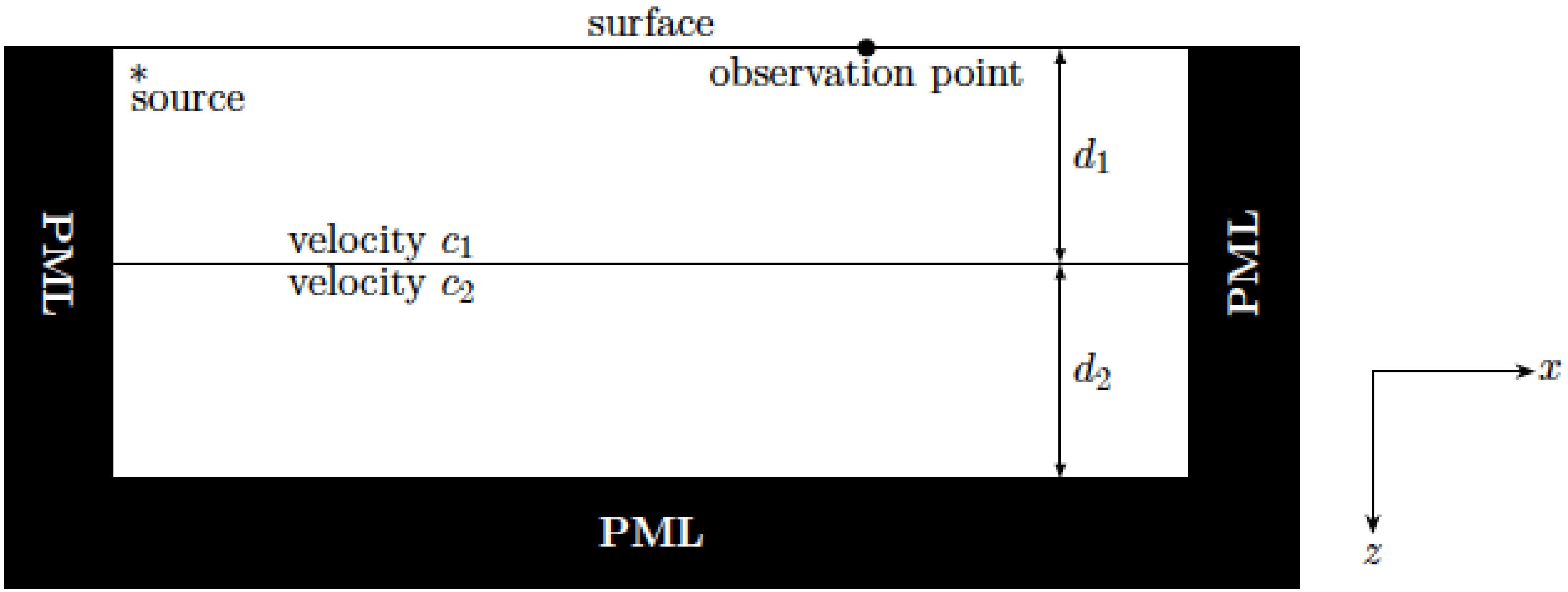
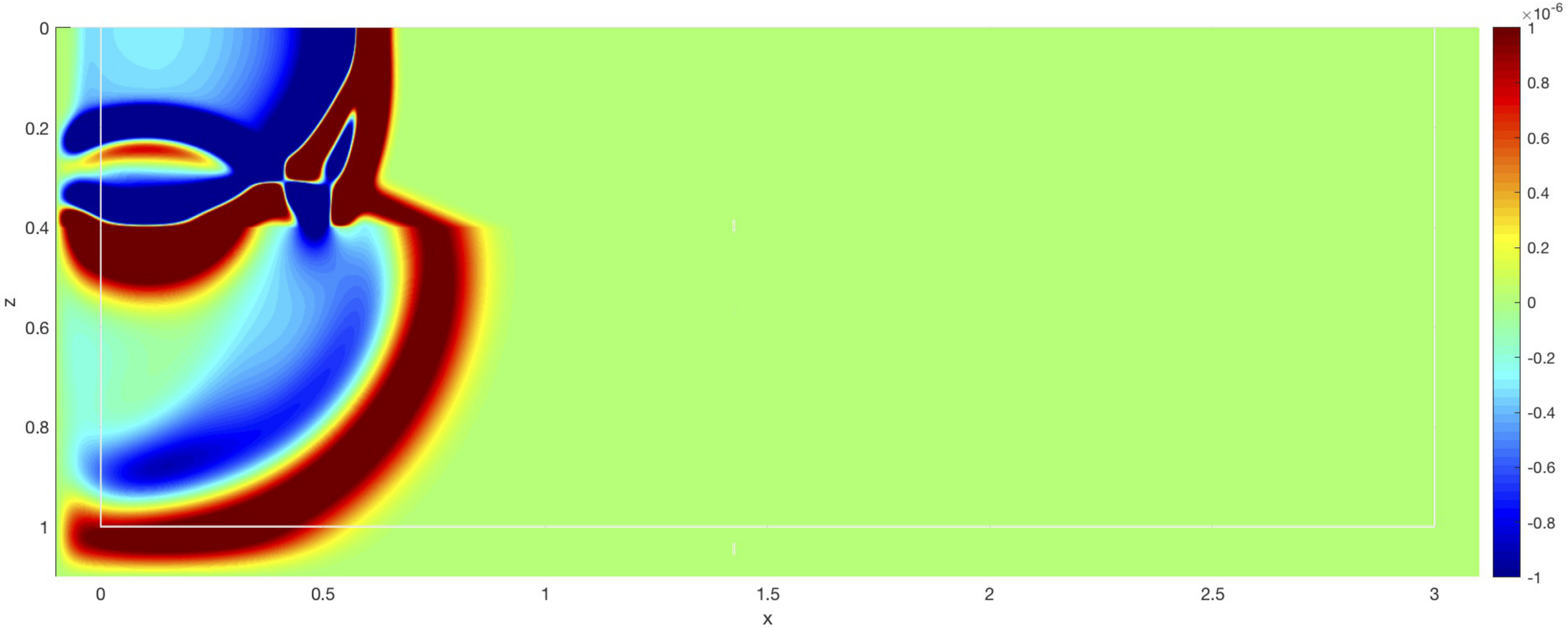


Illustration of the two-layers model. The regular computation domain Ω is surrounded by absorbing layers in which the plane waves decay rapidly as they approach the boundary.

