Fourier interpolation problems and modular forms

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Introduction

Recall that the Fourier transform of $f \in L^1(\mathbb{R}^d)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx, \qquad \xi \in \mathbb{R}^d.$$

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This formula requires knowing $\hat{f}$ on (almost) all of $\mathbb{R}^d$. What can we say if have less information about $\hat{f}$ and a little more about f?

A general framework

Let $\mathcal{S}$ be a space of continuous functions on $\mathbb{R}^d$ (or any LCAG) and $N, \widehat{N} \subset \mathbb{R}^d$ subsets. Consider the linear map

$$R : \mathcal{S} \to C(N) \times C(\widehat{N}), \qquad R(f) := (f|_N, \widehat{f}|_{\widehat{N}}).$$

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- **Reconstruction** problem: Can we recover f from $R(f)$? Can we find $a_n, \tilde{a}_n$ so that

$$f(x) = \int_{n \in N} a_n(x) f(n) + \int_{n \in \widehat{N}} \tilde{a}_n(x) \widehat{f}(n),$$

  for all $f \in \mathcal{S}$ and all $x \in \mathbb{R}^d$?

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  for all $f \in \mathcal{S}$ and all $x \in \mathbb{R}^d$?
- **Interpolation** problem: What is the image of R? Given $(g, \tilde{g}) \in C(N) \times C(\widehat{N})$, does

$$x \mapsto \int_{n \in N} a_n(x) g(n) + \int_{n \in \widehat{N}} \tilde{a}_n(x) \tilde{g}(n),$$

  define an element of $\mathcal{S}$ and map to $(g, \tilde{g})$ via R?

Example 0

The classical Whittaker–Shannon interpolation formula.

Theorem

For all continuous $f \in L^2(\mathbb{R})$ with $\operatorname{supp}(\hat{f}) \subset [-1/2, 1/2]$, one has

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)},$$

with uniformly point-wise convergence and convergence in L^2.

Thus, in an appropriate Paley–Wiener space, it is enough to have information only about $f|_{\mathbb{Z}}$!

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Theorem (Radchenko, Viazovksa)

There exist even Schwartz functions $a_n \in \mathcal{S}(\mathbb{R})$ such that

$$f(x) = \sum_{n=0}^{\infty} f(\sqrt{n}) a_n(x) + \sum_{n=0}^{\infty} \hat{f}(\sqrt{n}) \widehat{a_n}(x) \qquad (1)$$

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- $R : f \mapsto ((f(\sqrt{n}))_{n \in \mathbb{N}_0}, (\hat{f}(\sqrt{n}))_{n \in \mathbb{N}_0})$ defines an isomorphism (of Fréchet spaces) between $\mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ and a subspace of $\mathcal{S}(\mathbb{N}_0) \times \mathcal{S}(\mathbb{N}_0)$ of codimension 1.

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- Similar result holds for odd Schwartz functions.
- Formula (1) makes sense if $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^d)$.

Example 2

Theorem (Cohn, Kumar, Miller, Radchenko, Viazovska)

For $d \in \{8, 24\}$, there are $a_n, b_n \in \mathcal{S}_{rad}(\mathbb{R}^d)$ so that

$$f = \sum_{n=n_0}^{\infty} f(\sqrt{2n})a_n + f'(\sqrt{2n})b_n + \sum_{n=n_0}^{\infty} \widehat{f}(\sqrt{2n})\widehat{a_n} + \widehat{f}'(\sqrt{2n})\widehat{b_n},$$

for all $f \in \mathcal{S}_{rad}(\mathbb{R}^d)$. Here, $n_0 = 1$ if $d = 8$ and $n_0 = 2$ if $d = 24$.

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- The map "R" gives $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \cong \mathcal{S}(\mathbb{N})^4$.
- Used to prove universal optimality of the E_8- and the Leech lattice as energy minimizing point configurations.

Example 3

Theorem (Bondarenko, Radchenko, Seip)

There exist even entire functions $U_n : \mathbb{C} \to \mathbb{C}$ *and* $V_{\rho,j} : \mathbb{C} \to \mathbb{C}$, *indexed by pairs* (ρ, j), *consisting of:*

- *a non-trivial zero* ρ *of the Riemann zeta function,*
- *an integer* $j, 0 \leq j < m(\rho) =$ *the multiplicity of* ρ,

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and positive real numbers $T_k > 0$ such that: for all $\varepsilon > 0$ and all even holomorphic $f : \{|\operatorname{Im}(z)| < 1/2 + \varepsilon\} \to \mathbb{C}$ having some decay and all $z \in \mathbb{C}$ with $|\operatorname{Im}(z)| < 1/2$ we have

$$f(z) = \sum_{n=1}^{\infty} U_n(z) \widehat{f}\left(\frac{\log(n)}{4\pi}\right) + \lim_{k\to\infty} \sum_{0 \leq \operatorname{Im}(\rho) \leq T_k} \sum_{j \leq 0 < m(\rho)} V_{\rho,j}(z) f^{(j)}\left(\frac{\rho - 1/2}{i}\right),$$

