

Lösung 8

1. $\lim_{h \rightarrow 0} \frac{2(x+h)^2 - 4 - 2x^2 + 4}{h} = \lim_{h \rightarrow 0} \frac{4xh + h^2}{h} = \lim_{h \rightarrow 0} (4x + h) = 4x$
2. (a) $\lim_{h \rightarrow 0} \frac{c-c}{h} = 0,$
 (b) $\lim_{h \rightarrow 0} \frac{m(x+h) + b - mx - b}{h} = \lim_{h \rightarrow 0} \frac{mh}{h} = m,$
 (c) $\lim_{h \rightarrow 0} \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h} = \lim_{h \rightarrow 0} \frac{2axh + ah^2 + bh}{h} = 2ax + b$
3. (a)

$$\begin{aligned} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \xrightarrow{h \rightarrow 0} \frac{1}{2\sqrt{x}} \end{aligned}$$

(b)

$$\begin{aligned} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} &= \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x+h}\sqrt{x})} \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \\ &= \frac{x - (x+h)}{h(\sqrt{x+h}\sqrt{x})(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{(\sqrt{x+h}\sqrt{x})(\sqrt{x} + \sqrt{x+h})} \xrightarrow{h \rightarrow 0} \frac{-1}{2x\sqrt{x}} \end{aligned}$$

4. (a)

$$\begin{aligned} (f(x) + g(x))' &= \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x) \end{aligned}$$

$$(b) \quad c \cdot f(x)' = \lim_{h \rightarrow 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = c \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = c \cdot f'(x)$$

5. Es gilt

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h} &= \lim_{h \rightarrow 0} \frac{(x+h+1)(x-1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)} = \lim_{h \rightarrow 0} \frac{-2h}{h(x-1)(x+h-1)} \\ &= -2 \lim_{h \rightarrow 0} \frac{1}{(x-1)(x+h-1)} = -\frac{2}{(x-1)^2} \end{aligned}$$

6. (a) überall differenzierbar, Ableitung: $x \mapsto 3x^2 - 2$, (b) differenzierbar für $x > 0$, Ableitung: $x \mapsto \frac{1}{2\sqrt{x}}$, (c) differenzierbar für $x \geq 0$, Ableitung: $x \mapsto \frac{3}{2}\sqrt{x}$ und (d) differenzierbar für $x \neq 0$, Ableitung: $x \mapsto \begin{cases} 1 & \text{falls } x > 0; \\ -1 & \text{falls } x < 0. \end{cases}$

$$7. (a) f'(x) = \frac{x(6x^2+6x) - (2x^3+3x^2-3)}{x^2} = \frac{6x^3+6x^2-2x^3-3x^2+3}{x^2} = 4x + 3 + \frac{3}{x^2}, f''(x) = 4 - \frac{6}{x^3} \text{ und}$$

$$(b) g'(t) = \sin(4\pi) - 4\sin(4t), g''(t) = -16\cos(4t)$$

$$8. (a) f'(x) = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{(e^x + e^{-x} - e^x + e^{-x})(e^x + e^{-x} + e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} \text{ und}$$

$$(b) f'(x) = \frac{1}{x} \cdot \frac{1}{\log x} = \frac{1}{x \log x}.$$

9.

$$\begin{aligned} (\tan x)' &= \left(\frac{\sin x}{\cos x} \right)' = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \left(\frac{\sin x}{\cos x} \right)^2 = 1 + \tan^2 x \end{aligned}$$