The forward problem
Numerical experiments



Numerical solution of the two-layers model in $\Omega = (0, 3) \times (0, 1)$ with wave velocities $c_1 = 1$ in the upper layer, $c_2 = 3$ in the lower layer and the depth of the upper layer $d_1 = 0.4$

Different misfit functionals

Least-squares (L^2) norm

For observed data *g* and simulated data *f* recorded in the time interval *(0, T)* at receiver position x_r, the conventional full waveform inversion defines L^2 norm misfit functional as

$$L^2(f(m), g) = \int_0^T |f(x_r, t, m) - g(x_r, t)|^2 dt,$$

where *m* is the model parameter.

For numerical approximation of the L^2 norm, we subdivide the interval *(0, T)* in *N* equal subintervals $[t_i, t_{i+1}]$ of the length *h*. In each subinterval we use the Simpson's formula

$$S[\tilde{f}] = \frac{h}{6}\left(\tilde{f}(t_i) + 4\tilde{f}\frac{t_i + t_{i+1}}{2} + \tilde{f}(t_{i+1})\right) \simeq \int_{t_i}^{t_{i+1}} \tilde{f}(t)dt,$$

where $\tilde{f}(t) = (f(x_r, t, m) - g(x_r, t))^2.$

Different misfit functionals
Quadratic Wasserstein metric W_2

Wasserstein metric computes the lowest cost of rearranging one distribution into another given a cost function.

For two probability density functions *f,g* the quadratic Wasserstein metric is given by

$$W_2^2(f,g) = \inf_{T\in\mathcal{M}} \int_X |x - T(x)|^2 f(x)dx,$$

where *M* is the set of all maps that rearrange *f* into *g*.

$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad G(y) = \int_{-\infty}^{y} g(t)dt$$

Theorem 3.4 *Let* $0 < f, g < \infty$ *be two probability density functions, each supported on a connected subset of* $\mathbb{R}$*. Then the optimal map from* f *to* g *is* $T = G^{-1} \circ F$.

$$W_2^2(f,g) = \int_0^T |t - G^{-1}(F(t))|^2 f(t)dt.$$

Different misfit functionals
Quadratic Wasserstein metric W_2

Convexity of W_2 metric with respect to shift:

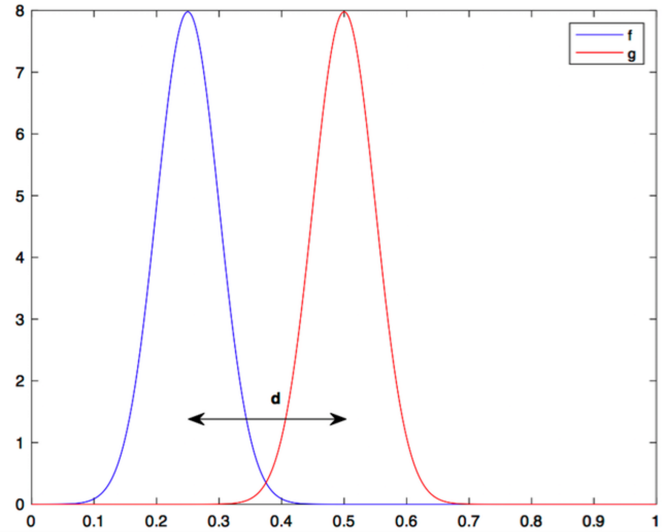
Theorem 3.7 [9] *Let $T : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^n$ be an optimal transference plan rearanging f into g, where f and g are two density functions. Then the optimal transference plan from a shifted density function $f_s(x) = f(x - s\eta)$, $\eta \in \mathbb{R}^n$ into g is $T_s = T(x - s\eta)$. Moreover, $W_2^2(f_s, g)$ is convex with respect to the shift size s.*

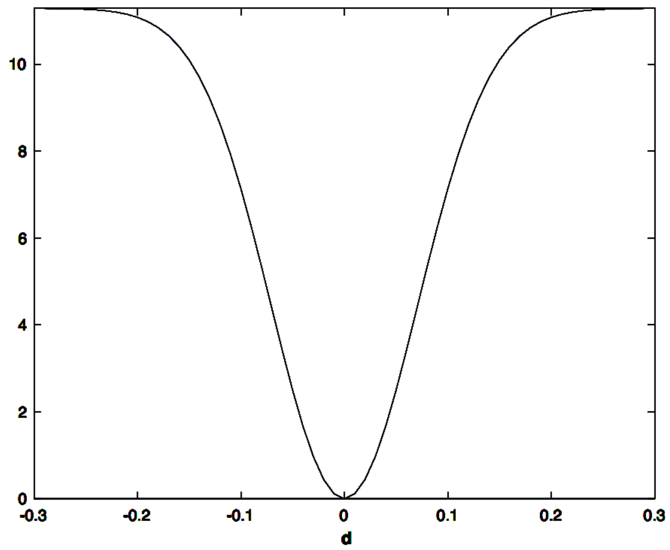
Data normalisation:

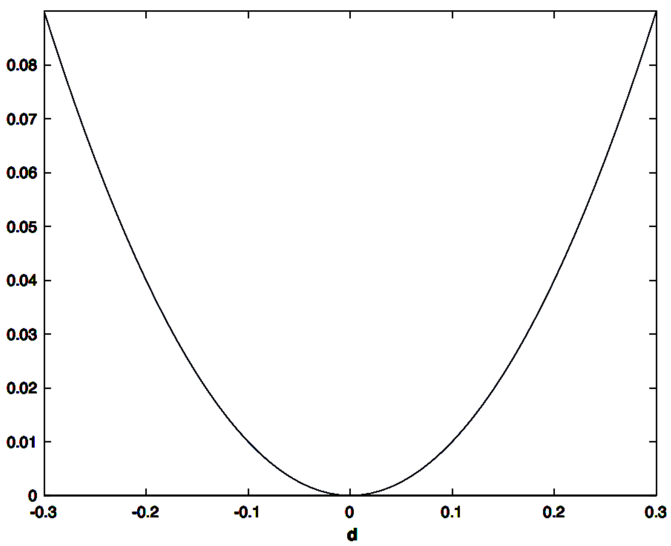
- $$f^+ = \max\{f, 0\}, \qquad f^- = \max\{-f, 0\}, \qquad \langle f \rangle = \int_X f(x)dx$$
  $$W_2^2(f,g) \approx W_2^2\left(\frac{f^+}{\langle f^+ \rangle}, \frac{g^+}{\langle g^+ \rangle}\right) + W_2^2\left(\frac{f^-}{\langle f^- \rangle}, \frac{g^-}{\langle g^- \rangle}\right)$$

- $$W_2^2(f,g) \approx W_2^2\left(\frac{f+c}{\langle f+c \rangle}, \frac{g+c}{\langle g+c \rangle}\right)$$

