University of Zurich[UZH]

Mod-ϕ approximation schemes: theory and applications to credit risk

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Plan of the talk

1. Mod-ϕ convergence
2. Mod-ϕ approximation schemes
3. Mod-Poisson approximation for credit risk models
4. Estimation of risk measures
5. CDO pricing

Mod-ϕ convergence

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Mod-ϕ convergence[1]:

- has been successfully applied in number theory, combinatorics, random graphs, random matrices, etc,
- gives *precise* large deviation and local limit theorems, deviations at all scales (from central limit theorem to large deviations), full treatment of "normality zone",
- comes in many flavors, but relies fundamentally on Fourier analysis tools,
- in this talk we focus on mod-ϕ approximation schemes[2] on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

[1] Féray, Méliot, Nikeghbali, *Mod-φ Convergence: Normality Zones and Precise Deviations*, Springer, 2016

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Remark: Let $(Y_k)_{k\in\mathbb{N}}$ be a sequence of centered iid rvs with variance σ^2 and set $X_n := \sum_{k=1}^n Y_k$, then

$$\frac{X_n}{\sqrt{n}\sigma} \xrightarrow{d} \mathcal{N}(0,1), \qquad \left(\hat{\mu}_{X_n}(t) \sim e^{-n(\frac{1}{2}t^2\sigma^2)}\right)$$

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More generally:

$$\hat{\mu}_{X_n}(t) \sim e^{\lambda_n \phi(t)},$$

where $\phi(t)$ is the Lévy-Khintchine exponent of an infinitely divisible (i.d.) law.

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Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{Z}$-valued rvs with laws μ_{X_n} and characteristic functions $\hat{\mu}_{X_n}$ (on $\mathbb{T}$) and let ϕ be the Lévy-Khintchine exponent of a *reference i.d. law*.

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Definition (Mod-ϕ convergence) We say that $(X_n)_{n\in\mathbb{N}}$ converges mod-ϕ with parameters $(\lambda_n)_{n\in\mathbb{N}}$ and limiting function ψ if $\lambda_n \to \infty$ and

$$\frac{\hat{\mu}_{X_n}(\xi)}{e^{\lambda_n \phi(\xi)}} =: \psi_n(\xi), \quad \xi \in \mathbb{T} \tag{1}$$

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$$\lim_{n\to\infty} \psi_n(\xi) = \psi(\xi) \quad \text{(e.g. in } \mathcal{C}^r(\mathbb{T})\text{)}.$$

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- Study the renormalized Fourier transforms, instead of the renormalized rvs.

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- By Wiener's theorem $e^{-\lambda_n \phi(\xi)}$ is in the Wiener algebra $\mathcal{A}(\mathbb{T})$, therefore $\psi_n(\xi) \in \mathcal{A}(\mathbb{T})$ and it can be thought of as *deconvolution residue*:

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- From a signal processing perspective, we have the decomposition:

$$X_n = (\lambda_n \text{ iid copies of reference i.d. law}) + \varepsilon_n$$

Theorem (Mod-Poisson convergence)
Let $X_n = \sum_{i=1}^{n} Y_i$ be a sum of n independent Bernoulli random variables, with $Y_i \sim \text{Bernoulli}(p_i)$ and let us assume that $\sum_{i=1}^{\infty} p_i = \infty$ and $\sum_{i=1}^{\infty} p_i^2 < \infty$.

Then $(X_n)_{n\in\mathbb{N}}$ converges mod-Poisson with parameters $\lambda_n = \sum_{i=1}^{n} p_i$.

Proof.

$$
\begin{aligned}
\psi_n(\xi) &= \hat{\mu}_{X_n}(\xi) e^{-\lambda_n(e^{i\xi}-1)} \\
&= \prod_{i=1}^{n} \left((1 + p_i(e^{i\xi} - 1)) e^{-p_i(e^{i\xi}-1)} \right) \to \prod_{i=1}^{\infty} \left((1 + p_i(e^{i\xi} - 1)) e^{-p_i(e^{i\xi}-1)} \right)
\end{aligned}
$$

where the converge is uniform on $\mathbb{T}$ because $\sum_{i=1}^{\infty} p_i^2 < \infty$.