Example 4

Theorem (S)

Let $d \geq 1$ and $n_0 + \hat{n}_0 = 1 + \lfloor d/4 \rfloor$. There are $a_n, \tilde{a}_n \in \mathcal{S}_{rad}(\mathbb{R}^d)$ so that

$$f = \sum_{n=n_0}^{\infty} f(\sqrt{n}) a_n + \sum_{n=\widehat{n}_0}^{\infty} \widehat{f}(\sqrt{n}) \tilde{a}_n,$$

for all $f \in \mathcal{S}_{rad}(\mathbb{R}^d)$. The map

$$R(f) = \Big((f(\sqrt{n}))_{n \in \mathbb{N}_0}, (\hat{f}(\sqrt{n}))_{n \in \mathbb{N}_0} \Big)$$

defines an isomorphism of Fréchet spaces between $\mathcal{S}_{rad}(\mathbb{R}^d)$ and a closed subspace of $\mathcal{S}(\mathbb{N}_0)^2$ defined as the pre-annihilator of the image of an injection $M_{d/2}(\Gamma(2)) \hookrightarrow (\mathcal{S}(\mathbb{N}_0)^2)^*$.

Interpolation bases

In the previous theorem, the interpolation functions a_n, $\tilde{a}_n$ satisfy, for all $n, m \geq n_0$ and all $i, j \geq \hat{n}_0$,

$$a_n(\sqrt{m}) = \delta_{nm} \qquad \widehat{a}_i(\sqrt{j}) = 0$$
$$\tilde{a}_n(\sqrt{m}) = 0 \qquad \widehat{\tilde{a}}_i(\sqrt{j}) = \delta_{ij}$$

All of the previous examples have similar properties, which makes these formulas non-redundant, in the sense that the set of interpolation nodes is minimal.

Example $4 + \varepsilon$

Since we have some control over the growth/decay of the interpolation basis $a_n, \tilde{a}_n$, we can apply functional-analytic methods by Ramos–Sousa [4] to obtain perturbed interpolation formulas. For example, for all sequences of sufficiently small real numbers $\varepsilon_n, \widehat{\varepsilon}_n$, there are $c_n(r), \tilde{c}_n(r)$, so that for all $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^4)$,

$$f(r) = \sum_{n=1}^{\infty} f(\sqrt{n + \varepsilon_n}) c_n(r) + \sum_{n=1}^{\infty} \widehat{f}(\sqrt{n + \widehat{\varepsilon}_n}) \tilde{c}_n(r).$$

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Theorem (S)

Fix $d \geq 2$. There exist $A_n, \tilde{A}_n \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$ and tempered distributions T_x, $\tilde{T}_x$ such that

$$f(x) = T_x(f) + \sum_{n=1}^{\infty} \int_{S^{d-1}} A_n(x,\zeta) f(\sqrt{n}\zeta) d\zeta$$
$$+ \tilde{T}_x(\hat{f}) + \sum_{n=1}^{\infty} \int_{S^{d-1}} \tilde{A}_n(x,\zeta) \widehat{f}(\sqrt{n}\zeta) d\zeta$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$. Can take $T_x = \tilde{T}_x = 0$ if $d \geq 4$.

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$$f(x) = T_x(f) + \sum_{n=1}^{\infty} \int_{S^{d-1}} A_n(x,\zeta) f(\sqrt{n}\zeta) d\zeta + \tilde{T}_x(\hat{f}) + \sum_{n=1}^{\infty} \int_{S^{d-1}} \tilde{A}_n(x,\zeta) \widehat{f}(\sqrt{n}\zeta) d\zeta$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$. Can take $T_x = \tilde{T}_x = 0$ if $d \geq 4$.

Using harmonic analysis on spheres

If for all $p \in \{d, d+2, d+4, \dots\}$, there exist $a_{p,n}(r), \tilde{a}_{p,n}(r)$ such that

$$g(v) = \sum_{n=0}^{\infty} g(\sqrt{n}) a_{p,n}(|v|) + \sum_{n=0}^{\infty} \widehat{g}(\sqrt{n}) \tilde{a}_{p,n}(|v|),$$

for all $g \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^p)$ and all $v \in \mathbb{R}^p$,

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for all $g \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^p)$ and all $v \in \mathbb{R}^p$, **then**

$$f(x) = \sum_{m=0}^{\infty} \Bigg(\sum_{n=0}^{\infty} a_{d+2m,n}(|x|) \frac{1}{\sqrt{n}^m} \int_{S^{d-1}} f(\sqrt{n}\zeta) Z_m^d(x,\zeta) d\zeta$$
$$+ \sum_{n=0}^{\infty} \tilde{a}_{d+2m,n}(|x|) \frac{i^m}{\sqrt{n}^m} \int_{S^{d-1}} \hat{f}(\sqrt{n}\zeta) Z_m^d(x,\zeta) d\zeta \Bigg),$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$. To get the $A_n, \tilde{A}_n, T_x, \tilde{T}_x$, rearrange these sums.