Different misfit functionals
Numerical experiments

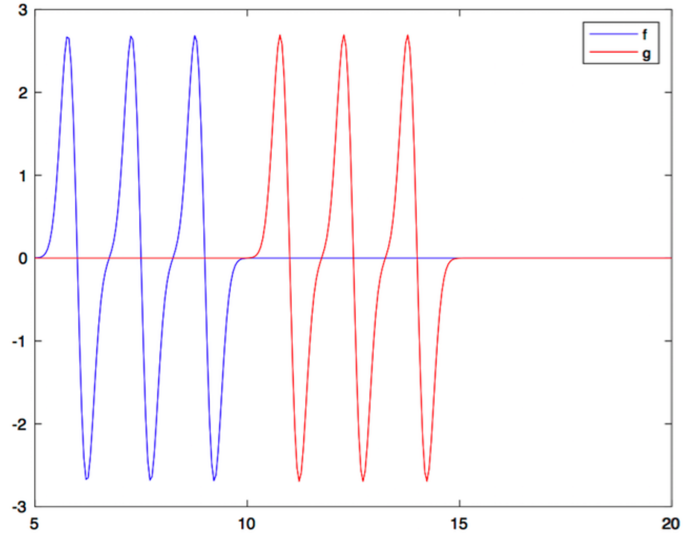


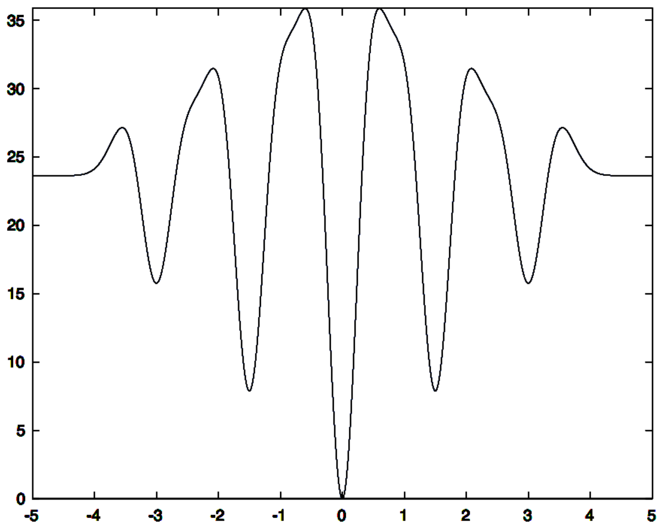


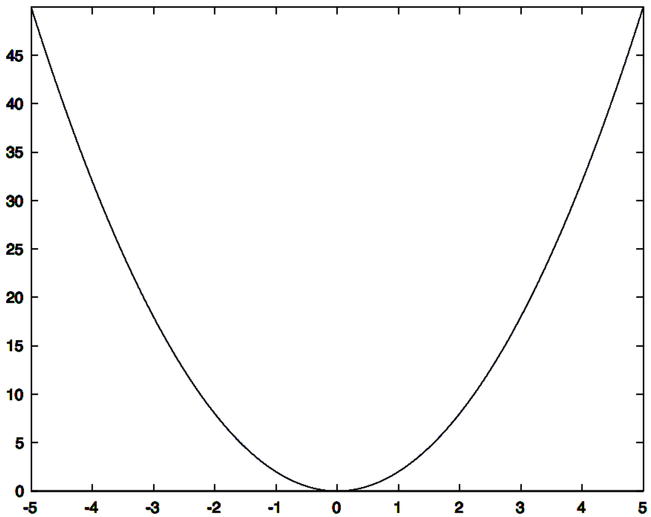


Different misfit functionals

Numerical experiments







Minimisation methods

The Method of Steepest Descent

Algorithm 1: Method of Steepest Descent

Input: $x_0 \equiv 0$

Output: $x_k \in \mathbb{R}^n$

Initialize $k \leftarrow 0$

while $\nabla f(p) \neq 0$ **do**

  1. Determine a direction of steepest descent $d_k \in \mathbb{R}^n$, so that $\nabla f(x_k)^\top d_k < 0$

  2. Solve the problem $\alpha_k \approx \underset{\alpha \in \mathbb{R}}{\operatorname{argmin}} f(x_k + \alpha d_k)$

  actualise $x_{k+1} \leftarrow x_k + \alpha_k d_k$

  set $k \leftarrow k + 1$

return x_k

Armijo-Goldstein rule:

$$f(p + \alpha \delta p) \leq f(p) + \alpha \sigma \nabla f(p)^\top \delta p$$

Minimisation methods
Newton's Method

Algorithm 3: Newton algorithm

Input: function $f : \mathbb{R}^n \to \mathbb{R}$ and start approximation $\mathrm{x}_0 \in \mathbb{R}^n$

Output: sequence of iterations $\{\mathrm{x}_k\}_{k \in \mathbb{N}}$

1 Initialisation: set $k := 0$

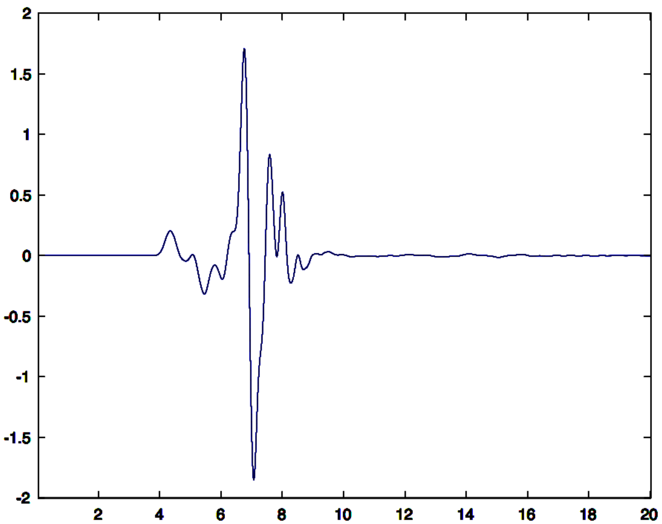
2 Calculate the Newton-direction by solving the linear equation system

$$\nabla^2 f(\mathrm{x}_k)\mathrm{d}_k = -\nabla f(\mathrm{x}_k)$$

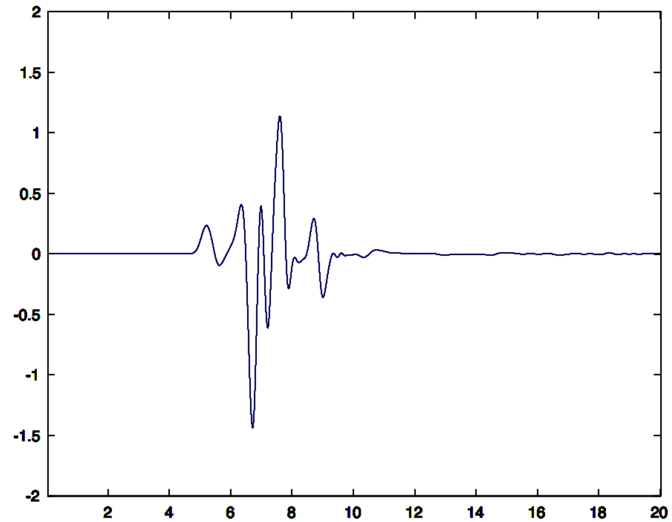
3 Set $\mathrm{x}_{k+1} := \mathrm{x}_k + \mathrm{d}_k$