□

Mod-ϕ approximation schemes

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- Approximation idea:

  $\Longrightarrow$ replace $\psi_n(\xi)$ on $\mathbb{T}$ by its truncated Laurent series around $\xi = 0$ up to order r:

$$\chi_n^{(r)}(\xi) = 1 + \sum_{k=1}^{r} b_{k,n}(e^{i\xi} - 1)^k + \sum_{k=1}^{r} c_{k,n}(e^{-i\xi} - 1)^k,$$

  $\Longrightarrow$ approximate $\hat{\mu}_{X_n}(\xi)$ by

$$\hat{\nu}_n^{(r)}(\xi) := e^{\lambda_n \phi(\xi)} \chi_n^{(r)}(\xi).$$

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- The sequence of signed measures $(\nu_n^{(r)})_{n \in \mathbb{N}}$ is the mod-ϕ approximation scheme of order r of $(X_n)_{n \in \mathbb{N}}$.
- The Wiener algebra $\mathcal{A}(\mathbb{T})$ is the right setting to study these approximations because:

$$\|\hat{\mu}\|_{\mathcal{A}(\mathbb{T})} = \sum_{n \in \mathbb{Z}} |c_n(\hat{\mu})| = \sum_{n \in \mathbb{Z}} |\mu(n)| = \|\mu\|_{TV},$$

  where $\| \cdot \|_{TV}$ is the total variation norm of μ.

Main results:

$$d_{TV}(\mu_{X_n},\nu_n^{(r)}) = \sum_{k\in\mathbb{Z}} |\mu(\{k\}) - \nu(\{k\})| = \frac{|\beta|}{\sqrt{2\pi}(\sigma^2\lambda_n)^{\frac{r+1}{2}}} \int_{\mathbb{R}} |G_{r+1}(\alpha)| d\alpha + o\left(\frac{1}{(\lambda_n)^{\frac{r+1}{2}}}\right)$$

$$d_L(\mu_{X_n},\nu_n^{(r)}) = \sup_{k\in\mathbb{Z}} |\mu(\{k\}) - \nu(\{k\})| = \frac{|\beta||G_{r+1}(z_{r+2})|}{\sqrt{2\pi}(\sigma^2\lambda_n)^{\frac{r}{2}+1}} + o\left(\frac{1}{(\lambda_n)^{\frac{r}{2}+1}}\right)$$

$$d_K(\mu_{X_n},\nu_n^{(r)}) = \sup_{k\in\mathbb{Z}} |\mu((-\infty,k]) - \nu((-\infty,k])| = \frac{|\beta||G_r(z_{r+1})|}{\sqrt{2\pi}(\sigma^2\lambda_n)^{\frac{r+1}{2}}} + o\left(\frac{1}{(\lambda_n)^{\frac{r+1}{2}}}\right)$$

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Remarks:

- Asymptotic convergence for $\lambda_n \to \infty$.
- For fixed n, get better convergence rate as r increases, unlike "classical" asymptotic series (e.g. saddlepoint method, Hedgeworth expansions, large deviations expansions, Chen-Stein method, etc.).

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Then $(X_n)_{n\in\mathbb{N}}$ converges mod-Poisson with parameters $\lambda_n = \sum_{i=1}^n p_i$.

The residues $\psi_n(\xi)$ admit the following Laurent series expansion:

$$\psi_n(\xi) = 1 + \sum_{k=1}^{r} b_{k,n}(e^{i\xi} - 1)^k,$$

with

$$b_{k,n} = \frac{1}{k!}\left(\sum_{\pi\in\Pi(k)} \mu(\hat{0},\pi) \prod_{B\in\pi} \mathfrak{p}_{|B|,n}\right),$$

where

- $\Pi(k)$ *is the lattice of set partitions of* $\{1,2,\ldots,k\}$,
- $\mu(\hat{0},\pi) = (-1)^{k-|\pi|}\prod_{B\in\pi}(|B|-1)!$ *is the Möbius function for the incidence algebra of the lattice* $\Pi(k)$,
- $\mathfrak{p}_{k,n} := \sum_{i=1}^n p_i^k$, *with the convention* $\mathfrak{p}_{1,n} := 0$.