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TFAE for all fixed dimensions $d \geq 1$ and fixed radii $r \geq 0$.

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(2) There exist holomorphic functions $F, \tilde{F} : \mathbb{H} \to \mathbb{C}$ of moderate growth satisfying

$$F(\tau + 2) = F(\tau), \quad \tilde{F}(\tau + 2) = \tilde{F}(\tau),$$
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The connection between $a_{d,n}(r)$ and $F(\tau, r)$ is

$$F(\tau, r) = \sum_{n=0}^{\infty} a_n(r) e^{\pi i n \tau}, \quad a_n(r) = \frac{1}{2} \int_{iy-1}^{iy+1} F(\tau, r) e^{-\pi i n \tau} \, d\tau$$

Functions with modularity

A piece of convenient notation: For $k \in 2\mathbb{Z}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $f : \mathbb{H} \to \mathbb{C}$, define

$$(f|_k M)(\tau) := (c\tau + d)^{-k} f(M\tau).$$

Extend notation to the group ring $\mathbb{C}[\mathrm{PSL}_2(\mathbb{R})]$, e.g. $f|_k(M-1) = 0$ means $f|_k M - f = 0$.

Definition

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index and let $f : \mathbb{H} \to \mathbb{C}$ be a function. Then f is *modular of weight k for* Γ, if

$$f|_k M = f \quad \text{for all } M \in \Gamma$$

and f is called a *modular form of weight k for* Γ, if it is in addition holomorphic on $\mathbb{H}$ and holomorphic at all the cusps of Γ.

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Idea: Take φ which is already modular with respect to a subgroup $\Gamma_\infty \leq \Gamma$ and average over $\Gamma_\infty \backslash \Gamma$:

$$P_k(\tau, \varphi) = \sum_{M \in \Gamma_\infty \backslash \Gamma} \varphi|_k M,$$

If $k > 2$, can take $\varphi = 1$ (Eisenstein series) or $\varphi(\tau) = e^{2\pi i m \tau}$ for $m \in \mathbb{Z}_{\geq 1}$ (Poincaré series with parameter m)

Rewriting the functional equations for F and $\tilde{F}$

Let

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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be the "standard generators" of $\mathrm{PSL}_2(\mathbb{Z})$. To solve the radial interpolation problem in dimension d, we want, for each radius $r \geq 0$, two holomorphic functions $F, \tilde{F}$, such that

$$F|_k(T^2 - 1) = 0, \quad \tilde{F}|_k(T^2 - 1) = 0, \quad F + \tilde{F}|_k S = \varphi,$$

Here[1] , $k = d/2$ and $\varphi(\tau) = g_\tau(r)$.

[1] Assume $8|d$ for simplicity

After eliminating $\tilde{F}$ from these equations, we have abstracted our problem to the following: Given $\varphi : \mathbb{H} \to \mathbb{C}$, we want $F : \mathbb{H} \to \mathbb{C}$ satisfying

$$F|_k(T^2 - 1) = 0, \qquad F|_k(ST^2S - 1) = \varphi|_k(ST^2S - 1).$$

If φ were zero (homogenous equations), then we need a *modular form* F with respect to the subgroup $\Gamma_2 \leq \mathrm{PSL}_2(\mathbb{Z})$ generated by the elements

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad ST^2S = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

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But our equations are inhomogeneous. Its solutions are called *modular integrals*. There is related work by M. Eichler and M. Knopp on this subject, which can be adapted thanks to D. Radchenko and M. Viazovska.

Constructing the modular integral F

Can use a construction similar to Eichlers generalized Poincaré series, as done in [5]; the method requires dimension $d > 4$. Alternatively, we can use

$$F(\tau) = \int_{\gamma_\tau} K(\tau, z)\varphi(z)dz,$$

for a suitable singular, separately modular integration kernel $K : \mathbb{H} \times \mathbb{H} \to \mathbb{P}^1(\mathbb{C})$.

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$$K(\tau, z) = \frac{\lambda'(z)}{\lambda(z) - \lambda(\tau)} \frac{\Theta(\tau)^d}{\Theta(z)^d} \frac{\lambda(\tau)^{n_0}(1-\lambda(\tau))^{\widehat{n}_0}}{\lambda(z)^{n_0}(1-\lambda(z))^{\widehat{n}_0}},$$

where $n_0 + \hat{n}_0 = 1 + [d/4]$ and

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}, \quad \lambda(\tau) = \frac{\Theta(-1/(\tau+1))^4}{\Theta(\tau)^4}$$

are Jacobi's theta function and the modular lambda invariant; and γ_τ is a suitable path in $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$.

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