4 Increase $k := k + 1$ and go to **2**.

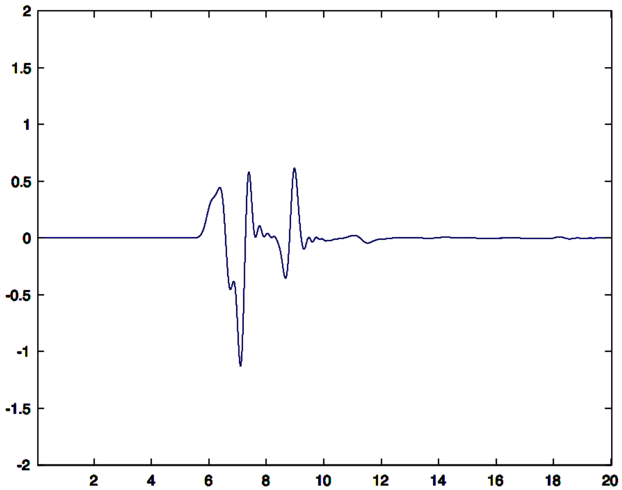
Computational results
FWI with L^2 and W_2 misfit functionals



$d_1 = 0.5$

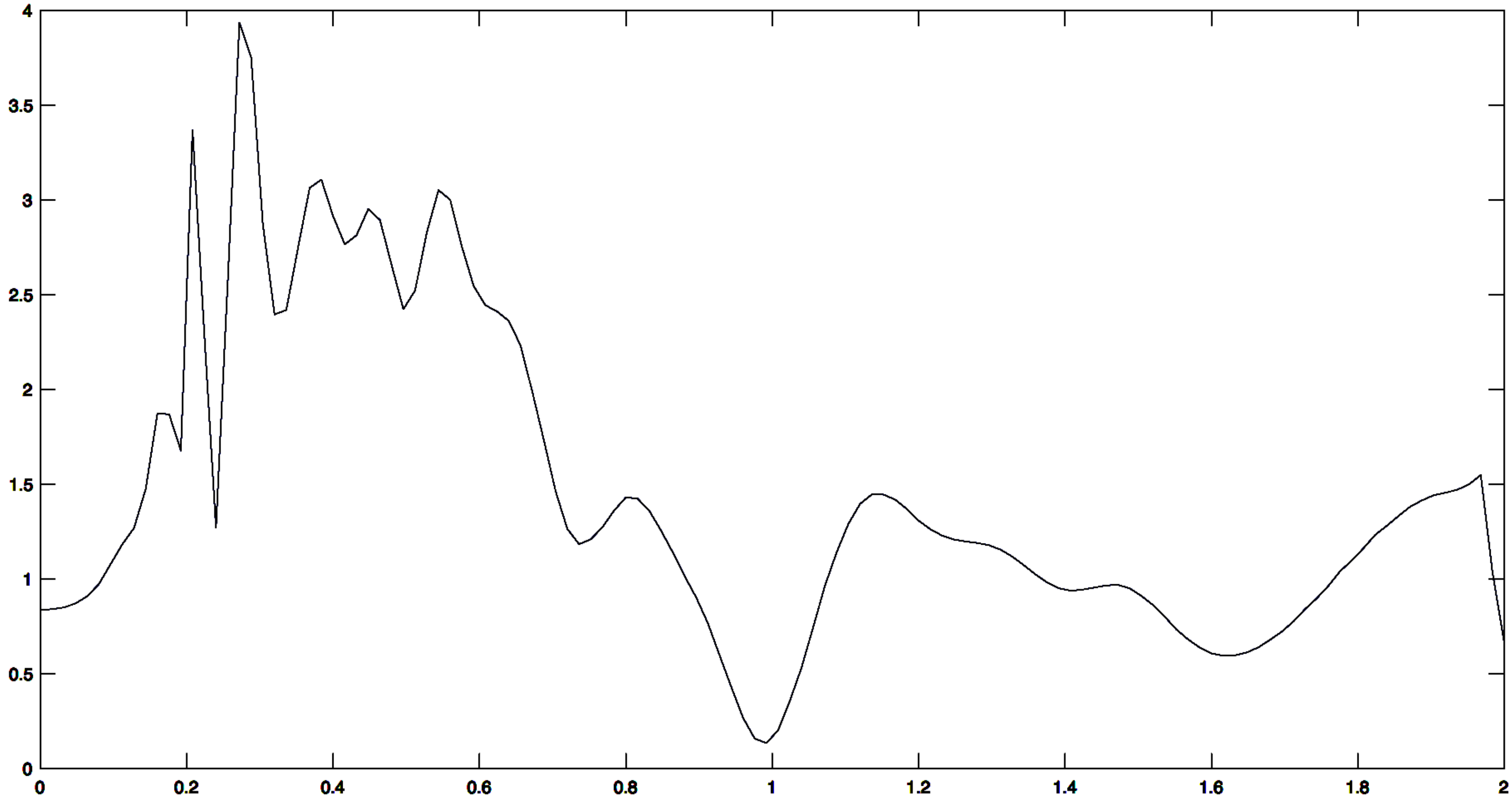


$d_1 = 1.0$



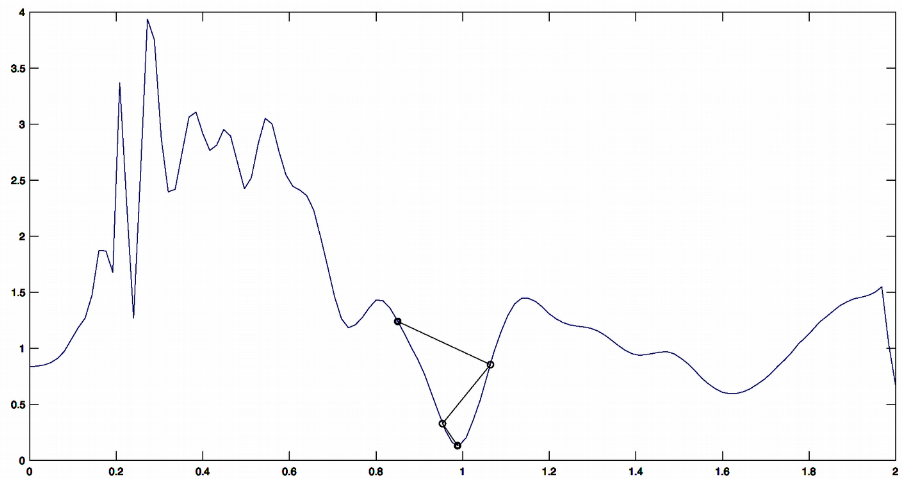
$d_1 = 1.5$

Computational results