Mod-Poisson approximation

The first few coefficients of $\psi_n(\xi)$ are:

$$\psi_n(\xi) = 1 - \frac{1}{2}\left(\sum_{i=1}^{n} p_i^2\right)(e^{i\xi} - 1)^2 + \frac{1}{3}\left(\sum_{i=1}^{n} p_i^3\right)(e^{i\xi} - 1)^3 + o(|\xi|^3)$$

- $r = 0 \Rightarrow$ classical Poisson approximation[3] ($X_n \approx$ Poisson $\left(\sum_{i=1}^{n} p_i\right)$).
- $r = 2 \Rightarrow$ Stein-Chen's method[4]
- Higher order coefficients ($r \geq 3$) are unique to the mod-ϕ approach.

[3] Prohorov, *Asymptotic behavior of the binomial distribution*, Uspehi Mat. Nauk., 8(3):135:142, 1953.
Le Cam, *An approximation theorem for the Poisson binomial distribution*, Pacific J. Math, 10(4):1181-1197, 1960.
Kerstan, *Verallgemeinerung eines Satzes von Prochorow und Le Cam*, Z. Wahrsch. Werw. Gebiete, 2:173-179, 1964.

[4] Chen, *On the convergence of Poisson binomial to Poisson distributions*, Ann. Probab., 2:178-180, 1974.
Chen, *Poisson approximation for dependent trials*, Ann. Probab., 3:534-545, 1975.

Mod-ϕ approximation schemes

- Even with full knowledge of the Laurent series coefficients $(b_{k,n}, c_{k,n})_{k=1}^{r}$ of $\psi_n(\xi)$, the measures $(\nu_n^{(r)})_{n\in\mathbb{N}}$ may be difficult to compute.

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- But expectations under the measures $\nu_n^{(r)}$ admit a natural functional interpretation.
- For any square-integrable function f we have:

$$\sum_{k\in\mathbb{Z}} f(j)\,\nu_n^{(r)}(\{j\}) = \mathbb{E}\left[f_r(Y_n)\right],$$

where Y_n follows the reference i.d. law with exponent $\lambda_n\phi$ and f_r is given by:

$$f_r(j) = f(j) + \sum_{k=1}^{r} b_{k,n}(\Delta_+^k(f))(j) + \sum_{k=1}^{r} c_{k,n}(\Delta_-^k(f))(j),$$

where Δ_+^k (resp. Δ_-^k) is the k-th order forward (resp. backward) finite difference operator:

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- Therefore we have the approximation $\mathbb{E}\left[f(X_n)\right] \approx \mathbb{E}\left[f_r(Y_n)\right]$.

Mod-Poisson approximation for credit risk models

Portfolio credit risk models

We want to study the distribution of the total portfolio losses:

$$L_n = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} E_i \cdot \mathbb{1}_{D_i}$$

where

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Mixture/threshold/copula models

Key assumption: obligors default independently conditionally on a mixing variable Ψ.

$$\mathbb{P}(D_i|\Psi) = p_i(\Psi),$$

for example in the (infamous) Gaussian copula model one has

$$p_i(\Psi) = \Phi\left(\frac{d_i - \sqrt{\rho}\cdot\Psi}{\sqrt{1-\rho}}\right),$$

where $\Psi \sim \mathcal{N}(0,1)$, Φ denotes the standard Gaussian c.d.f., $\rho \in (0,1)$ is an equicorrelation coefficient.

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i.e. you need VaR(99%) in capital to cover total portfolio losses in 99% of cases.

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- *Expected Shortfall* (ES):

$$\mathrm{ES}(\alpha) := \mathbb{E}\left[L_n \mid L_n \geq \mathrm{VaR}(\alpha)\right] = \frac{1}{1-\alpha} \int_0^1 \mathrm{VaR}(\alpha) du$$

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Pricing of credit derivatives:

- The price of a CDO tranche can be expressed as a sum of expectations of *call functions*:

$$\mathbb{E}\left[\max\{L_n - K, 0\}\right], \quad K \in \mathbb{R}.$$

$\implies$ requires good estimates for $\mathbb{E}[f(L_n)]$.

Mod-Poisson approximation

Mod-Poisson approximation for credit portfolios:

1. Approximate $L_n|\Psi$ using the mod-Poisson approximation scheme of order r.
2. Obtain estimates $\mathbb{P}(L_n > t|\Psi)$ and $\mathbb{E}[f(L_n)|\Psi]$.
3. Integrate numerically over Ψ.

