

Tamm's theorem for log-analytic functions

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Set-theoretical definition of an o-minimal structure

Definition

A structure \mathcal{S} on a set R is a sequence $(S_n)_{n \in \mathbb{N}}$ of sets such that the following conditions hold:

- (1) S_n contains subsets of the set R^n for all $n \in \mathbb{N}$,
- (2) If $A, B \in S_n$, then $A \cap B \in S_n$, $A \cup B \in S_n$ and $A \setminus B \in S_n$ für all $n \in \mathbb{N}$,
- (3) If $A \in S_n$ and $B \in S_m$, then $A \times B \in S_{n+m}$ for all $n, m \in \mathbb{N}$,
- (4) If $A \in S_{n+1}$, then $\pi(A) \in S_n$, where π denotes the projection on the first n coordinates for all $n \in \mathbb{N}$,
- (5) $\emptyset \in S_n$ and $R^n \in S_n$ for all $n \in \mathbb{N}$.

Set-theoretical definition of an o-minimal structure

Definition

A o-minimal structure \mathcal{M} is a structure on a set R with a dense linear order without endpoints " \leq " with the following two conditions:

- (1) $\{(x, y) \mid x \leq y\} \in S_2$,
- (2) If $A \in S_1$, then A is a finite union of points and intervals.

Definition

Let \mathcal{M} be as above. let $n \in \mathbb{N}$.

- (1) We call a set $A \subset R^n$ definable in \mathcal{M} if $A \in S_n$ holds.
- (2) Let $B \subseteq R^n$. Let $m \in \mathbb{N}$. We call a function $f : B \rightarrow R^m$ definable in \mathcal{M} if

$$\Gamma(f) := \{(x, y) \in B \times R^m \mid f(x) = y\} \subseteq R^n \times R^m$$

is definable in \mathcal{M} .

Important o-minimal structures on the real field

- \mathbb{R} : The pure real field.
- \mathbb{R}_{an} : The real field augmented by restricted analytic functions. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called restricted analytic if it is of the form

$$f(x) := \begin{cases} p(x), & \text{if } x \in [-1, 1]^n, \\ 0 & \text{else.} \end{cases},$$

where $p(x)$ is a power series which converges on a neighbourhood of $[-1, 1]^n$. The definable sets and functions are exactly the globally subanalytic ones.

- $\mathbb{R}_{\text{an,exp}}$: The structure \mathbb{R}_{an} augmented by the exponential function \exp .

A parametric result of Tamm's theorem for \mathbb{R}_{an}

L. van den Dries and C. Miller have shown the following theorem:

Theorem (A parametric version of Tamm's theorem)

- *Let $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ be definable in \mathbb{R}_{an} . Then there exists $N \in \mathbb{N}$ such that for all $(x_0, y_0) \in \mathbb{R}^{n+m}$, if $y \mapsto f(x_0, y)$ is C^N in a neighbourhood of y_0 , then $y \mapsto f(x_0, y)$ is real analytic in a neighbourhood of y_0 .*
- *The set $\{(x, y) \in \mathbb{R}^{n+m} \mid f(x, -) \text{ is real analytic at } y\}$ is definable in \mathbb{R}_{an} .*

Question: Do these two parts of the theorem hold in $\mathbb{R}_{\text{an,exp}}$?

\Rightarrow In general not.

Counterexample to the first claim in $\mathbb{R}_{\text{an},\text{exp}}$.

- Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(x, y) := \begin{cases} xe^{-\frac{1}{y^2}}, & \text{if } y > 0, \\ 0 & \text{else.} \end{cases}$$

- For $x \neq 0$, $f(x, -)$ is C^∞ , but not real analytic at $y = 0$.

Counterexample to the second claim in $\mathbb{R}_{\text{an},\text{exp}}$

- Consider the function

$$f(x, y) := \begin{cases} |y|^{\frac{1}{x}}, & \text{if } x > 0 \text{ and } y \neq 0, \\ 0 & \text{else.} \end{cases}$$

and the set

$$A := \{(x, y) \in \mathbb{R}^2 \mid f(x, -) \text{ is real analytic at } y\}.$$

- We see that

$$M := \{x \in \mathbb{R} \mid f(x, -) \text{ is real analytic at } y = 0\}$$

is not definable in $\mathbb{R}_{\text{an},\text{exp}}$. So A isn't definable in $\mathbb{R}_{\text{an},\text{exp}}$ as well.