FWI with L^2 and W_2 misfit functionals

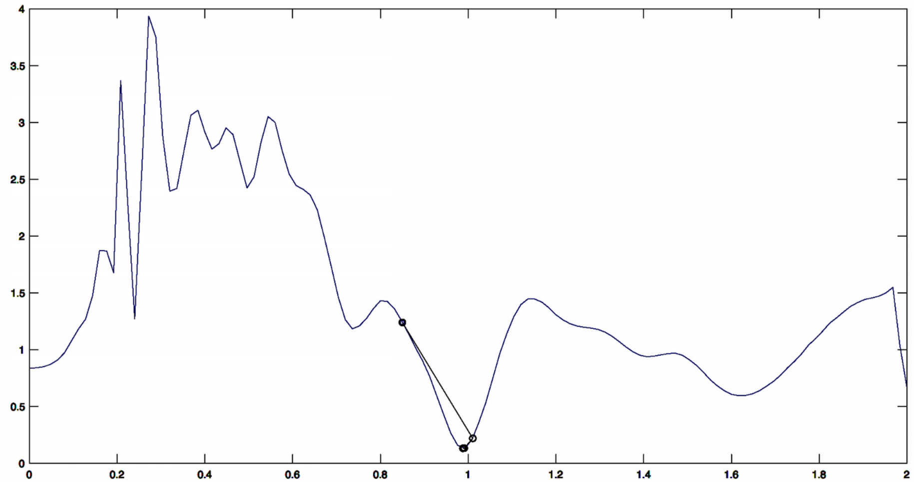


Misfit function in L^2 for varying layer thickness d_1

Computational results
FWI with L^2 and W_2 misfit functionals

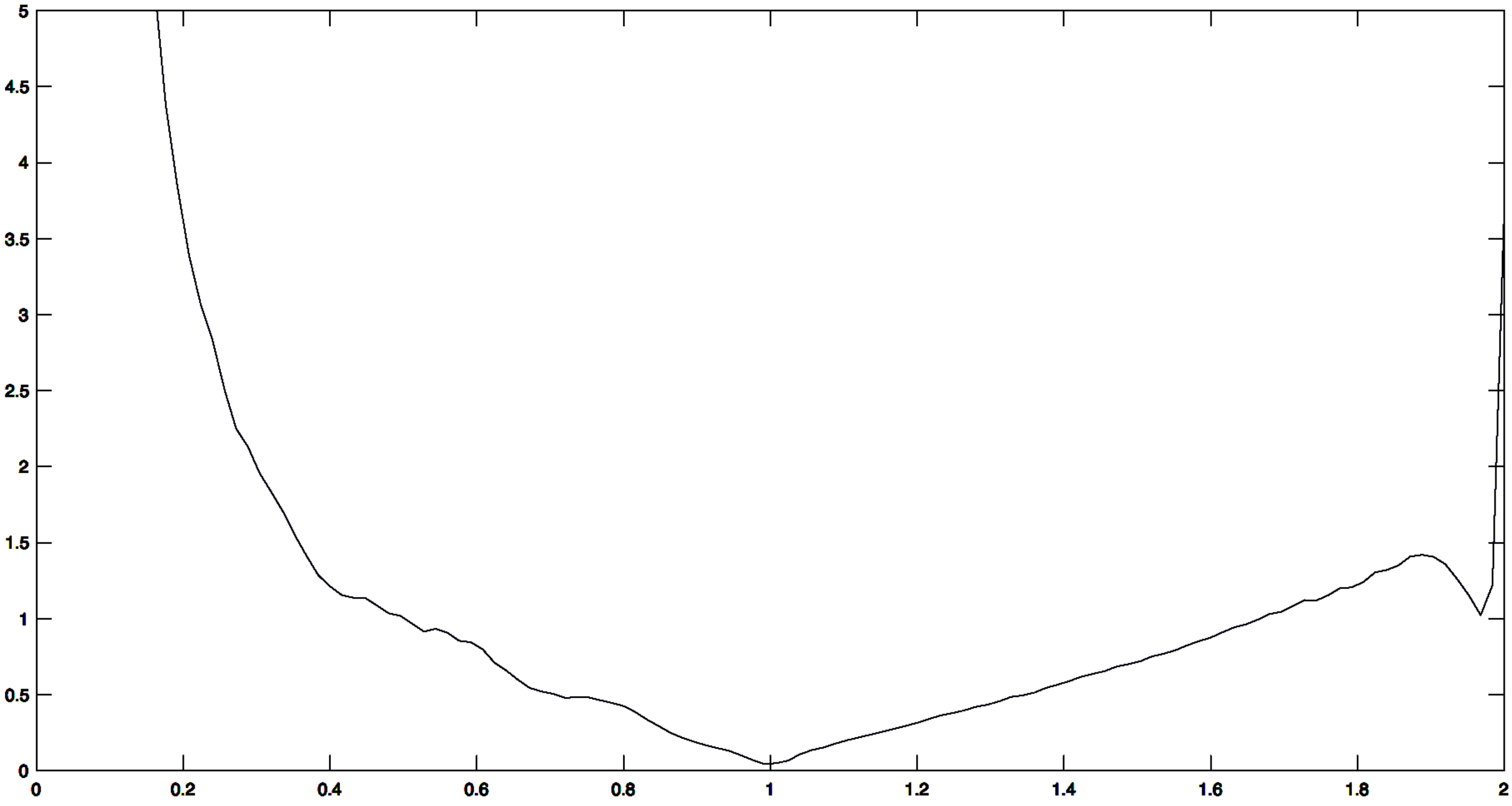


Using the Method of Steepest Descent