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If $f(x) = \mathbb{1}_{\{x \geq t\}}$ or $f(x) = \max\{x - t, 0\}$, with $t \in \mathbb{R}$, we have:

$$f_r(j) = f(j) + \underbrace{\sum_{k=1}^{r} b_{k,n}(\Delta_+^k(f))(j)}_{=:\Delta(x)}$$

with $\Delta(x) \neq 0$ only on $[t - r, r] \cap \mathbb{N}$.

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Therefore if $Y_n = \text{Poisson}\ (\lambda_n)$, with $\lambda_n = \sum_{i=1}^{n} p_i$, one has:

$$\mathbb{P}(L_n \geq t) \approx \mathbb{P}(Y_n \geq t) + \mathbb{E}[\Delta(Y_n)]$$

$$\mathbb{E}[\max\{L_n - t, 0\}] \approx \lambda_n \mathbb{P}(Y_n \geq \lceil t \rceil - 1) - t\mathbb{P}(Y_n \geq \lceil t \rceil) + \mathbb{E}[\Delta(Y_n)]$$

where

- $\mathbb{P}(Y_n \geq t)$ can be computed efficiently as the (upper) incomplete Gamma function,
- $\mathbb{E}[\Delta(Y_n)]$ is numerically inexpensive, because $\Delta(x)$ is finitely supported (on at most r points).

Estimation of risk measures

Overview of other methods

Commonly used methods in credit risk:

- Panjer recursion,
- Monte Carlo simulation,
- Importance sampling,
- Large deviations theory.

Panjer recursion (benchmark):

- General recursive technique for compound distributions $L = \sum_{i=1}^{N} X_i$ with compounding distribution of Panjer class, i.e.

$$\mathbb{P}\left(N = k\right) = \left(a + \frac{b}{k}\right) \mathbb{P}\left(N = k - 1\right).$$

- Conditionally on $\Psi = \psi$:
    1. For $m = 1$, set $\mathbb{P}^{(1)}(L = 0) = 1 - p_1$
    2. For $m = 2, 3, \ldots, n$, set $\mathbb{P}^{(m)}(L = k) = \mathbb{P}^{(m-1)}(L = k)(1 - p_m) + \mathbb{P}^{(m-1)}(L = k - 1)p_m$, for $k = 0, 1, \ldots, m$.
- **Pros**: for $L_i \sim \mathit{Bernoulli}(p_i)$, the result is exact,
- **Cons**: computationally expensive ($O(n^2)$), infeasible for high values of n.

Overview of other methods

Monte Carlo simulation:

- Draw m simulations, $\{\psi_1, \ldots, \psi_m\}$, of $\Psi \sim N(0,1)$.
- For each draw ψ_i simulate $L_j^{(i)} \sim \text{Bernoulli}(p_j(\psi_i))$ and compute $L_n^{(i)} = \sum_{j=1}^{n} L_j^{(i)}$.
- Compute estimate $\mathbb{P}(L_n > x) \approx \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} \mathbb{1}(\{L_n^{(j)} > x\})$
- **Pros**: simple, additional sources of randomness (e.g. random exposures, E_i) are easy to implement, good for all copula models.
- **Cons**: computationally expensive, both in time and memory (probabilities of order 10^{-m} require 10^m simulations).

Overview of other methods

Importance Sampling[5]

- Draw m simulations, $\{\psi_1, \ldots, \psi_m\}$, of $\Psi \sim N(\mu, 1)$, where μ is (an approximation to) the exponential tilting that minimizes $\text{Var}(\mathbb{E}[\hat{p}_x|\Psi])$.
- For each draw ψ_i, find optimal exponential tilting $s_x(\psi_i)$, which solves $\frac{\partial}{\partial s}\phi(s|\psi_i) = x$ and sample $\tilde{L}_n^{(i)}$ from this tilted distribution.
- Compute estimate

$$\mathbb{P}(L_n > x) \approx \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}(\{\tilde{L}_n^{(j)} > x\}) \exp\left(-s_x(\psi_j)\tilde{L}_n^{(j)} + \phi(s_x(\psi_i)|\psi_i) - \mu\psi_i + \frac{1}{2}\mu^2\right)$$

- **Pros**: reasonably accurate for the whole tail with only $m \approx 10000$ simulations.
- **Cons**: available for Gaussian copula only (and even in the Gaussian case optimization of an approximated objective function only), estimate is affected by sampling noise.