Remark

Tamm's theorem doesn't hold for C^∞ instead of real analytic in general.

Main Question: Is there a natural class of functions which are definable in the structure $\mathbb{R}_{\text{an},\text{exp}}$ such that the parametric version of Tamm's theorem holds?

Log-analytic functions

Let $X \subseteq \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}$ be definable in $\mathbb{R}_{\text{an}, \text{exp}}$.

Definition (log-analytic functions)

- f is log-analytic of type 0, if f is piecewise the restriction of a globally subanalytic function on X .
- Let $r \in \mathbb{N}$. We call a function $f : X \rightarrow \mathbb{R}$ log-analytic of type $\leq r$, if there is a decomposition \mathcal{C} of X in analytic cells definable in $\mathbb{R}_{\text{an}, \text{exp}}$ such that for all $C \in \mathcal{C}$

$$f(x) = F(g_1(x), \dots, g_l(x), \log(g_{l+1}(x)), \dots, \log(g_m(x))),$$

where F is globally subanalytic and $g_1, \dots, g_m : C \rightarrow \mathbb{R}$ are log-analytic functions of type $\leq r - 1$.

Main theorem

Main theorem

- Let $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ be a log-analytic function. Then there exists $N \in \mathbb{N}$ such that for all $(x_0, y_0) \in \mathbb{R}^{n+m}$ if $y \mapsto f(x_0, y)$ is C^N in a neighbourhood of y_0 then $y \mapsto f(x_0, y)$ is real analytic in a neighbourhood of y_0 .
- The set $M := \{(x, y) \in \mathbb{R}^{n+m} \mid f(x, -) \text{ is real analytic}\}$ is definable in $\mathbb{R}_{\text{an}, \text{exp}}$.

Convention

"Definable" means definable in $\mathbb{R}_{\text{an},\text{exp}}$.

Proof of the "Main theorem"

Theorem (Lion/Rolin preparation for log-analytic functions)

Let $X \subseteq \mathbb{R}^{n+1}$ definable in $\mathbb{R}_{\text{an},\text{exp}}$ and $f : X \rightarrow \mathbb{R}$ a log-analytic function. Then there is a $r \in \mathbb{N}$ and a decomposition \mathcal{C} of X into definable analytic cells such that for all $C \in \mathcal{C}$

$$f(x, y) = A(x)(|y_0|)^{q_0} \cdot \dots \cdot (|y_r|)^{q_r} U(x, y_0, \dots, y_r),$$

where

$$y_0 = y - \Theta_0(x), y_1 = \log(|y_0|) - \Theta_1(x), \dots, y_r = \log(|y_{r-1}|) - \Theta_r(x)$$

such that $A, \Theta_0, \dots, \Theta_r$ are log-analytic functions on the base of \mathcal{C} , $y_i \neq 0$ on C for all $i \in \{0, \dots, r\}$, U is a special unit in y_0, \dots, y_r , $q_i \in \mathbb{Q}$ and $\Theta_i \equiv 0$ or $y_i \leq M_i |\Theta_i|$ for a $M_i \in \mathbb{R}$ and all $i \in \{0, \dots, r\}$.

Proof steps of the "Main Theorem"

Conclusion

Let $f : A \times (0, 1) \rightarrow \mathbb{R}$ a log-analytic function with $A \subseteq \mathbb{R}^m$. There are definable sets A_1, \dots, A_m with $A = \bigcup_{i \in \{1, \dots, m\}} A_i$, definable functions h_1, \dots, h_m with $h_i : A_i \rightarrow (0, 1)$ and log-analytic functions f_1, \dots, f_m such that f_i is prepared and $f = f_i$ holds on $A_i \times (0, h_i)$. Here

$$A_i \times (0, h_i) := \{(x, y) \mid x \in A_i, 0 < y < h_i(x)\} \text{ for } i \in \{1, \dots, M\}.$$

Proof of the "Main theorem"

Reduction of real analyticity to dimension one: Let $U \subseteq \mathbb{R}^n$ and $f : U \rightarrow \mathbb{R}$ be a log-analytic function. Let $x \in U$ and $y \in \mathbb{R}^n$. We consider the function $t \mapsto f(x + yt)$ in a small interval around zero.