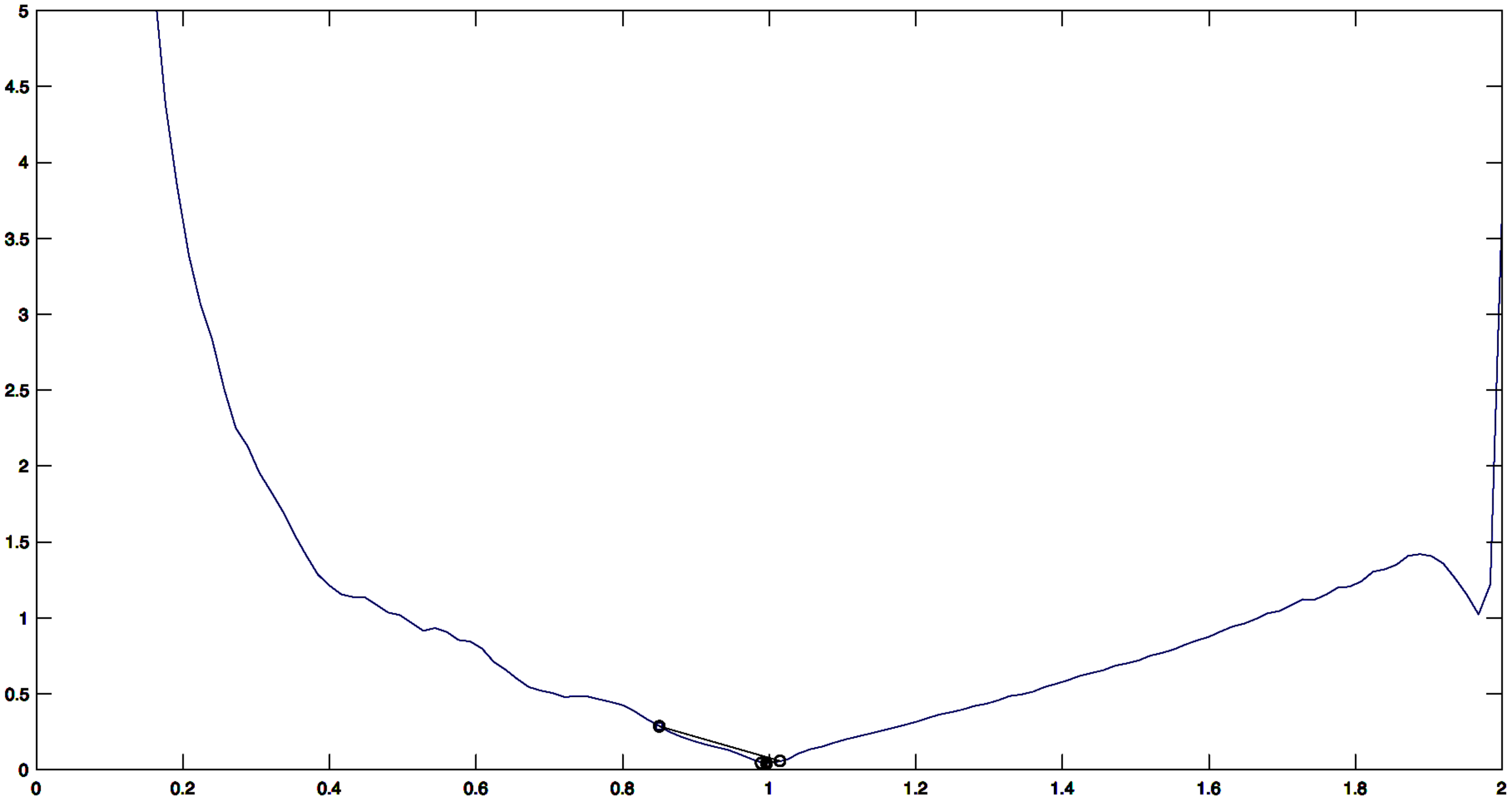
Using the Newton's Method

Computational results
FWI with L^2 and W_2 misfit functionals



Lower value region of misfit function in W_2 for varying layer thickness d_1

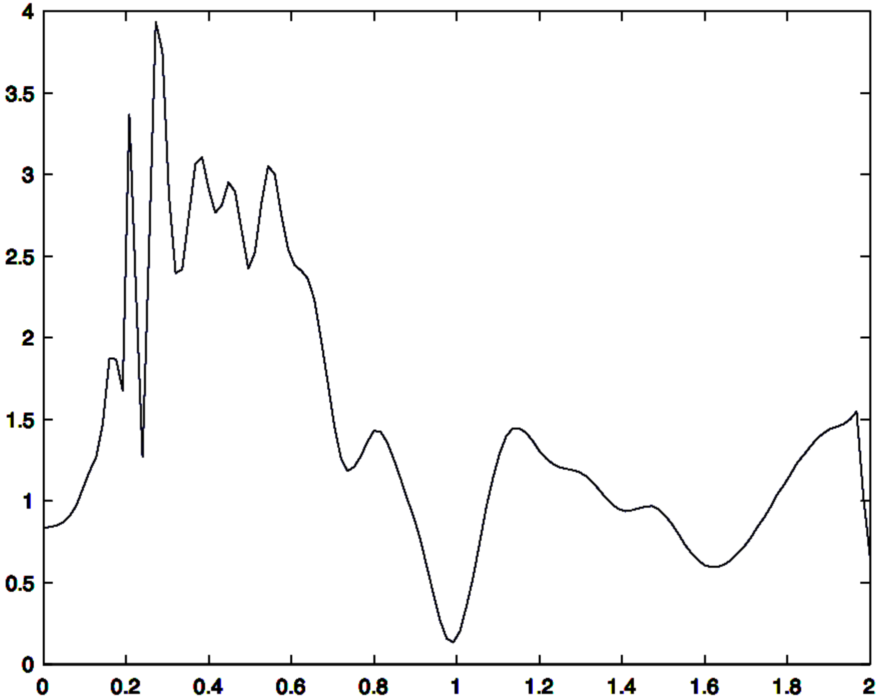
Computational results
FWI with L^2 and W_2 misfit functionals



