Lösung 8

1. $\lim_{h\to 0} \frac{2(x+h)^2-4-2x^2+4}{h} = \lim_{h\to 0} \frac{4xh+h^2}{h} = \lim_{h\to 0}(4x+h) = 4x$
2. (a) $\lim_{h\to 0} \frac{c-c}{h} = 0$,

   (b) $\lim_{h\to 0} \frac{m(x+h)+b-mx-b}{h} = \lim_{h\to 0} \frac{mh}{h} = m$,

   (c) $\lim_{h\to 0} \frac{a(x+h)^2+b(x+h)+c-ax^2-bx-c}{h} = \lim_{h\to 0} \frac{2axh+ah^2+bh}{h} = 2ax+b$
3. (a)

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+h}+\sqrt{x}} \overset{h\to 0}{\longrightarrow} \frac{1}{2\sqrt{x}}\end{aligned}$$

   (b)

$$\begin{aligned}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h} &= \frac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x+h}\sqrt{x})}\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\\ &= \frac{x-(x+h)}{h(\sqrt{x+h}\sqrt{x})(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{(\sqrt{x+h}\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \overset{h\to 0}{\longrightarrow} \frac{-1}{2x\sqrt{x}}\end{aligned}$$

4. (a)

$$\begin{aligned}(f(x)+g(x))' &= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)\end{aligned}$$

   (b) $c\cdot f(x))' = \lim_{h\to 0} \frac{c\cdot f(x+h)-c\cdot f(x)}{h} = c\cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = c\cdot f'(x)$
5. Es gilt

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}}{h} &= \lim_{h\to 0}\frac{(x+h+1)(x-1)-(x+1)(x+h-1)}{h(x-1)(x+h-1)} = \lim_{h\to 0}\frac{-2h}{h(x-1)(x+h-1)}\\ &= -2\lim_{h\to 0}\frac{1}{(x-1)(x+h-1)} = -\frac{2}{(x-1)^2}\end{aligned}$$

6. (a) überall differenzierbar, Ableitung: $x \mapsto 3x^2-2$, (b) differenzierbar für $x>0$, Ableitung: $x \mapsto \frac{1}{2\sqrt{x}}$, (c) differenzierbar für $x \geq 0$, Ableitung: $x \mapsto \frac{3}{2}\sqrt{x}$ und (d) differenzierbar für $x \neq 0$, Ableitung: $x \mapsto \begin{cases} 1 & \text{falls } x>0; \\ -1 & \text{falls } x<0. \end{cases}$
7. (a) $f'(x) = \frac{x(6x^2+6x)-(2x^3+3x^2-3)}{x^2} = \frac{6x^3+6x^2-2x^3-3x^2+3}{x} = 4x+3+\frac{3}{x^2}$, $f''(x) = 4-\frac{6}{x^3}$ und (b) $g'(t) = \sin(4\pi)-4\sin(4t)$, $g''(t) = -16\cos(4t)$
8. (a) $f'(x) = \frac{(\mathrm{e}^x+\mathrm{e}^{-x})^2-(\mathrm{e}^x-\mathrm{e}^{-x})^2}{(\mathrm{e}^x+\mathrm{e}^{-x})^2} = \frac{(\mathrm{e}^x+\mathrm{e}^{-x}-\mathrm{e}^x+\mathrm{e}^{-x})(\mathrm{e}^x+\mathrm{e}^{-x}+\mathrm{e}^x-\mathrm{e}^{-x})}{(\mathrm{e}^x+\mathrm{e}^{-x})^2} = \frac{4}{(\mathrm{e}^x+\mathrm{e}^{-x})^2}$ und

   (b) $f'(x) = \frac{1}{x}\cdot\frac{1}{\log x} = \frac{1}{x\log x}$.
9.

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cos x+\sin x\sin x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}\\ &= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = 1+\left(\frac{\sin x}{\cos x}\right)^2 = 1+\tan^2 x\end{aligned}$$