[5] Glasserman, Paul, and Jingyi Li. "Importance sampling for portfolio credit risk." *Management science* 51.11 (2005): 1643-1656.

Large deviations approximation[6]:

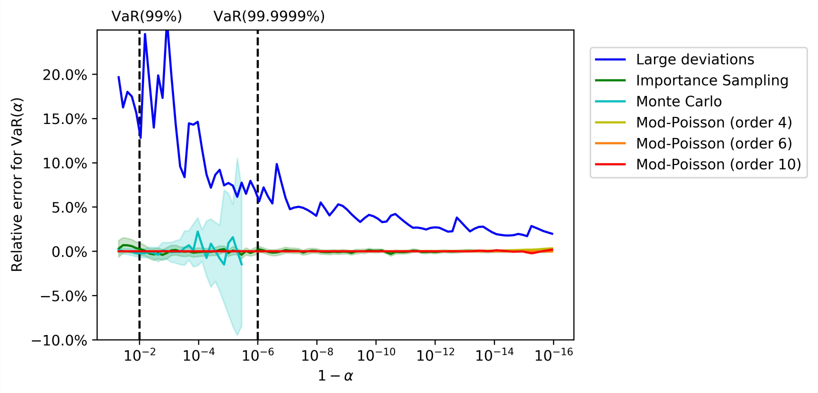
- Find the optimal exponential tilting, $s_x(\Psi)$ that solves $\frac{\partial}{\partial s}\phi(s|\Psi) = x$, where ϕ is the cumulant generating function of n independent Bernoulli rvs with probabilities $p_i(\Psi)$.
- Integrate numerically over Ψ the classical large deviations approximation:

$$\begin{aligned}\hat{p}_x &= \int p_n(\psi) F_\Psi(d\psi) \\ &= \int \frac{1}{\sqrt{2\pi n s_x(\psi)\phi''(s_x(\psi))}} \exp\left(-n\left(s(x,\psi)x - \phi(s_x(\psi)|\psi)\right)\right) F_\Psi(d\psi)\end{aligned}$$

- **Pros**: approximation is analytical $\Rightarrow$ computationally fast, no sampling noise
- **Cons**: numerically unstable (integrability issues with $\phi''(s_x(\psi))^{-1/2}$), convergence is slow in n, especially early in the tail.

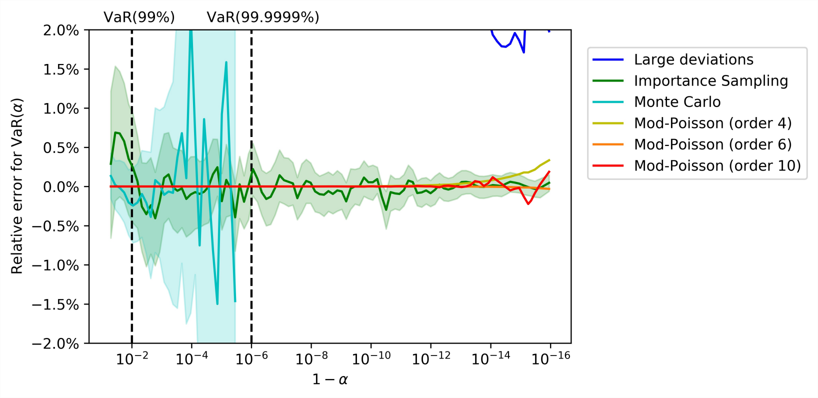
[6] Dembo, Amir, Jean-Dominique Deuschel, and Darrell Duffie. "Large portfolio losses." *Finance and Stochastics* 8.1 (2004): 3-16.