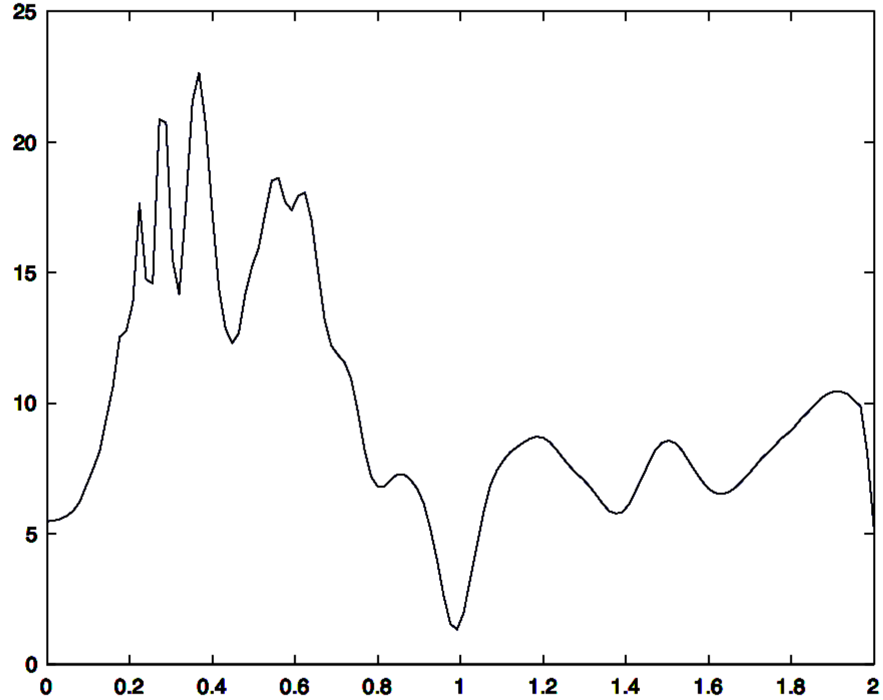
Using the Method of Steepest Descent

Computational results
Extension to many observation points

Misfit function in L^2



One observation point

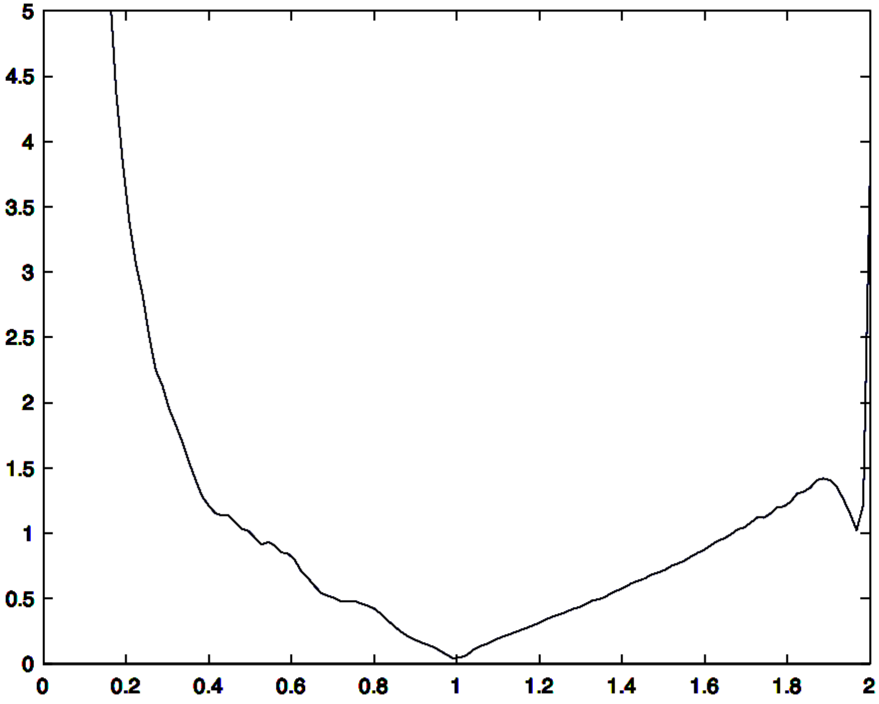


Six observation points

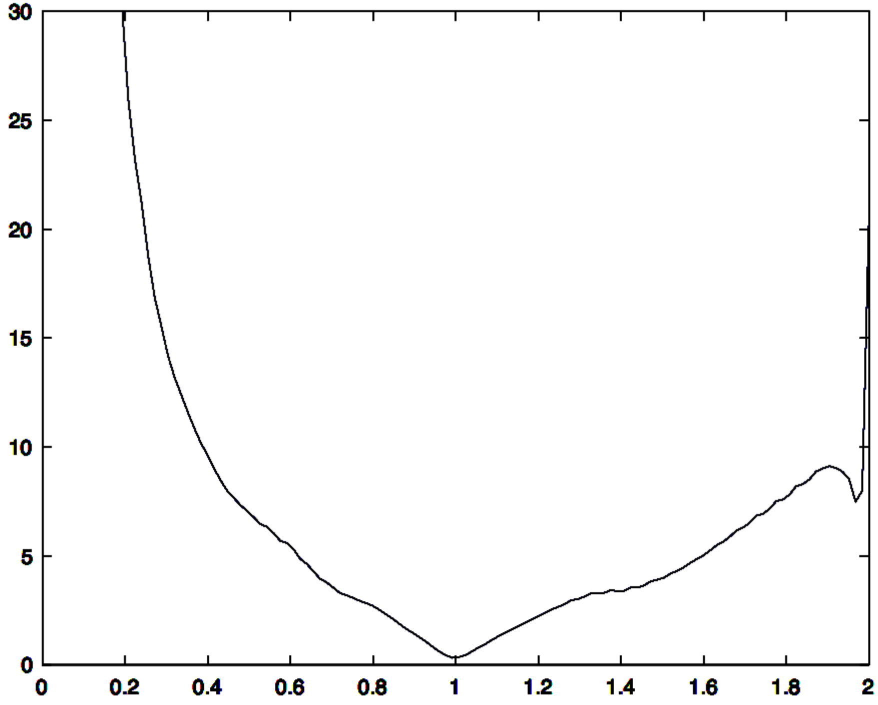
Computational results

Extension to many observation points

Misfit function in W_2

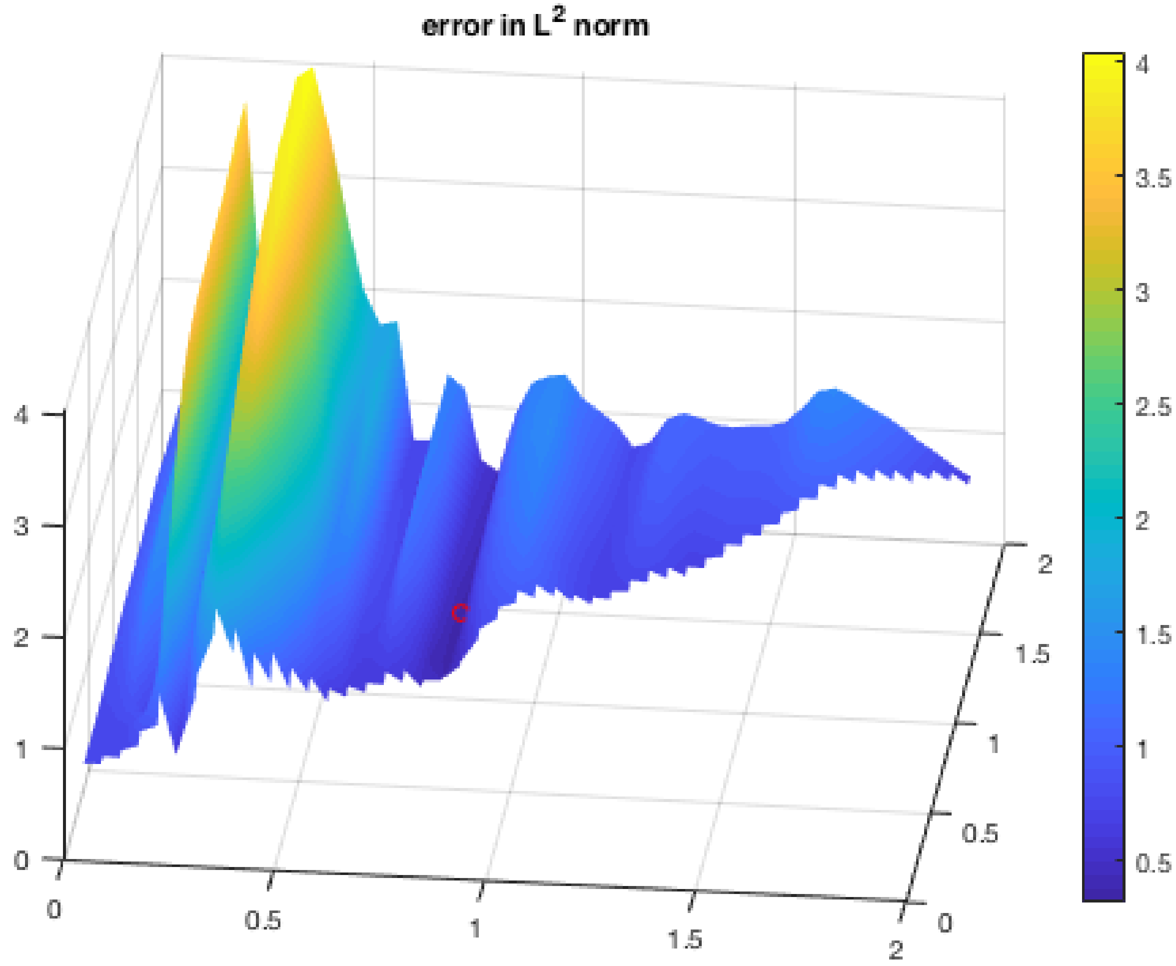


One observation point