- We call f G^k at x , if $t \mapsto f(x + yt)$ is C^k at $t = 0$ for all $y \in \mathbb{R}^n$ and $y \mapsto \frac{d^k f(x + yt)}{dt^k}(0) : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by a homogeneous polynomial in y of degree k .
- If f is G^k for all $k \in \mathbb{N}$, then f is called G^∞ .

Lemma (van den Dries/Miller)

f is real analytic at x if and only if f is G^∞ at x and there exists $\epsilon > 0$ such that for all $y \in \mathbb{R}^n$ with $|y| < 1$ the function $t \mapsto f(x + yt)$ is real analytic on $(-\epsilon, \epsilon)$.

Proof of the "Main theorem"

Definition (Flatness)

Let $U \subseteq \mathbb{R}^n$ be open and $a \in U$. Let $f : U \rightarrow \mathbb{R}$ be a function. We call f N -flat at a if f is C^N at a and all partial derivatives of f of order less than N vanish at a . We call f flat at a if all partial derivatives vanish at a .

Lemma

Let $A \subseteq \mathbb{R}^n$ definable and $f : A \rightarrow \mathbb{R}$ be a log-analytic function. Then there exists $N \in \mathbb{N}$ such that for all $(x, y) \in A$ with $y \in \text{int}(A_x)$ the following holds:

f is N -flat at $a \Rightarrow f(x, -) \equiv 0$ in a neighbourhood of y on A_x .

Proof of the "Main theorem"

Conclusions

Let $U \subseteq \mathbb{R}^n$ be open and definable connected and $(f_i)_{i \in \mathbb{N}}$ C^∞ functions which are log-analytic on U . Then the following holds:

- (1) f_i is flat at $a_0 \in U \Rightarrow f_i \equiv 0$ on U .
- (2) Let $Z(f_i) := \{x \in U \mid f_i(x) = 0\}$. Then there exists $M \in \mathbb{N}$ such that $\bigcap_{i \in \mathbb{N}} Z(f_i) = \bigcap_{i \leq M} Z(f_i)$ holds.

- Reduce the property G^k on the zero set of a definable function: $f(x, -)$ is G^k at y if and only if there is a certain definable C^∞ log-analytic function $w_k : \mathbb{R}^{n+m} \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $w_k(x, y, z) = 0$ for all $z \in \mathbb{R}^n$.
- There exists $N \in \mathbb{N}$ such that for all $(x, y) \in \mathbb{R}^{n+m}$ the log-analytic function $f(x, -)$ is G^N at y if and only if $f(x, -)$ is G^∞ at y .

Proof of the "Main theorem"

Main step: Consider the definable function $F : \mathbb{R}^{n+m} \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$,

$$F(x, y, z, t) := f(x, y + tz)$$

Set $v := (x, y, z)$.

Lemma

There is a $N \in \mathbb{N}$, so that for all $v \in \mathbb{R}^{m+n} \times \mathbb{R}^n$ holds: $F(v, -)$ is C^N at $t = 0$ if and only if $F(v, -)$ is real analytic at $t = 0$.

Proof of the "Main theorem"

Sketch of proof:

- Apply the Lion/Rolin preparation theorem on F in the variable t :

$$F(v, -) := A(v)t_0^{q_0} \cdot \dots \cdot t_r^{q_r} U(v, t_0, \dots, t_r)$$

- We get a multidimensional Puiseux-series in the Variables t_1, \dots, t_r .
- Big step: We can find a function g which is real analytic at $t = 0$ such that $F - g$ is N -flat for all $N \in \mathbb{N}$. $\Rightarrow F - g \equiv 0$ at a neighbourhood of 0. $\Rightarrow F(v, -)$ is real analytic at $t = 0$.