Estimation of risk measures: results



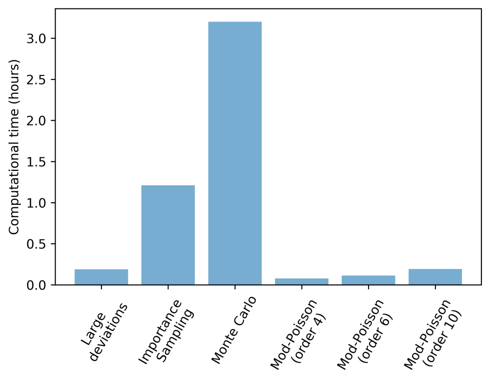
- All relative errors computed with respect to exact value (Panjer recursion),
- Large deviations converges too slowly
- Monte Carlo estimator variance diverges

Estimation of risk measures: results



Estimation of risk measures: results

Computational time



- Semi-analytical methods (e.g. mod-Poisson and large deviations) are faster,
- For mod-Poisson the time scales linearly in the approximation order r.

CDO pricing

CDO pricing

Important application in credit risk: pricing of derivatives underwritten on the credit portfolio.

CDO pricing requires the estimation of two different cashflows:

- **Premium leg**

$$V^{\text{Premium}}(x) = x \sum_{n=1}^{N} e^{-rt_n}(t_n - t_{n-1}) \mathbb{E}\left[(K_{j-1} - L_n(t_n))^+ - (K_j - L_n(t_n))^+\right]$$

- **Default leg**

$$V^{\text{Default}} = \mathbb{E}\left[\int_0^T e^{-rt} dL_t^{[K_{j-1},K_j]}\right] \approx \sum_{n=1}^{N} e^{-rt_n} \mathbb{E}\left[L_{t_n}^{[K_{j-1},K_j]} - L_{t_{n-1}}^{[K_{j-1},K_j]}\right]$$

The **fair spread** of the CDO can be found by solving for x:

$$V^{\text{Premium}}(x) = V^{\text{Default}}$$

$\Longrightarrow$ We need to estimate well $\mathbb{E}[(L_n - K)^+]$, for many values of K in the range of L_n.

Overview of methods

Stein's method approximation[7]:

- **First-order Gaussian approximation:** conditionally on Ψ approximate $\tilde{L}_n = \sum_{j=1}^n (L_j - \mathbb{E}[L_j])$ by Gaussian rv:

$$\mathbb{E}[h(\tilde{L}_n)] \approx \mathbb{E}[h(Z)] + \frac{\sum_{j=1}^n \mathbb{E}[L_j^3]}{2\sigma^4} \mathbb{E}[\tilde{h}(Z)]$$

  where $\sigma^2 = \mathrm{Var}(L_n)$, $Z \sim N(0, \sigma^2)$ and $\tilde{h}(x) = \left(\frac{x^2}{3\sigma^2} - 1\right) x h(x)$.

- **First-order Poisson approximation:** conditionally on Ψ approximate $L_n = \sum_{j=1}^n L_j$ by Poisson rv:

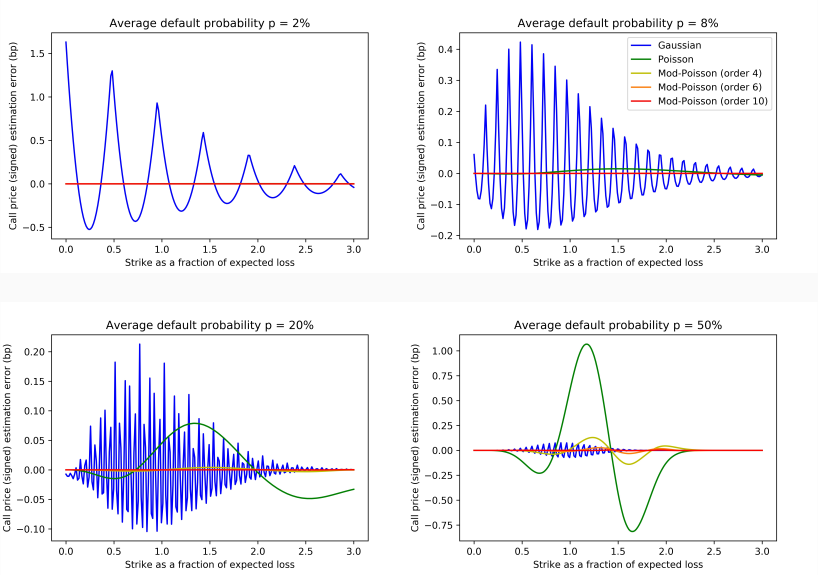
$$\mathbb{E}[h(L_n)] \approx \mathbb{E}[h(Z)] + \frac{\sigma^2 - \lambda}{2} \mathbb{E}[\tilde{h}(Z)]$$

  where $\sigma^2 = \mathrm{Var}(L_n)$, $\lambda = \mathbb{E}[L_n]$, $Z \sim \mathrm{Poi}(\lambda)$ and $\tilde{h}(x) = h(x+1) - h(x)$. Both estimates require a numerical integration over Ψ.