Six observation points

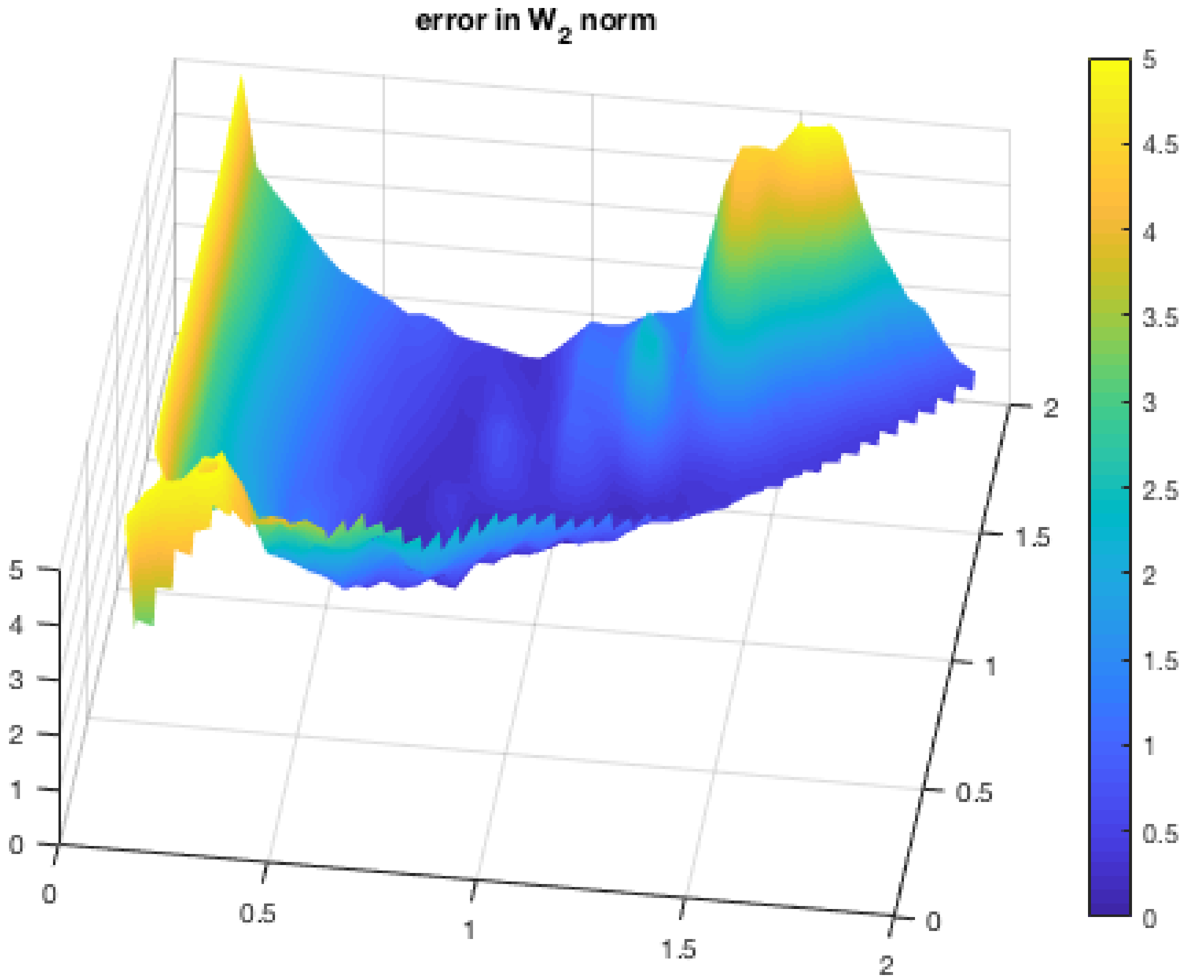
Computational results
Extension to three-layers model



Misfit function in L^2 norm

Computational results

Extension to three-layers model



Misfit function in W_2 norm

Conclusion

Wasserstein metric is a good candidate to replace the commonly used L_2 norm in the full waveform inversion

Acknowledgements

- Prof. Dr. Markus Grote

- Jet Hoe Tang

University of Basel

Thank you
for your attention.