- **Pros**: approximation is analytical $\Rightarrow$ computationally fast, no sampling noise, not just for tail estimations
- **Cons**: $h \in C^2$, only first order.

[7] El Karoui, Nicole, and Ying Jiao. "Stein's method and zero bias transformation for CDO tranche pricing." *Finance and Stochastics* 13.2 (2009): 151-180.

CDO pricing: results



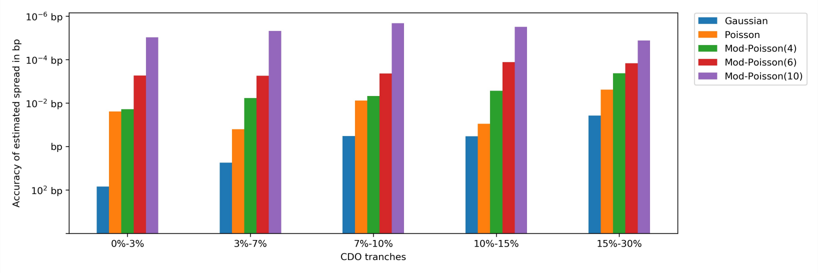
- The mod-Poisson approximation is accurate outside the domain of the Poisson theorem.

CDO pricing: results

Attachment points		Benchmark (recursive)	Gaussian approximation	Poisson approximation	Mod-Poisson (order=4)	Mod-Poisson (order=6)
0% - 3%	Default leg	232.5975 bp	228.8759 bp	232.5996 bp	232.5979 bp	232.5974 bp
	Premium leg	452.2145 bp	451.0626 bp	452.2208 bp	452.2137 bp	452.2145 bp
	Fair spread	5143.5210 bp	5074.1488 bp	5143.4961 bp	5143.5404 bp	5143.5204 bp
3% - 7%	Default leg	200.2722 bp	200.7338 bp	200.2540 bp	200.2716 bp	200.2723 bp
	Premium leg	1364.6971 bp	1362.7014 bp	1364.7217 bp	1364.6987 bp	1364.6971 bp
	Fair spread	1467.5213 bp	1473.0575 bp	1467.3613 bp	1467.5153 bp	1467.5218 bp
7% - 10%	Default leg	62.8105 bp	62.7749 bp	62.8088 bp	62.8099 bp	62.8104 bp
	Premium leg	1248.7606 bp	1248.8878 bp	1248.7468 bp	1248.7608 bp	1248.7606 bp
	Fair spread	502.9824 bp	502.6464 bp	502.9747 bp	502.9777 bp	502.9820 bp
10% - 15%	Default leg	33.6304 bp	33.5575 bp	33.6500 bp	33.6310 bp	33.6304 bp
	Premium leg	2204.4540 bp	2204.5755 bp	2204.4246 bp	2204.4529 bp	2204.4540 bp
	Fair spread	152.5566 bp	152.2176 bp	152.6473 bp	152.5594 bp	152.5565 bp
15% - 30%	Default leg	7.2444 bp	7.2698 bp	7.2461 bp	7.2447 bp	7.2445 bp
	Premium leg	6738.6074 bp	6738.5758 bp	6738.6165 bp	6738.6076 bp	6738.6073 bp
	Fair spread	10.7506 bp	10.7883 bp	10.7531 bp	10.7511 bp	10.7508 bp

Table 1: Default leg, premium leg and fair spread for five tranches computed using different techniques. Benchmark values are exact and computed using the recursive methodology.

CDO pricing: results



- Accuracy for the estimation of the fair spread grows exponentially with the order of the mod-Poisson approximation.
- Poisson approximation via mod-Poisson remains good across all tranches (not just senior ones).

Conclusions

Recap of results:

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Thanks for your attention!